Magic Moore–Penrose inverses & philatelic magic squares, with special emphasis on the Daniels–Zlobec magic matrix

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We study magic matrices in which all the numbers in the rows and columns and in the 2 main diagonals add up to the same *magic sum*.

Our interest focuses on such magic matrices for which the Moore–Penrose inverse is also magic. Special attention is given to the Daniels–Zlobec magic square

$$\mathbf{Z} = \begin{pmatrix} 24 & 11 & 22 & 17 \\ 21 & 18 & 23 & 12 \\ 15 & 20 & 13 & 26 \\ 14 & 25 & 16 & 19 \end{pmatrix}$$

introduced by the British magician and television performer Paul Daniels (b. 1938) and introduced to me by Sanjo Zlobec in 2001.

The matrix $\mathbf{Z}$ has magic sum 74.

If we subtract 1 from each element of the Daniels–Zlobec magic matrix $\mathbf{Z}$ we obtain the *adjusted Daniels–Zlobec magic matrix*

$$\mathbf{Z}_1 = \begin{pmatrix} 23 & 10 & 21 & 16 \\ 20 & 17 & 22 & 11 \\ 14 & 19 & 12 & 25 \\ 13 & 24 & 15 & 18 \end{pmatrix}$$

with magic sum 70. Happy birthday, Sanjo!

We define a "philatelic magic square" as a square arrangement of images of postage stamps so that the associated nominal values form a magic square. Three philatelic magic squares with stamps especially chosen for Sanjo Zlobec are presented in celebration of his 70th birthday.

Sanjo Zlobec (*Image* 2001) commented that:

> *In my undergraduate lectures on magic squares I sometimes mention Paul Daniels's magic square—Paul Daniels is a well-known British magician and TV personality who I often saw on Canadian TV in the 1970s.*
> *I copied the matrix for his magic square once from TV. It is:*

$$\mathbf{Z} = \begin{pmatrix} 24 & 11 & 22 & 17 \\ 21 & 18 & 23 & 12 \\ 15 & 20 & 13 & 26 \\ 14 & 25 & 16 & 19 \end{pmatrix}.$$

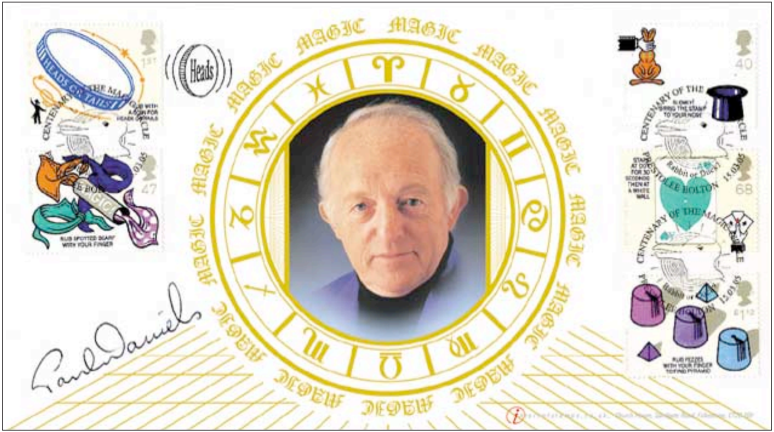
We call **Z**, which has magic sum 74, the *Daniels–Zlobec magic matrix* and

$$\mathbf{Z}_1 = \begin{pmatrix} 23 & 10 & 21 & 16 \\ 20 & 17 & 22 & 11 \\ 14 & 19 & 12 & 25 \\ 13 & 24 & 15 & 18 \end{pmatrix}$$

with magic sum $\underline{70}$ we call the adjusted Daniels–Zlobec magic matrix.

Many happy returns, Sanjo!

Paul Daniels (born Newton Edward Daniels on 6 April 1938) is a British magician and television performer who achieved international fame with his TV series *The Paul Daniels Magic Show*, which ran on the BBC from 1979 to 1994. He was depicted on a first-day cover issued by Great Britain in 2005 with 4 postage stamps for the Centenary of the Magic Circle, the "premier magical society in the world of magic and illusion".



