# Homework 4 and 5
## Solutions

(Q1)

#2 p. 293-294

Obtain the Laurent expansion wrt z.

a) State the 1st 4 non-zero terms.

b) State explicitly the $n^{th}$ term in the series, and c) state the largest possible annular domain in which your series is a valid rep'n of the fcn,

$$\frac{\cos(1/z)}{z^3}.$$

a) Use $\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots + \frac{(-1)^n z^{2n}}{2n!} + \cdots$

~~about z=0~~.

So for $z \neq 0$,

$$\frac{\cos(1/z)}{z^3} = \frac{1}{z^3}\left[1 - \frac{1/z^2}{2!} + \frac{1/z^4}{4!} + \cdots + \frac{(-1)^n z^{-2n}}{2n!} + \cdots\right]$$

$$= z^{-3} - \frac{z^{-5}}{2!} + \frac{z^{-7}}{4!} - \frac{z^{-9}}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \left(z^{-2n-3}\right)}{2n!}$$

$$\underset{n \mapsto -1-m}{\text{under}} = \sum_{m=-\infty}^{-1} (-1)^{m+1} \frac{z^{2m-1}}{(-2-2m)!}$$

Thus, $U_n = (-1)^{n+1} \dfrac{z^{2n-1}}{(-2-2n)!}$ and this expansion is valid for $|z|>0$. (ps. sol'n manuals aren't always correct!)

Q2

#4 Same 3 parts as for #2, but for $\text{Log}\left[1 + \dfrac{1}{z-1}\right]$ expanded wrt $z-1$.

~~power~~ Recall that

$$-\text{Log}(1-\tau) = \tau + \frac{\tau^2}{2} + \frac{\tau^3}{3} + \cdots + \frac{\tau^n}{n} + \cdots$$

for $|\tau|<1$

So $\text{Log}\left(1+\dfrac{1}{z-1}\right) = \text{Log}\left(1 - \dfrac{1}{1-z}\right)$

$$= -\left(-\text{Log}\,(1-(-\tau))\right)$$

for $-\tau = \dfrac{1}{1-z}$, $+\tau = \dfrac{1}{z-1}$, $|\tau| < 1$.

$$= (-1) \sum_{n=1}^{\infty} \frac{(-\tau)^n}{n}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\tau^n}{n}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(z-1)^{-n}}{n}$$

under $n \mapsto -n$
$$= \sum_{n=-\infty}^{-1} (-1)^{n} \frac{(z-1)^{n}}{n}$$

Thus, a) $= \frac{1}{(z-1)} - \frac{(z-1)^{-2}}{2} + \frac{(z-1)^{3}}{3} - \frac{(z-1)^{4}}{4} + \dots$

b) $u_n = (-1)^n \frac{(z-1)^n}{n}$

c) $\left|\frac{1}{z-1}\right| < 1$ or equivalently

$|z-1| > 1$

Q3

#9 Expand $\frac{1}{z+2}$ in a Laurent series wrt $z-i$. a) State the center + inner radius of the domain. b) Give the $n^{th}$ term of the series.

Note that,

$$\frac{1}{z+2} = \frac{1}{(z-i)+(i+2)} = \frac{1}{z-i}\left(\frac{1}{1+\left(\frac{i+2}{z-i}\right)}\right)$$

using the Laurent expansion of $\frac{1}{1-\omega} = 1+\omega+\omega^2+\dots$ for $|\omega|<1$ with $\omega = (-1)\left(\frac{i+2}{z-i}\right)$

We get that

$$\frac{1}{z+2} = \frac{1}{z-i}\left(1 - \frac{i+2}{z-i} + \left(\frac{i+2}{z-i}\right)^2 - \cdots\right)$$

for $\left|\frac{i+2}{z-i}\right| < 1$ ~~or~~ in other words, for

(a) $|z-i| > |i+2| = \sqrt{5}$.

(b) In terms of n,

$$\frac{1}{z-i} - \frac{i+2}{(z-i)^2} + \frac{(i+2)^2}{(z-i)^3} - \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(2+i)^n}{(z-i)^{n+1}}$$

$-m = n+1$

$$= \sum_{m=-\infty}^{-1} (-1)^{m+1} (2+i)^{-1-m} (z-i)^m$$

So the $m^{th}$ coefficient in the series is

$$(-1)^{m+1}(2+i)^{-1-m}.$$

Q4

~~#3~~ #23(a)

Show that in the expansion

$$\frac{1}{\sin z} = \sum_{n=-1}^{\infty} c_n z^n \quad , \quad 0 < |z| < \pi$$

we can get $c_n$ from the recursion

$$c_n = \left[ \frac{c_{n-2}}{3!} - \frac{c_{n-4}}{5!} + \frac{c_{n-6}}{7!} + \cdots \pm \frac{c_{-1}}{(n+2)!} \right]$$

when $n$ is odd. Recall that $c_{-1} = 1$.

