M366 HW3 Solutions Sketch

Prob 1. Since $\sin\pi z = \frac{1}{2i}(e^{i\pi z} - e^{-i\pi z})$, so

$\sin\pi z = 0 \Leftrightarrow e^{i\pi z} = e^{-i\pi z} \Leftrightarrow e^{2i\pi z} = 1$

$\Leftrightarrow 2i\pi z = 2\pi k$ for $k \in \mathbb{Z}$ $\Leftrightarrow$ $z \in \mathbb{Z}$.

Thus, the zeros of $\sin\pi z$ are exactly at the integers.

~~By considering the expansion~~ We may expand $\sin\pi z$ at k, for any $k \in \mathbb{Z}$.

So, $\sin\pi z = \frac{(-1)^k}{2i}\left(e^{i\pi(z-k)} - e^{-i\pi(z-k)}\right)$

$= \frac{(-1)^k}{2i} 2i\pi(z-k) + O((z-k)^2)$

Hence, each $k \in \mathbb{Z}$ is a zero of order 1 for $\sin\pi k$.

Finally, $res_k\left(\frac{1}{\sin\pi z}\right) = \lim_{z\to k} \frac{z-k}{\sin\pi z} = \frac{(-1)^k}{\pi}$, $k \in \mathbb{Z}$.

2. Let $f(z) = \frac{1}{1+z^4}$. The simple poles of $f(z)$ are at

$1 + z^4 = 0 \Leftrightarrow z = e^{i\pi(\frac{1}{4}+\frac{k}{2})}$, $k = 0, 1, 2, 3$.

Let γ_R be the semi-circle of radius R centered at the origin in the upper half plane.

Then only $e^{i\pi/4}$ and $e^{3i\pi/4}$ are inside γ, while the other two are in the lower half plane. Then

$$\int_{\gamma_R} f(z)\,dz = 2\pi i\left(res_{e^{i\pi/4}} f(z) + res_{e^{3i\pi/4}} f(z)\right)$$

$$= 2\pi i\left(\lim_{z\to e^{i\pi/4}} \frac{z-e^{i\pi/4}}{1+z^4} + \lim_{z\to e^{3i\pi/4}} \frac{z-e^{3i\pi/4}}{1+z^4}\right)$$

$$= 2\pi i\left(\tfrac{1}{4}e^{-3i\pi/4} + \tfrac{1}{4}e^{-9i\pi/4}\right)$$

$$= 2\pi i\left(-\tfrac{\sqrt{2}}{8}(1+i) - \tfrac{\sqrt{2}}{8}(1-i)\right)$$

$$= \frac{\pi}{\sqrt{2}}$$

Let C_R be the upper half-circle of radius R, then

$$\left|\int_{C_R} f(z)\,dz\right| \le \pi R \cdot \sup_{z\in C_R}|f(z)| \le \pi R \cdot \frac{1}{R^4-1} \to 0 \text{ as } R\to\infty.$$

Since $\int_{\gamma_R} f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(z)\,dz$, so letting $R\to\infty$, $\int_{-\infty}^{\infty} f(z)\,dz = \frac{\pi}{\sqrt{2}}$

3. Let $f(z) = \frac{e^{iz}}{z^2+a^2} = \frac{e^{iz}}{(z-ia)(z+ia)}$, $a>0$

So $f(z)$ has simple poles at ia and $-ia$.

Let γ_R be the upper half plane semi-circle with radius R, then only ia is enclosed in γ.

We have
$$\int_{\gamma_R} f(z)\,dz = 2\pi i \operatorname{res}_{ia} f(z)$$
$$= 2\pi i \lim_{z\to ia} \frac{e^{ia}}{z+ai} = \frac{\pi}{ae^a}.$$

Let C_R be the upper half circle, then
$$\left|\int_{C_R} f(z)\,dz\right| \le \pi R \cdot \sup_{z\in C_R} |f(z)| \le \pi R \cdot \frac{1}{R^2-a^2}$$
which goes to 0 as $R\to\infty$.

So, letting $R\to\infty$ we have
$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+a^2}\,dx = \frac{\pi}{ae^a}.$$

Taking the real part gives
$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,dx = \frac{\pi}{ae^a}$$

Prob. 6. Let $f(z) = \frac{1}{(1+z^2)^{2n+1}}$. Then $f(z)$ has poles of order $n+1$ at i and $-i$.

Let γ_R be the upper half plane semi circle with radius R which encloses ~~the~~ pole i.