In *A Lifetime of Puzzles: A Collection of Puzzles in Honor of Martin Gardner's 90th Birthday* (AK Peters 2008), we find these words from Persi Diaconis:

> *Martin Gardner (1914–2010) has turned hundreds of mathematicians into magicians and hundreds of magicians into mathematicians!*

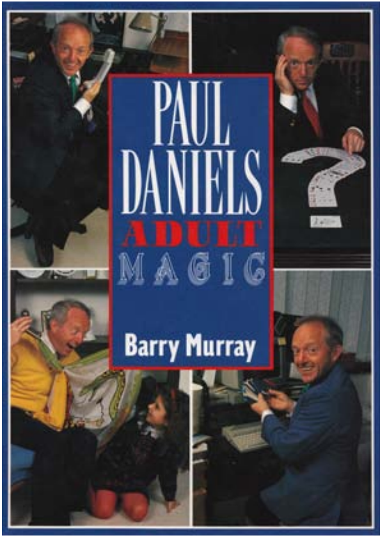
Many thanks to Sanjo Zlobec for introducing me to the magic of Paul Daniels.

In an $n \times n$ magic matrix, the numbers in all the rows and columns and in the two main diagonals add up to the same number, the *magic sum*.

The $n \times n$ array defined by a magic matrix is called a *magic square*. When the entries are consecutive positive integers, then it is said to be *classic* (or *natural* or *normal*).

Magic squares are over 1000 years old, but the term "magic matrix" seems to have originated just over 50 years ago with the 1956 paper by Charles Fox (1897–1977), Professor of Mathematics at McGill University from 1949–1967. I joined McGill in 1969 and Sanjo came a year later in 1970; Sam Drury joined McGill in 1972 and Ka Lok Chu received his Ph.D. degree from McGill in 2004.

While there is an enormous body of literature on matrix squares, relatively little has been published about magic <u>matrices</u>.



In his *Paul Daniels Adult Magic*, Murray (1989, p. 30) gives the *Daniels–Murray magic matrix* with parameter d defined by

$$\mathbf{M}_d = \begin{pmatrix} d & 1 & 12 & 7 \\ 11 & 8 & d-1 & 2 \\ 5 & 10 & 3 & d+2 \\ 4 & d+1 & 6 & 9 \end{pmatrix}.$$

Let $\mathbf{E}$ denote the 4×4 matrix with every element equal to 1. Then the Daniels–Zlobec magic matrices

$$\mathbf{Z} = \begin{pmatrix} 24 & 11 & 22 & 17 \\ 21 & 18 & 23 & 12 \\ 15 & 20 & 13 & 26 \\ 14 & 25 & 16 & 19 \end{pmatrix} = \mathbf{M}_{14} + 10\mathbf{E},$$

$$\mathbf{Z}_1 = \begin{pmatrix} 23 & 10 & 21 & 16 \\ 20 & 17 & 22 & 11 \\ 14 & 19 & 12 & 25 \\ 13 & 24 & 15 & 18 \end{pmatrix} = \mathbf{M}_{14} + 9\mathbf{E}$$

$$= \mathbf{Z} - \mathbf{E}.$$

We find that the determinant of the Daniels–Murray magic matrix

$$\det \mathbf{M}_d = (d-14)(d-6)(d+2)(d+20)$$

and so $\mathbf{M}_d$ is singular if and only if $d = 14,\ 6,\ -2 \text{ or } -20$.

While $\mathbf{M}_d$ is magic for all d, it is classic magic only when $d = 14$, confirming that the Daniels–Zlobec magic matrices $\mathbf{Z} = \mathbf{M}_{14} + 10\mathbf{E}$ and $\mathbf{Z}_1 = \mathbf{M}_{14} + 9\mathbf{E}$ are both classic.

Moreover, the Moore–Penrose inverse $\mathbf{M}_d^+$ is magic only when $d = 14$ and so only when it is classic magic, and hence the Moore–Penrose inverse

$$\mathbf{Z}^+ = \frac{1}{38480}\begin{pmatrix} 1277 & 167 & -55 & -869 \\ -1979 & 1055 & -647 & 2091 \\ 315 & 1129 & -1017 & 93 \\ 907 & -1831 & 2239 & -795 \end{pmatrix} = (\mathbf{M}_{14} + 10\mathbf{E})^+$$

of the Daniels–Zlobec matrix $\mathbf{Z}$ is also magic, as observed by Zlobec (*Image* 2001).

