M366 HW 4 Solutions Sketch

P127. 1 (a)-(c), 3, 6, P133. 1, P156. 1, P248. 1, 4, 5, 10

P127. 1. (a) $A(\xi) - B(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi\xi(x-t)} dx = e^{i2\pi\xi t} \hat{f}(\xi) = 0$

(b) Since f is continuous and of moderate decrease, so
$\left|\int_{-\infty}^{t} f(x) e^{-i2\pi z(x-t)} dx\right| \le \int_{-\infty}^{t} \frac{A}{1+x^2} \le \int_{-\infty}^{\infty} \frac{A}{1+x^2} dx = A\pi$
for some $A > 0$. Hence $A(z)$ is bounded.

Next, let $F_n(z) = \int_{-n}^{t} f(x) e^{-i2\pi z(x-t)} dx$, where the integrand is holomorphic ~~[illegible]~~ and $(z, x) \mapsto f(x) e^{-i2\pi z(x-t)}$ is continuous on $H \times [-n, t]$.

So by Chapter 2 Thm 5.4, $F_n(z)$ is holomorphic in H.

Now, $|F_n(z) - A(z)| \le \int_{-\infty}^{-n} \frac{A}{1+x^2} dx = A\left(\frac{\pi}{2} - \arctan\frac{n}{2}\right)$ which goes to 0 as $n \to \infty$ for all $z \in \mathbb{C}$.

Hence, $F_n(z) \to A(z)$ uniformly as $n \to \infty$, so $A(z)$ is also holomorphic in H.

In summary, $A(z)$ is bounded & holomorphic in H.
Therefore, by Liouville Thm, $A(z)$ is a constant.
Same for $B(z)$.

Finally, fix $t > 0$, and let $\tilde{x} > 0$, then
$|A(i\tilde{x})| = \left|\int_{-\infty}^{t} f(x) e^{-i2\pi(i\tilde{x})\cdot(x-t)} dx\right| \le e^{-2\pi\tilde{x}t} \cdot A$
which goes to 0 as $\tilde{x} \to \infty$.
Since $A(z)$ is a constant, so $A(z) \equiv 0$.

(c) $A(0) = \int_{-\infty}^{t} f(x) dx = 0$, $\forall t$.
Also, $\frac{d}{dt}\int_{-\infty}^{t} f(x) dx = f(t) = 0$, $\forall t$.

3. Let $f(z) = \frac{a}{z^2+a^2} e^{-i2\pi\xi z}$ and let γ_R be upper half plane semi-circle with radius R. Then γ_R encloses pole ia.
Then $\int_{\gamma_R} f(z) dz = 2\pi i \cdot \mathrm{Res}_{ia}(f(z)) = \pi e^{-2\pi a|\xi|}$.
With similar arguments as in HW 3's solutions, we may conclude.
Finally, $\int_{-\infty}^{\infty} e^{-2\pi a|\xi|} e^{i2\pi\xi x} dx = \int_{-\infty}^{0} e^{2\pi a\xi + i2\pi\xi x} dx + \int_{0}^{\infty} \cdots dx$
$= \frac{1}{2\pi(a+ix)} + \frac{1}{2\pi(a-ix)} = \frac{a}{\pi(a^2+x^2)}$.

6. By P128.3, $f(x) = \frac{1}{\pi}\frac{a}{a^2+x^2}$, $\hat{f}(\xi) = e^{-2\pi a|\xi|}$.

Poisson summation formula gives

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$$

Hence, $\frac{1}{\pi}\sum_{n\in\mathbb{Z}} \frac{a}{a^2+n^2} = \sum_{n\in\mathbb{Z}} e^{-2\pi a|n|}$

Since $\sum_{n=-\infty}^{\infty} e^{-2\pi a|n|} = 2\sum_{n=0}^{\infty} e^{-2\pi a|n|} - 1$

$$= \frac{2}{1-e^{-2\pi a}} - 1 = \frac{e^{\pi a}+e^{-\pi a}}{e^{\pi a}-e^{-\pi a}} = \coth \pi a$$

so the result follows.

P153.1. We establish Jensen's formula for functions without zeros in the unit disc. Suppose f is analytic in $D(0,1)$ and has zeros at $z_1, \dots, z_N$, counted with multiplicity. Then $g(z) = \frac{f(z)}{\psi_1(z)\cdots\psi_N(z)}$ is analytic in $D(0,1)$ without zeros, where $\psi_k \equiv \psi_{z_k}$ for simplicity.

