Northwestern University

Eigenfunction restriction bounds for Neumann data along hypersurfaces

J. Toth (joint work with H. Christianson and A. Hassell)

March 10, 2014

Background

- (M^n, g) compact, smooth manifold (with or without boundary).

- $H \subset M^n$ an orientable smooth hypersurface. In some cases, H can be a higher-codimension submanifold.

Cauchy data along H

- Consider the eigenvalue problem on M

$$-\Delta_g \phi_j = \lambda_j^2 \phi_j, \quad \langle \phi_j, \phi_k \rangle = \delta_{jk}$$

$$B\phi_j = 0 \text{ on } \partial M,$$

where $\langle f, g \rangle = \int_M f \bar{g} dV$ (dV is the volume form of the metric) and where B is the boundary operator, e.g. $B\phi = \phi|_{\partial M}$ in the

Dirichlet case or $B\phi = \partial_\nu \phi|_{\partial M}$ in the Neumann case. We also allow $\partial M = \emptyset$.

- Let $h_j = \lambda_j^{-1}$ and ϕ_{h_j} be a corresponding orthonormal basis of eigenfunctions with eigenvalue h_j^{-2}, so that the eigenvalue problem takes the semi-classical form,

$$(-h^2 \Delta_g - 1)\phi_h = 0,$$

$$B\phi_h = 0 \text{ on } \partial M$$

where $B = I$ or $B = hD_\nu$ in the Dirichlet or Neumann cases respectively.

- Semiclassical Cauchy data along H:

$$CD(\phi_h) := \{(\phi_h|_H,\ hD_\nu \phi_h|_H)\}.$$

- **Problem**: Upper (and lower) bounds for

$$\|\phi_h\|_{L^2(H)} \ (Dirichlet),$$

$$\|hD_\nu \phi_h\|_{L^2(H)} \ (Neumann).$$

- A lot of recent work on upper bounds for Dirichlet data along H. We considered Neumann data (closely linked with Dirichlet via Rellich identity).

- **Theorem 1 [Christianson-Hassell-T]** Let $H \subset M$ be any oriented, smooth separating hypersurface with $H \cap \partial M = \emptyset$. Then,

$$\|h\partial_\nu \phi_h\|_{L^2(H)} = O(1).$$

- Result holds for eigenfunctions ϕ_h of general Schrödinger operators $P(h) = -h^2\Delta_g + V(x)$ with $V \in C^\infty(M; \mathbf{R})$ and

$$P(h)\phi_h = E(h)\phi_h, \ E(h) = E + O(h),$$

E regular energy value and $H \subset \{V(x) < E\}$.

Cauchy data along H: Rellich identity

- Let $H \subset M$ be an oriented separating hypersurface with exterior unit normal ν bounding smooth domain $M_H \subset M$.

- **Rellich identity.** Self-adjointness of Δ_g, the eigenfunction equation $-h^2 \Delta_g \phi_h = \phi_h$ and an easy application of Green's formula gives with *any* $A(h) \in \Psi_h^m$ and $V \in C^\infty(M, \mathbf{R})$,

$$\frac{i}{h} \langle [-h^2 \Delta_g + V, A(h)] \phi_h, \phi_h \rangle_{M_H}$$

$$= \langle A(h) \phi_h, h D_\nu \phi_h \rangle_H + \langle h D_\nu A(h) \phi_h, \phi_h \rangle_H. \quad (*)$$

- Key identity relating Dirichlet and Neumann data along H to interior eigenfunctions on M.

- Let (x_n, x') be Fermi coordinates near H with $H = \{x_n = 0\}$ and $\chi \in C_0^\infty([-\delta, \delta])$ equal to 1 near origin. Idea of proof of [CHT] is to apply Rellich with $A(h) = \chi(x_n) h D_{x_n}$.

- In this case, Rellich gives

$$\frac{i}{h} \langle [-h^2 \Delta_g + V, \chi h D_\nu] \phi_h, \phi_h \rangle_{M_H}$$

$$= \langle h D_\nu \phi_h, h D_\nu \phi_h \rangle_H + \langle (I + h^2 \Delta_H) \phi_h, \phi_h \rangle_H.$$

- Formula has other applications such as in the case where (ϕ_h) is quantum ergodic (QE); that is, for any $B(h) \in \Psi_h^0(M)$,

$$\langle B(h) \phi_h, \phi_h \rangle_{L^2(M)} \sim_{h \to 0^+} \int_{B^*M} b(x, \xi) dx d\xi.$$

Quantum ergodic restriction for Cauchy data (QERCD)

