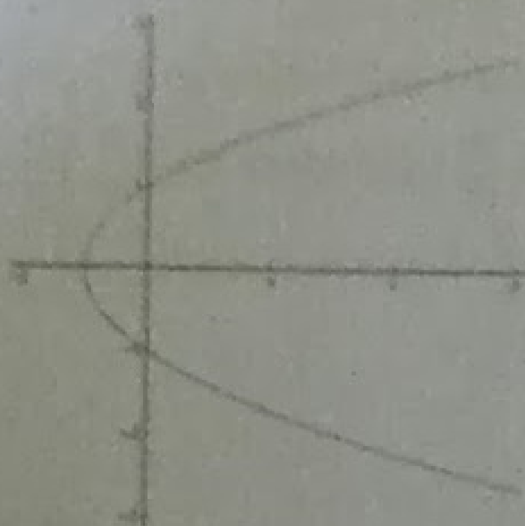
excerpts from the excellent work by students

1 Question 1 (f)

Let $z = x + iy$ for $x, y \in \mathbb{R}$. Then,

$$\begin{aligned} |z| = \Re(z) + 1 &\Leftrightarrow \sqrt{x^2+y^2} = x+1 \\ &\Leftrightarrow x^2+y^2 = x^2+2x+1 \\ &\Leftrightarrow y^2 = 2x+1 \end{aligned}$$

Thus, this set is the parabola $x = \frac{1}{2}(y^2-1)$ which looks like this



2 Question 8

Let

$$\begin{aligned} f(x,y) &= u(x,y) + iv(x,y) \\ g(u,v) &= \xi(u,v) + i\eta(u,v) \end{aligned}$$

Then, we have the following:

$$\begin{aligned} \frac{\partial g}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)(\xi(u,v) + i\eta(u,v)) = \frac{1}{2}(\xi_u + \eta_v - i(\xi_v - \eta_u)) \\ \frac{\partial f}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u(x,y) + iv(x,y)) = \frac{1}{2}(u_x + v_y - i(u_y - v_x)) \\ \frac{\partial g}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)(\xi(u,v) + i\eta(u,v)) = \frac{1}{2}(\xi_u - \eta_v + i(\xi_v + \eta_u)) \end{aligned}$$

$$\frac{\partial \bar{f}}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u(x,y) - iv(x,y)) = \frac{1}{2}(u_x - v_y - i(u_y + v_x))$$

Thus,

$$\begin{aligned}
\frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial z} &= \frac{1}{4}[(\xi_u + \eta_v)(u_x + v_y) + (-\xi_v + \eta_u)(u_y - v_x)] \\
&+ \frac{i}{4}[(\xi_u + \eta_v)(-u_y + v_x) + (-\xi_v + \eta_u)(u_x + v_y)] \\
&+ \frac{1}{4}[(\xi_u - \eta_v)(u_x + v_y) + (\xi_v + \eta_u)(u_y - v_x)] \\
&+ \frac{i}{4}[(\xi_u - \eta_v)(-u_y - v_x) + (\xi_v + \eta_u)(u_x - v_y)] \\
&= \frac{1}{2}(\xi_u u_x + \eta_v v_y + \xi_v v_x + \eta_u u_y) \\
&+ \frac{i}{2}(-\xi_u u_y + \eta_v v_x - \xi_v v_y + \eta_u u_x) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x}\xi(u(x,y), v(x,y)) + \frac{\partial}{\partial y}\eta(u(x,y), v(x,y))\right) \\
&+ \frac{i}{2}\left(-\frac{\partial}{\partial y}\xi(u(x,y), v(x,y)) + \frac{\partial}{\partial x}\eta(u(x,y), v(x,y))\right)
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{\partial h}{\partial z} &= \frac{\partial}{\partial z}g(f(x,y)) \\
&= \frac{\partial}{\partial z}g(u(x,y), v(x,y)) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(\xi(u(x,y), v(x,y)) + i\eta(u(x,y), v(x,y))) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x}\xi(u(x,y), v(x,y)) + \frac{\partial}{\partial y}\eta(u(x,y), v(x,y))\right) \\
&+ \frac{i}{2}\left(-\frac{\partial}{\partial y}\xi(u(x,y), v(x,y)) + \frac{\partial}{\partial x}\eta(u(x,y), v(x,y))\right) \\
&= \frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial z}
\end{aligned}$$