Thus, $$\log\left|\frac{f(0)}{\psi_1(0)\cdots\psi_N(0)}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{f(e^{i\theta})}{\psi_1(e^{i\theta})\cdots\psi_N(e^{i\theta})}\right| d\theta$$

$$\Rightarrow \log|f(0)| - \sum_{k=1}^{N} \log|\psi_k(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|\,d\theta - \frac{1}{2\pi}\sum_{k=1}^{N}\int_0^{2\pi} \log|\psi_k(e^{i\theta})|\,d\theta$$

Since $\psi_k(0) = z_k$ and $$|\psi_k(e^{i\theta})| = \left|\frac{z_k - e^{i\theta}}{-e^{i\theta}(\overline{z_k} - e^{-i\theta})}\right| = 1$$

so $$\log|f(0)| = \sum_{k=1}^{N} \log|z_k| + \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|\,d\theta.$$

P156.1. By Jensen's formula, for each $R<1$,
$$\sum_{|z_k|<R} \log\left|\frac{R}{z_k}\right| = \int_0^{2\pi} \log|f(Re^{i\theta})| \frac{d\theta}{2\pi} - \log|f(0)|,$$
which is bounded since f is so. Then $\exists M$ s.t.

for all $R<1$, $\sum_{|z_k|<R} \log\left|\frac{R}{z_k}\right| < M$. Letting $R \uparrow 1$ gives
$$\sum_{\text{all } k} \log\left|\frac{1}{z_k}\right| \le M \Rightarrow \sum_{\text{all } k}(1-|z_k|) \le \sum_{\text{all } k} \log\left|\frac{1}{z_k}\right| < M < \infty$$

P248.1. Suppose $f'(z_0)=0$ for some $z_0 \in U$. Then we can write
$f(z) - f(z_0) = a(z-z_0)^k + G(z)$ for z in a disc around z_0, $a \neq 0$, and $k \geq 2$, with G ~~vanishing~~ of order at least $k+1$.

To show that for some disc around z_0, f isn't injective, we pick w small enough s.t. $\exists \varepsilon > 0$ and $|G(z)| < |a(z-z_0)^k - w|$ for all $z \in D(z_0, \varepsilon) \setminus \{z_0\}$

By Rouche's Thm, $f(z) - f(z_0) - w$ has two distinct roots.

Next, if f is not locally bijective, then $\exists$ two distinct continuous paths γ_1 & $\gamma_2 : [0,1] \to \mathbb{C}$ s.t. $\gamma_1(0) = \gamma_2(0) = z_0$ and $f(\gamma_1(x)) = f(\gamma_2(x))$ $\forall x \in [0,1]$ with $\gamma_1(x) \neq \gamma_2(x)$. Hence, $f(\gamma_1(x)) - f(\gamma_2(x)) = f'(\xi)(\gamma_1(x) - \gamma_2(x))$ gives $f'(\xi) = 0$ for some ξ

4. Let F be given in the textbook. Use F to find a map from $D(0,1)$ to H. Then shift H down to ~~the lower half~~ by using the map $z \mapsto z-i$ and compose with squaring to get the map $z \mapsto (F(z)-i)^2$, which fills $\mathbb{C}$ since squaring will double the argument.

5. Follow the hint. Solving $f(z)=w$ gives $z_\pm = -w \pm \sqrt{w^2-1}$.
Since $z_+ \cdot z_- = 1$ so one of them is in $\overline{D(0,1)}$, wlog say z_+.
Now note that $\operatorname{Im}(f(z)) = -\frac{1}{2}[\operatorname{Im}(z) + \operatorname{Im}(\frac{1}{z})] = -\frac{1}{2}(r - \frac{1}{r})\sin\theta$,
where $z = re^{i\theta}$. Also, if $f(z_+) = w \in H$, then $\operatorname{Im}(f(z_+)) > 0$.
Since $z_+ \in \overline{D(0,1)}$, so if r_+ is its radius then $r_+ - \frac{1}{r_+} \leq 0$.
Hence, $\sin\theta_+ > 0$, where $z_+ = r_+ e^{i\theta_+}$. $\Rightarrow$ $z_+ \in \overline{D(0,1)} \cap H$.

10. Let $G: D(0,1) \to H$ be $G(z) = i\frac{1+z}{1-z}$. Then $F \circ G : D(0,1) \to D(0,1)$ and is holomorphic in $D(0,1)$ with $F \circ G(0) = F(i) = 0$.
By Schwarz Lemma, $|F \circ G(z)| \leq |z| \quad \forall z \in D(0,1)$.
Hence, $|F(z)| = |F \circ G \circ G^{-1}(z)| \leq |G^{-1}(z)| = \left|\frac{z-i}{z+i}\right|$.