- **Theorem 1 [Christianson-Zelditch-T]** Suppose $H \subset M$ is a smooth, codimension 1 embedded orientable separating hypersurface and assume $H \cap \partial M = \emptyset$ if $\partial M \neq \emptyset$. Assume that $\{\phi_h\}$ is an interior QE sequence. Then the appropriately renormalized Cauchy data $d\Phi_h^{CD}$ of ϕ_h is quantum ergodic in the sense that for any $a^w \in \Psi^0(H)$, there exists a sub-sequence of eigenvalues of density one so that as $h_j \to 0^+$,

$$\langle a^w hD_\nu \phi_h|_H, hD_\nu \phi_h|_H \rangle_{L^2(H)}$$

$$+\langle a^w (1 + h^2 \Delta_H)\phi_h|_H, \phi_h|_H \rangle_{L^2(H)}$$

$$\to_{h\to 0^+} \frac{4}{\mu(S^*M)} \int_{B^*H} a_0(x', \xi')(1-|\xi'|^2)^{1/2} d\sigma.$$

- Here, $a_0(x', \xi')$ is the principal symbol of a^w, $-h^2\Delta_H$ is the induced tangential (semi-classical) Laplacian with principal symbol $|\xi'|^2$ and $d\sigma$ is the standard symplectic volume form on B^*H.

- Result holds for *all* interior hypersurfaces and generalizes results of Hassell-Zelditch and Burq in the boundary case (ie. $H = \partial\Omega$.)

- Idea of Proof in [CTZ]: Apply Rellich with $A(h) = \chi(x_n) h D_\nu$ and compute LHS with the commutator $\frac{i}{h}[-h^2\Delta_g, A(h)] \in \Psi_h^0$ applying QE assumption on the eigenfunction sequence (ϕ_h).

Dirichlet Data

- **Problem:** Estimate the L^2 restrictions

$$\int_H |\phi_\lambda(s)|^2 \, d\sigma(s). \qquad (1)$$

- **Rationale:** Key to understanding the large$-\lambda$ behaviour of the ϕ_λ's. Pointwise L^∞ results are very hard; difficult to improve on the bound

$$\|\phi_\lambda\|_{L^\infty(M)} = \mathcal{O}(\lambda^{\frac{n-1}{2}}).$$

The problem in (1) is easier but still very non-trivial.

2) Restriction bounds naturally arise in study of eigenfunction nodal sets. In particular. *lower* bounds of the form

$$\int_H |\phi_\lambda|^2 ds \geq e^{-C\lambda}, \; C > 0$$

are central to this problem.

3) Quantum ergodicity: Recent results (Zelditch-T, Dyatlov-Zworski) on **Quantum Ergodic Restriction (QER)** show that for generic H's satisfying a geodesic asymmetry condition relative to H and a density-one subsequence of eigenfunctions ϕ_λ,

$$\lim_{\lambda \to \infty} \langle Op_H(a)\phi_\lambda|_H, \phi_\lambda|_H \rangle_{L^2(H)}$$

$$= 2 \int_{B^*H} a(s,\sigma)(1 - |\sigma|^2)^{1/2} ds d\sigma.$$

- Most results extend to semiclassical Schrödinger operators $P(h) = -h^2\Delta + V(x)$ with eigenfunctions ϕ_h satisfying $P(h)\phi_h = E(h)\phi_h$, $|E(h) - E| = o(1)$, E a regular energy level.

General Results for Dirichlet data

- For general Laplace eigenfunctions with $\|\phi_\lambda\|_{L^2(M)} = 1$, Burq-Gérard-Tzvetkov [BGT] prove that

$$\int_H |\phi_\lambda|^2 \, d\sigma(s) = \mathcal{O}(\lambda^{\frac{1}{2}}), \ (n = 2). \qquad (2)$$

- The universal bound (2) is achieved on S^2 with $H = \{(x, y, z) \in S^2; z = 0\}$ the equator and $\phi_n(x, y, z) = c_0 n^{\frac{1}{4}}(x+iy)^n; n = 1, 2, 3, ...$, the highest-weight harmonics.

- In the case where H has positive geodesic curvature, the bound (2) improves to

$$\int_H |\phi_\lambda|^2 \, d\sigma(s) = \mathcal{O}(\lambda^{\frac{1}{3}}); \ (n = 2).$$

BGT also obtain sharp general L^p bounds for $p \neq 2$ in any dimension and Hu generalized the positively-curved results to any dimension. Hassell-Tacy have extended these L^p bounds to the semiclassical case where $P(h) = -h^2\Delta + V(x)$.