Therefore, we have

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial z}$$

We now prove the second statement. We have

$$\frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u(x,y) + iv(x,y)) = \frac{1}{2}\left(u_x - v_y + i(u_y + v_x)\right)$$
$$\frac{\partial \bar f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u(x,y) - iv(x,y)) = \frac{1}{2}\left(u_x + v_y + i(u_y - v_x)\right)$$

Thus,

$$\begin{aligned}
\frac{\partial g}{\partial z}\frac{\partial f}{\partial \bar z} + \frac{\partial g}{\partial \bar z}\frac{\partial \bar f}{\partial \bar z} &= \frac{1}{4}\left[(\xi_u + \eta_v)(u_x - v_y) + (\xi_v - \eta_u)(u_y + v_x)\right] \\
&\quad + \frac{i}{4}\left[(\xi_u + \eta_v)(u_y + v_x) + (-\xi_v + \eta_u)(u_x - v_y)\right] \\
&\quad + \frac{1}{4}\left[(\xi_u - \eta_v)(u_x + v_y) + (-\xi_v - \eta_u)(u_y - v_x)\right] \\
&\quad + \frac{i}{4}\left[(\xi_u - \eta_v)(u_y - v_x) + (\xi_v + \eta_u)(u_x + v_y)\right] \\
&= \frac{1}{2}\left(\xi_u u_x - \eta_v v_y + \xi_v v_x - \eta_u u_y\right) \\
&\quad + \frac{i}{2}\left(\xi_u u_y + \eta_v v_x + \xi_v v_y + \eta_u u_x\right) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x}\xi(u(x,y),v(x,y)) - \frac{\partial}{\partial y}\eta(u(x,y),v(x,y))\right) \\
&\quad + \frac{i}{2}\left(\frac{\partial}{\partial y}\xi(u(x,y),v(x,y)) + \frac{\partial}{\partial x}\eta(u(x,y),v(x,y))\right) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\xi(u(x,y),v(x,y)) + i\eta(u(x,y),v(x,y))\right) \\
&= \frac{\partial}{\partial \bar z} g(f(x,y)) \\
&= \frac{\partial h}{\partial \bar z}
\end{aligned}$$

Therefore, we also have

$$\frac{\partial h}{\partial \bar z} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial \bar z} + \frac{\partial g}{\partial \bar z}\frac{\partial \bar f}{\partial \bar z}$$

3 Question 10

We have,

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z} = 4\left[\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\right]\left[\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right]$$

$$= \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$
$$= \frac{\partial^2}{\partial x^2} + i\frac{\partial^2}{\partial x\partial y} - i\frac{\partial^2}{\partial y\partial x} + \frac{\partial^2}{\partial y^2}$$

By Clairaut's theorem, $\frac{\partial^2}{\partial x\partial y} = \frac{\partial^2}{\partial y\partial x}$, so

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Similarly, by Clairaut's theorem again,

$$4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} = 4\left[\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\right]\left[\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right]$$
$$= 4\left[\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\right]\left[\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right]$$
$$= 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}$$
$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

4 Question 14

$$\sum_{n=M}^{N} a_n b_n = \sum_{n=M}^{N} a_n(B_n - B_{n-1})$$
$$= \sum_{n=M}^{N} a_n B_n - \sum_{n=M}^{N} a_n B_{n-1}$$
$$= \left(a_N B_N + \sum_{n=M}^{N-1} a_n B_n\right) - \left(a_M B_{M-1} + \sum_{n=M+1}^{N} a_n B_{n-1}\right)$$
$$= a_N B_N - a_M B_{M-1} + \sum_{n=M}^{N-1} a_n B_n - \sum_{n=M}^{N-1} a_{n+1} B_n$$
$$= a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n$$

5 Question 16

(b) We will first show that $\lim_{n\to\infty}(n!)^{1/n} = \infty$. Note that

$$(n!)^{1/n} = e^{\frac{1}{n}\ln n!}$$

Now,

$$\begin{aligned}\frac{1}{n}\ln n! &= \frac{1}{n}\sum_{k=1}^{n}\ln k \geq \frac{1}{n}\int_1^n \ln x\,dx\\ &= \frac{1}{n}(n(\ln n - 1) + 1) = \ln n - 1 + \frac{1}{n}\\ &> \ln n - 1\end{aligned}$$

Thus,

$$(n!)^{1/n} = e^{\frac{1}{n}\ln n!} \geq e^{\ln n - 1} = \frac{n}{e}$$

But $\frac{n}{e} \to \infty$ as $n \to \infty$, so $(n!)^{1/n} \to \infty$ as $n \to \infty$. Therefore

$$\limsup_{n\to\infty}(n!)^{1/n} = \infty$$

so by Hadamard's formula, the radius of convergence is $R = 0$.