Then
$$\int_{\gamma_R} f(z)\,dz = 2\pi i \operatorname{res}_i f(z) = 2\pi i \lim_{z\to i} \frac{1}{n!}\frac{d^n}{dz^n}(z+i)^{-n-1}$$
$$= 2\pi i \cdot \frac{(2n)!}{(n!)^2} \cdot \frac{1}{2i\,2^{2n}} = \frac{\pi\,(2n)!}{2^{2n}(n!)^2}$$

Let C_R be the upper half circle, then
$$\left|\int_{C_R} f(z)\,dz\right| \le \pi R \cdot \sup_{z\in C_R} \frac{1}{|1+z^2|} \le \frac{\pi R}{R^4-1} \to 0 \text{ as } R\to\infty.$$

Letting $R\to\infty$ we have
$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{2n+1}}\,dx = \pi\frac{(2n)!}{2^{2n}(n!)^2} = \frac{\pi\, 1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$$

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7. Let $z = e^{i\theta}$ then $d\theta = \frac{dz}{iz}$. Let $f(z) = \frac{1}{(a+\frac{z+z^{-1}}{2})^2 iz}$.
Then the poles of $f(z)$ are at $z^2 + 2az + 1 = 0$,
that is, at $-a \pm \sqrt{a^2-1}$, $a > 1$. Their orders are 2.
Let γ_R be the upper half plane semicircle, then
it encloses the only pole $-a+\sqrt{a^2-1}$
Then $\int_{\gamma_R} f(z)\,dz = 2\pi i \cdot \mathrm{res}_{-a+\sqrt{a^2-1}} f(z)$

$$= 2\pi i \cdot \lim_{z \to -a+\sqrt{a^2-1}} \frac{d}{dz}\left(z - (\sqrt{a^2-1}-a)\right)^2 f(z)$$

$$= \frac{2\pi a}{(a^2-1)^{3/2}}$$

With ~~similar~~ argument as before, we obtain the result.

8. Similarly, Let $f(z) = \frac{1}{a+b(\frac{z+z^{-1}}{2})} \cdot \frac{1}{iz}$
Then the poles of $f(z)$ are at $\frac{-a \pm \sqrt{a^2-b^2}}{b}$
Let γ_R be the upper half plane semi-circle, the it only
encloses ~~the~~ pole $\frac{-a+\sqrt{a^2-b^2}}{b}$.
Then $\int_{\gamma_R} f(z)\,dz = 2\pi i \cdot \mathrm{res}_{\frac{-a+\sqrt{a^2-b^2}}{b}} f(z) = \frac{2\pi}{\sqrt{a^2-b^2}}$.

Using similar argument as before we obtain the result.

Prob. 13. ~~[illegible]~~
~~[illegible]~~
Let $g(z) = (z-z_0) f(z)$, $z \in D(z_0, r) \setminus \{z_0\}$.
Then $|g(z)| = |z-z_0||f(z)| \le A|z-z_0|^{\epsilon} \to 0$ as $z \to z_0$.
So g is bounded and holomorphic in $D(z_0, r) \setminus \{z_0\}$.
By Riemann's Thm g extends to a holomorphic function
in $D(z_0, r)$. Since z_0 is a pole for g, so we
can write $g(z) = (z-z_0) h(z)$ for some function h
holomorphic in $D(z_0, r)$. We can let h be the extended
f holomorphic across z_0.

14. Since $f(z)$ is entire, so $f(\frac{1}{z})$ only has one singularity at $z=0$.
Note that $\lim_{z\to 0} f(\frac{1}{z})$ cannot be bounded, otherwise $f(z)$ would be bounded and Liouville Thm would tell us that $f(z)$ was a constant, contradicting to $f(z)$ being injective.

Next we show that the singularity of $f(\frac{1}{z})$ at $z=0$ cannot be essential. Suppose otherwise, and consider a punctured disc $D(0,r)\setminus\{0\}$ which is open. By the Open Mapping Thm, $\{f(\frac{1}{z}) : z \in D(0,r)\setminus\{0\}\}$ is open; moreover, it is dense in the complex plane by the Casorati-Weierstrass Thm, so it includes any value $f(\frac{1}{z_0})$ for $z_0 \notin D(0,r)\setminus\{0\}$, contradicting to $f(z)$ being injective.

Hence $z=0$ is a pole for $f(\frac{1}{z})$. So we may write $f(\frac{1}{z}) = \sum_{j=0}^{n} a_j z^{-j} + G(z)$ for some $n \in \mathbb{Z}^+$, some holomorphic function $G(z)$ in $D(0,r)\setminus\{0\}$, and some a_j's.
Then $f(z) = \sum_{j=0}^{n} a_j z^j + G(\frac{1}{z})$ where $G(\frac{1}{z}) = \sum_{j=1}^{\infty} b_j z^{-j}$ for some b_j's. Since $f(z)$ is entire, so $G(\frac{1}{z})$ must be analytic across 0, hence, $b_j = 0\ \forall j$, i.e. $G \equiv 0$.
Note that $f(z)$ is injective, so $n \equiv 1$, i.e. $f(z) = a_0 + a_1 z$.