- Improvements for (M, g) non-positive curvature (Chen-Sogge, Sogge-Zelditch), flat tori with dim $= 2, 3$ (Bourgain-Rudnick), arithmetic surfaces (Jung, Rudnick-Sarnak, Ghosh-Reznikov-Sarnak), quantum completely integrable case (T), ...

Neumann Data along H

- Starting point is the Rellich identity with test operator $A(h) = \chi(x_n) h D_{x_n}$. We recall it here (with $V = 0$ for simplicity):

$$\frac{i}{h} \langle [-h^2 \Delta_g, \chi h D_\nu] \phi_h, \phi_h \rangle_{M_H}$$

$$= \langle h D_\nu \phi_h, h D_\nu \phi_h \rangle_H + \langle (I + h^2 \Delta_H) \phi_h, \phi_h \rangle_H.$$

- Let $\gamma_H : C^0(M) \to C^0(H)$ be restriction. Consider spectral projector $N(h) = \sum_j \chi(h^{-1} - h_j^{-1}) \phi_j(x) \overline{\phi_j(y)}$ with supp $\hat{\chi} \subset [\epsilon, 2\epsilon] > 0$. The kernel

$$N(x, y, h) = (2\pi h)^{-(n-1)/2} e^{i r(x,y)/h} a(x, y, h)$$

$$+ O(h^\infty)_{L^2 \to L^2}.$$

Writing

$$\gamma_H \phi_h = \gamma_H N(h) \phi_h,$$

it is not hard to show that

$$WF_h(\phi_h) \subset B^*H = \{(s,\eta) \in T^*H; |\eta|_{g(s)} \leq 1\}. \; (*)$$

- **Heuristic:** On the RHS of Rellich, the Dirichlet term $\langle (I + h^2\Delta_H)\phi_h, \phi_h \rangle_H$ looks like it should be "essentially" non-negative in view of $(*)$ since

$$\sigma(I + h^2\Delta_H)(s,\eta) = 1 - |\eta|^2_{g(s)}.$$

- If that is the case, we are done and would simply get

$$\|hD_\nu \phi_h\|_H^2 \leq \frac{i}{h} \langle [-h^2\Delta_g, \chi D_\nu \phi_h] \phi_h, \phi_h \rangle_{M_H} = O(1)$$

  where the last estimate follows by L^2-boundedness of $\frac{i}{h}[-h^2\Delta_g, \chi D_\nu \phi_h] \in \Psi_h^0(M)$.

- Unfortunately, we cannot quite prove this. Subtlety lies in mass concentration of $\phi_h|_H$ near glancing set S^*H on h^δ-scales with $\delta > 1/2$.

The example of the disc

- Consider Dirichlet eigenfunctions $\phi_\lambda(r,\theta)$ in the unit disc with eigenvalue λ. They are of the form

$$\phi_{\lambda,n}(r,\theta) = c_n J_n(\lambda r) e^{in\theta}, \;\; J_n(\lambda) = 0.$$

- Let $H = \{r = \frac{1}{2}\}$, so that

$$\phi^H_{\lambda,n}(\theta) = c_n J_n \left(\frac{\lambda}{2}\right) e^{in\theta}.$$

- Consider pairs (λ, n) with

$$\lambda = 2nzn^{1/3}, \; z \in [z_1, z_2].$$

These eigenfunctions peak near the caustic $H = \{r = \frac{1}{2}\}$ and can contain semiclassical frequencies $1 + cn^{-2/3}$ with

$$\lambda^{-1} \partial_r \phi_{\lambda,n} \approx n^{-1/6} Ai(2^{1/3} z).$$

Sketch of proof of Theorem 1

- Choose $a^w \in \Psi_h^*$ with principal symbol principal symbol

$$a(x,\xi) = \chi(x_n)\xi_n,$$

- We recall Rellich formula

$$\frac{i}{h}\int_{M_-}[-h^2\Delta - 1, a^w]\phi_h\overline{\phi_h}dx$$

$$= \int_H (hD_n a^w \phi_h)|_H \, \overline{\phi_h}|_H \, d\sigma_H$$

$$+ \int_H (a^w \phi_h)|_H \, \overline{hD_n\phi_h}|_H d\sigma_H.$$

- It follows that

$$\int_H (1+h^2\Delta_H)\phi_h^H \overline{\phi_h^H} d\sigma_H + \int_H |\phi_h^{H,\nu}|^2 d\sigma_H = O(1).$$

- In order to bound the Neumann data from above, need to show the first term on the left hand side of (**??**) is semi-bounded below.