(c) We have

$$\lim_{n\to\infty}\frac{n^2}{4^n + 3n} = \lim_{n\to\infty}\frac{1}{\frac{4^n}{n^2} + \frac{3}{n}}$$

Now, $\frac{4^n}{n^2} \to \infty$ and $\frac{3}{n} \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty}\frac{n^2}{4^n + 3n} = 0$$

Thus,

$$\limsup_{n\to\infty}\frac{n^2}{4^n + 3n} = 0$$

$\lim x_n = 0$
doesn't imply

so we get by Hadamard's formula that the radius of convergence is $R = \cancel{\infty}$

(d) By using Stirling's formula, i.e.

$$n! \sim c n^{n+\frac{1}{2}} e^{-n}$$

for some $c > 0$, we get

$$a_n = \frac{(n!)^3}{(3n)!} \sim \frac{c^3 n^{3n+\frac{3}{2}} e^{-3n}}{c(3n)^{3n+\frac{1}{2}} e^{-3n}} = \frac{c^2}{\sqrt{3}}\frac{n}{27^n} \Rightarrow \limsup a_n^{\frac{1}{n}} = \frac{1}{27}$$

Now, $\frac{n}{27^n} \to 0$ as $n \to \infty$, so $a_n \to 0$ as $n \to \infty$. Thus,

$$\limsup_{n\to\infty} a_n = 0$$

so by Hadamard's formula, the radius of convergence is $R = \cancel{\infty}$

6 Question 19 (a)

Suppose z is on the unit circle. We must show that $\sum_{n=1}^{\infty} nz^n$ does not converge. Equivalently, we must show that the limit

$$\lim_{N\to\infty} S_N$$

of partial sums

$$S_N = \sum_{n=1}^{N} nz^n$$

does not exist. Since $|z| = 1$, we can write $z = e^{i\theta}$ for some $\theta \in [0, 2\pi)$. Thus,

$$S_N = \sum_{n=1}^{N} ne^{in\theta} = \sum_{n=1}^{N} n(\cos n\theta + i\sin n\theta) = \sum_{n=1}^{N} n\cos n\theta + i\sum_{n=1}^{N} n\sin n\theta$$

If $\lim_{N\to\infty} S_N$ exists, then both the real and imaginary part of S_N must converge. But, $\lim_{n\to\infty} n\cos n\theta$ does not exists, so in particular $\lim_{n\to\infty} n\cos n\theta \neq 0$ which implies that $\sum_{n=1}^{\infty} n\cos n\theta$ does not converge. Hence, $\lim_{N\to\infty} S_N$ does not exists, so we conclude that the series $\sum_{n=1}^{\infty} nz^n$ does not converge when $|z| = 1$.

7 Question 20

Let

$$f(z) = (1-z)^{-m}$$

We have

$$f'(z) = m(1-z)^{-m-1}$$
$$f''(z) = m(m+1)(1-z)^{-m-2}$$

and in general

$$f^{(n)}(z) = m(m+1)\cdots(m+n-1)(1-z)^{-m-n}$$

Thus,

$$f^{(n)}(0) = m(m+1)\cdots(m+n-1) = \frac{(m+n-1)!}{(m-1)!}$$

Hence,

$$f(x) = \sum_{n=0}^{k} \frac{(m+n-1)!}{n!(m-1)!} x^n + R_k(x)$$

Thus, if

$$(1-x)^{-m} = \sum_{n=0}^{\infty} a_n x^n$$

then

$$a_n = \frac{(m+n-1)!}{n!(m-1)!}$$

Now,

$$\begin{aligned}
\lim_{n\to+\infty} \frac{\left(\frac{(m+n-1)!}{n!(m-1)!}\right)}{\left(\frac{n^{m-1}}{(m-1)!}\right)} &= \lim_{n\to+\infty} \frac{(m+n-1)!}{n!n^{m-1}} \\
&= \lim_{n\to+\infty} \frac{(n+1)(n+2)\cdots(n+m-1)}{n^{m-1}} \\
&= \lim_{n\to+\infty} \left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{m-1}{n}\right) \\
&= 1
\end{aligned}$$