From Example 4,

$$1 = \left( z - \frac{z^3}{3!} + \cdots \right)\left( \sum_{n=-1}^{\infty} c_n z^n \right)$$

$$= \sum_{n=-1}^{\infty} c_n z^{n+1} - \sum_{n=1}^{\infty} \frac{c_{n-2}}{3!} z^{n+1} + \sum_{n=3}^{\infty} \frac{c_{n-4}}{5!} z^{n+1} + \cdots$$

Thus, for $n$ odd, $n > -1$, by looking at the $z^{n+1}$-th coefficient we get

$$0 = \left[ c_n - \frac{c_{n-2}}{3!} + \frac{c_{n-4}}{5!} - \cdots + (-1)^{\frac{n+1}{2}} \frac{c_{-1}}{(n+2)!} \right]$$

and hence,

$$c_n = \frac{c_{n-2}}{3!} - \frac{c_{n-4}}{5!} + \cdots + (-1)^{\frac{n-1}{2}} \frac{c_{-1}}{(n+2)!}$$

Q5

#~~6~~ p. 340-1

Evaluate $\oint \sum_{n=-\infty}^{\infty} \cosh\left(\frac{1}{z}\right) dz$ around the

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square with corners at $\pm(1 \pm i)$ using the method of residues and using the

a) Laurent expansions valid in deleted nbhds of the singular pts to b) get the residue.

Since $\cosh \tau = \sum_{k=0}^{\infty} \frac{\tau^{2k}}{(2k)!}$ is holomorphic,

a) $\cosh\left(\frac{1}{z}\right) = \sum_{k=0}^{\infty} \frac{z^{-2k}}{(2k)!}$ has an isolated pole @ $z=0$, and this expansion is valid for $|z| \neq 0$. Thus the residue at $z=0$ is zero and

b)

$$\oint \cosh\left(\frac{1}{z}\right) = 0$$

since $z=0$ is on the interior of the square.

Q6

#7 Evaluate $\oint z \sin\left(\frac{1}{z-1}\right) dz$ around $|z|=2$ using a), b) as in Q5.

$z\sin\left(\frac{1}{z-1}\right)$ has an isolated singularity @ $z=1$ in the interior of $|z|=2$, so we can evaluate the integral using the L. Exp. @ $z=1$.

(a) $z \sin\left(\frac{1}{z-1}\right) = (1+(z-1)) \sin\left(\frac{1}{z-1}\right)$

using $\sin \omega = \sum_{k=0}^{\infty} \frac{(-1)^k \omega^{2k+1}}{(2k+1)!}$

$$= \underbrace{\sum_{k=0}^{\infty} (-1)^k \frac{(z-1)^{-2k-1}}{(2k+1)!}}_{\text{odd terms}} + \underbrace{\sum_{k=0}^{\infty} (-1)^k \frac{(z-1)^{-2k}}{(2k+1)!}}_{\text{even terms}}$$

(b) So the coefficient of $(z-1)^{-1}$ is

$$\frac{(-1)^0}{(2(0)+1)!} = 1 = \text{residue } [f(z), 1]$$

Thus $\oint z \sin\left(\frac{1}{z-1}\right) = 2\pi i (1) = 2\pi i.$

Q7

#2 p.350-1

a) Using the Laurent expansion, show that the function $(z-1)^3 \cosh\left(\frac{1}{z-1}\right)$ has an essential singularity at $z=1$. b) State the residue and give $c_{-2}, c_{-1}, c_0, c_1$.

(a) As in Q5, we get

$$(z-1)^3 \cosh\left(\frac{1}{z-1}\right) = (z-1)^3 \sum_{k=0}^{\infty} \frac{(z-1)^{-2k}}{(2k)!}$$

$$= \sum_{k=0}^{\infty} \frac{(z-1)^{-2k+3}}{(2k)!}$$

$$= \frac{(z-1)^3}{} + \frac{(z-1)}{2!} + \frac{(z-1)^{-1}}{4!} + \frac{(z-1)^{-3}}{6!} + \cdots$$

Since the L.E. has $\infty$-many negative terms,

$\Rightarrow$ essential sing. @ $z=1$.

(b) Res $= c_{-1} = \frac{1}{4!}$, $c_{-2} = c_0 = 0$, $c_1 = \frac{1}{2!}$

Q8

#8 (a) Show $f(z) = \frac{e^{z}-1}{z}$ has a removable sing. @ $z=0$ by showing $\lim_{z \to z_0 = 0} f(z)$ exists and is finite.

(b) State how $f(z_0)$ should be defined in order to remove the singularity.

(a) $$\frac{e^z - 1}{z} = \left(\frac{1}{z}\right)\left(-1 + \sum_{k=0}^{\infty} \frac{z^k}{k!}\right)$$

$$= \sum_{k=1}^{\infty} \frac{z^{k-1}}{k!} = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots$$

Thus $\lim_{z \to 0} f(z) = 1$. $\Rightarrow$ removable sing. @ $z=0$.

(b) Define $f(0) = 1$.