- Use small scale decomposition of T^*H with $\chi_{in}, \chi_{tan}, \chi_{out}$ cutoffs supported in sets $|\sigma| < 1 - h^\delta, 1 - h^\delta < |\sigma| < 1 + h^\delta, |\sigma| > 1 + h^\delta$ respectively, satisfying

$$1 = (\chi_{in})^w_{h,\delta} + (\chi_{tan})^w_{h,\delta} + (\chi_{out})^w_{h,\delta}$$

- Here, we need to choose 2-microlocal scales with $\delta \in (1/2, 2/3)$.

- By Proposition on mass concentration of ϕ_h^H:

$$\int_H (1 + h^2 \Delta_H) \phi_h^H \overline{\phi_h^H} d\sigma_H$$

$$= \int_H (1 + h^2 \Delta_H)(\chi_{in})^w_{h,\delta} \phi_h^H \overline{\phi_h^H} d\sigma_H$$

$$+ \int_H (1 + h^2 \Delta_H)(\chi_{tan})^w_{h,\delta} \phi_h^H \overline{\phi_h^H} d\sigma_H + O(h^\infty).$$

- On the support of χ_{in}, we have $1 - |\sigma|^2 \geq h^\delta$ and Gårding inequality gives

$$\int_H (1 + h^2 \Delta_H)(\chi_{in})^w_{h,\delta} \phi_h^H \overline{\phi_h^H} d\sigma_H$$

$$\geq C_1 h^\delta \int_H (\chi_{in})^w_{h,\delta} \phi_h^H \overline{\phi_h^H} d\sigma_H.$$

- On the support of χ_{tan}, we have $|1-|\sigma|^2| \leq C_2 h^\delta$, so that

$$\left| \int_H (1+h^2\Delta_H)(\chi_{tan})^w_{h,\delta} \phi_h^H \overline{\phi_h^H} d\sigma_H \right|$$

$$\leq C_2 h^\delta \left| \int_H (\chi_{tan})^w_{h,\delta} \phi_h^H \overline{\phi_h^H} d\sigma_H \right|.$$

- Combining these two estimates,

$$\int_H (1+h^2\Delta_H) \phi_h^H \overline{\phi_h^H} d\sigma_H$$

$$\geq C_1 h^\delta \int_H (\chi_{in})^w_{h,\delta} \phi_h^H \overline{\phi_h^H} d\sigma_H$$

$$-C_2 h^\delta \left| \int_H (\chi_{tan})^w_{h,\delta} \phi_h^H \overline{\phi_h^H} d\sigma_H \right| + O(h^\infty)$$

$$\geq -C h^\delta \int_H |\phi_h^H|^2 d\sigma_H,$$

- exterior term is $O(h^\infty)$, so adding it back in is harmless.

- Use the $\|\phi_h^H\|_{L^2(H)} = O(h^{-1/4})$ bound of Burq-Gérard-Tzvetkov to get

$$\int_H (1 + h^2 \Delta_H) \phi_h^H \overline{\phi_h^H} d\sigma_H \geq -Ch^{\delta - 1/2}.$$

- Choosing $\delta > 1/2$ gives

$$-Ch^{\delta-1/2} + \int_H |hD_\nu \phi_h|^2 d\sigma_H = O(1)$$

and so,

$$\int_H |hD_\nu \phi_h|^2 d\sigma_H = O(1).$$

- Choosing $\delta \sim 2/3$ gives the best estimate

$$\|hD_\nu \phi_h\|_H^2 \leq \frac{1}{h} \left| \langle [-h^2 \Delta_g, \chi h D_n] \phi_h, \phi_h \rangle_{M_H} \right|$$

$$+ C_\epsilon h^{\frac{1}{6}+\epsilon}.$$

- **Corollary** If (ϕ_h) is QE sequence, then for any $\epsilon > 0$, and $h \in (0, h_0(\epsilon)]$,

$$\|hD_\nu \phi_h\|_H^2 \leq |S_H^* M| + \epsilon.$$

Open problems/questions

- When is it true that for $h < h_0$,

$$\|hD_\nu \phi_h\|_H^2 < |S_H^* M| ?$$

  An immediate corollary would be

$$\|\phi_h\|_H^2 \geq C > 0$$

  ie. strong unique continuation for the eigenfunction restrictions $\phi_h|_H$.

- Run Rellich with other test operators such as $A(h) = x_n \chi(x_n) h D_n$ with $H = \{x_n = 0\}$ to try to decouple the Dirichlet and Neumann data along H.