Thus,

$$a_n = \frac{(m+n-1)!}{n!(m-1)!} \sim \frac{n^{m-1}}{(m-1)!}$$

8 Question 23

We have

$$f(x) = \begin{cases} 0 & ,\text{if } x \le 0 \\ e^{-1/x^2} & ,\text{if } x > 0 \end{cases}$$

We will show that $f(x)$ is indefinitely differentiable on $\mathbb{R}$. Our first step will be to prove that $f(x)$ is differentiable for all $x \in \mathbb{R}$. This is clear for $x < 0$, we have $f'(x) = 0$ in that case. Also, it is clear for $x > 0$ with $f'(x) = \frac{2}{x^3} e^{-1/x^2}$. Now, for $x = 0$, we have

$$\lim_{h\to 0+} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0+} \frac{e^{-1/h^2}}{h} = \lim_{h\to 0+} \frac{1/h}{e^{1/h^2}}$$

Now, $1/h \to \infty$ and $e^{1/h^2} \to \infty$ as $h \to 0+$, so by applying l'Hospital's rule, we get

$$\lim_{h\to 0+} \frac{1/h}{e^{1/h^2}} = \lim_{h\to 0+} \frac{-1/h^2}{-\frac{2}{h^3}e^{1/h^2}} = \lim_{h\to 0+} \frac{he^{-1/h^2}}{2} = 0$$

On the other hand,

$$\lim_{h\to 0-} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0-} 0 = 0$$

Thus, we have

$$f'(0) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = 0$$

Hence, $f'(x)$ exists for all $x \in \mathbb{R}$ and we have

$$f'(x) = \begin{cases} 0 & ,\text{if } x \le 0 \\ \frac{2}{x^3} f(x) & ,\text{if } x > 0 \end{cases}$$

Our second step is to show that f is indefinitely differentiable, and that

$$f^{(n)}(x) = \begin{cases} 0 & ,\text{if } x \le 0 \\ g_n(x) f(x) & ,\text{if } x > 0 \end{cases}$$

for some functions $g_n(x)$ that are a sum of terms of the form $\frac{c_k}{x^k}$ for some constants c_k and $k \in \mathbb{N}$. We will show this by induction on n. We already proved the case $n = 1$. Now suppose the result is true for some $n \in \mathbb{N}$. The result for $n+1$ is then clear for $x < 0$, we have $f^{(n+1)}(x) = 0$ in that case. It is also clearly true for $x > 0$. Indeed if $x > 0$ then we have

$$f^{(n+1)}(x) = g_n'(x) f(x) + g_n(x)\frac{2}{x^3} f(x) = \left(g_n'(x) + \frac{2g_n(x)}{x^3}\right) f(x)$$

so that $g_{n+1}(x) = g_n'(x) + \frac{2g_n(x)}{x^3}$ which is still a sum of terms of the form $\frac{c_k}{x^k}$. Now, for $x = 0$, this is true only if we have

$$\lim_{h\to 0+} \frac{f^{(n)}(0+h) - f^{(n)}(0)}{h} = \lim_{h\to 0+} \frac{g_n(h) f(h)}{h} = 0$$

which is automatically true if we can show that for all $k \in \mathbb{N}$,

$$\lim_{h\to 0+} \frac{e^{-1/h^2}}{h^k} = 0$$

This is done by successively applying l'Hospital Rule as for the case $k = 1$
that we did. We get

$$\begin{aligned}
\lim_{h\to 0+} \frac{e^{-1/h^2}}{h^k} &= \lim_{h\to 0+} \frac{1/h^k}{e^{1/h^2}} = \lim_{h\to 0+} \frac{-\frac{k}{h^{k+1}}}{-\frac{2}{h^3}e^{1/h^2}} = \frac{k}{2}\lim_{h\to 0+} \frac{1/h^{k-2}}{e^{1/h^2}} \\
&= \frac{k(k-2)}{2^2}\lim_{h\to 0+} \frac{1/h^{k-4}}{e^{1/h^2}} = \frac{k(k-2)(k-4)}{2^3}\lim_{h\to 0+} \frac{1/h^{k-6}}{e^{1/h^2}} \\
&= \frac{k(k-2)\cdots(k-2(j-2))}{2^j}\lim_{h\to 0+} \frac{1/h^{k-2j}}{e^{1/h^2}}
\end{aligned}$$

Continuing this way, we reach a point where the exponent $k-2j$ on
becomes negative and the limit is then obviously equal to zero. Hence,
have $f^{(n+1)}(0) = 0$ so the result is true for $n+1$.

Therefore, we proved that f is indefinitely differentiable, and in part
ular that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Thus, the nth Taylor expansion of f
$x = 0$ is

$$f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \cdots + \frac{1}{n!}f^{(n)}(0)x^n + R_n(x) = R_n(x$$

Hence, the remainder $R_n(x)$ is $f(x)$ so it does not converge to zero
$x = 0$. Thus, f does not have a Taylor series expansion near $x = 0$.

Exce

another except

MATH 3x6
Assignment #1

1/. $|z| = Re(z)+1 \Rightarrow \sqrt{x^2+y^2} = x+1$
$\Rightarrow x^2+y^2 = x^2+2x+1 \Rightarrow x = \frac{1}{2}(y^2-1)$
So the set of points satisfying $|z| = Re(z)+1$ is the parabola $\{x = \frac{1}{2}(y^2-1)\}$.
We could also say that it is the set of points $\{(x,y) : x \in [-\frac{1}{2}, \infty),\ y = \pm\sqrt{2x+1}\}$.

8. We have that ① $h_z = \frac{1}{2}(h_x - ih_y)$, $h_{\bar{z}} = \frac{1}{2}(h_x + ih_y)$
Define u and v before using them.
Looking at g as a function of u and v we get
② $h_x = g_u u_x + g_v v_x$, $h_y = g_u u_y + g_v v_y$
Combining ② and ③ $g_z = \frac{1}{2}(g_u - ig_v)$, $g_{\bar{z}} = \frac{1}{2}(g_u + ig_v)$
gives ④ $g_u = g_z + g_{\bar{z}}$, $g_v = i(g_z - g_{\bar{z}})$
Combining ①, ② and ④ gives
$h_z = \frac{1}{2}\left(\left((g_z+g_{\bar{z}})u_x + i(g_z-g_{\bar{z}})v_x\right) - i\left((g_z+g_{\bar{z}})u_y + i(g_z-g_{\bar{z}})v_y\right)\right)$
$= \frac{1}{2}\left(g_z\left((u_x+iv_x) - i(u_y+iv_y)\right) + g_{\bar{z}}\left((u_x-iv_x) + i(u_y-iv_y)\right)\right)$
$= \frac{1}{2}\left(g_z(f_x - if_y) + g_{\bar{z}}(\bar{f}_x - i\bar{f}_y)\right)$

$$= g_z \tfrac{1}{2}(f_x - i f_y) + g_{\bar z} \tfrac{1}{2}(\bar f_x - i \bar f_y)$$
$$= g_z f_z + g_{\bar z} \bar f_z = \frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar z}\frac{\partial \bar f}{\partial z}$$

The same thing can be done for $\frac{\partial h}{\partial \bar z}$, changing the sign in the operation yields

$$\frac{\partial h}{\partial \bar z} = \frac{1}{2}\left(g_z(f_x + i f_y) + g_{\bar z}(\bar f_x + i \bar f_y)\right)$$
$$= g_z \tfrac{1}{2}(f_x + i f_y) + g_{\bar z} \tfrac{1}{2}(\bar f_x + i \bar f_y)$$
$$= g_z f_{\bar z} + g_{\bar z} \bar f_{\bar z} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial \bar z} + \frac{\partial g}{\partial \bar z}\frac{\partial \bar f}{\partial \bar z}$$

10. We know that

$$\frac{\partial}{\partial z} = \frac{1}{2}(\partial x - i\partial y), \quad \frac{\partial}{\partial \bar z} = \frac{1}{2}(\partial x + i\partial y)$$

Doing an elementary multiplication gives

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z} = \frac{1}{4}(\partial x\partial x + i\,\partial y\partial x - i\,\partial x\partial y - i^2\partial y\partial y)$$

the equality $\partial y\partial x = \partial y\partial x$ then implies

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z} = \frac{\partial}{\partial \bar z}\frac{\partial}{\partial z} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \frac{1}{4}\Delta$$

This is the desired result. ∎