The adjusted Daniels–Zlobec magic matrix $\mathbf{Z}_1 = \mathbf{Z} - \mathbf{E} = \mathbf{M}_{14} + 9\mathbf{E}$ is also classic magic and its Moore–Penrose inverse is also magic.

A main goal in this research is to identify magic matrices, like $\mathbf{Z}$ and $\mathbf{Z}_1$, that have magic Moore–Penrose inverses. A key result is

The Moore–Penrose inverse of a **V***-associated magic matrix is also* **V***-associated, and when the involutory matrix* **V** *is centrosymmetric, then the Moore–Penrose inverse is magic.*

We have not found a magic matrix with a magic Moore–Penrose inverse that is not **V**-associated.

The symmetric *involutory matrix* **V** satisfies $\mathbf{V}^2 = \mathbf{I}$ and all the row totals of **V** equal to 1. The $n \times n$ magic matrix **A** with magic sum m is ***V**-associated* whenever

$$\mathbf{AV} + \mathbf{VA} = 2m\bar{\mathbf{E}},$$

or equivalently

$$\mathbf{A} + \mathbf{VAV} = 2m\bar{\mathbf{E}},$$

where $\bar{\mathbf{E}} = \frac{1}{n}\mathbf{E}$ has every element equal to $\frac{1}{n}$.

The involutory matrix **V** is *centrosymmetric* whenever it commutes with the *flip matrix*,

$$\mathbf{FV} = \mathbf{VF} \quad \text{or equivalently} \quad \mathbf{FVF} = \mathbf{V}.$$

The symmetric *flip matrix*

$$\mathbf{F} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

is involutory with all row totals equal to 1.

F-associated magic squares have been widely studied and are often called (just) *associated* (or *associative*, *regular* or *symmetrical;* Heinz & Hendricks, *Magic Square Lexicon* 2000, pp. 8, 136).

We define the $2h \times 2h$ involutory matrix

$$\mathbf{H} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_h \\ \mathbf{I}_h & \mathbf{0} \end{pmatrix}$$

with $n = 2h$ even. We note that **H** has all row totals equal to 1.

When $n = 2h + 1$ is odd, then we define **H** similarly, but with an extra row and column in the middle with all elements 0 except for 1 in the centre.

The adjusted Daniels–Zlobec matrix

$$\mathbf{Z}_1 = \begin{pmatrix} \underline{23} & 10 & 21 & 16 \\ 20 & 17 & 22 & 11 \\ 14 & 19 & \underline{12} & 25 \\ 13 & 24 & 15 & 18 \end{pmatrix}$$

is **H**-associated and so, in particular, the underlined entries sum to $\frac{1}{2}m = 35$.

It is easy to show that the 4×4 magic matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{pmatrix}$$

with magic sum m is **H**-associated whenever

$$\mathbf{M}_{11} + \mathbf{M}_{22} = \mathbf{M}_{12} + \mathbf{M}_{21} = \tfrac{1}{2}m\mathbf{E},$$

where $\mathbf{M}_{11}, \mathbf{M}_{12}, \mathbf{M}_{21}, \mathbf{M}_{22}$ are all 2×2 and **E** is the 2×2 matrix with every element 1.

For the adjusted Daniels–Zlobec matrix $\mathbf{Z}_1$ we recall that $m = 70$ and so $\frac{1}{2}m = 35$.

We have shown (Chu *et al.* 2010a, 2010b) that all **V**-associated magic matrices of even order are singular. It follows at once that the Daniels–Murray magic matrix cannot be **H**-associated if it is nonsingular.

In their wonderful book *Most-Perfect Pandiagonal Magic Squares: Their Construction and Enumeration*, Ollerenshaw & Brée (1998, p. 20) define the $n \times n$ magic matrix $\mathbf{P} = \{p_{i,j}\}$ with $n = 4h$ and magic sum m to be *most-perfect* when $\mathbf{P}$ (1) is $\mathbf{H}$-associated and (2) has the "blocks of four" property (McClintock 1897),

$$p_{i,j} + p_{i,j+1} + p_{i+1,j} + p_{i+1,j+1} = m$$

for all $i, j = 1, ..., n = 4h$ with the subscripts taken modulo n.

A magic matrix is said to be *pandiagonal* whenever the numbers in the broken diagonals also all add up to the magic sum.

Pandiagonal magic matrices have been widely studied and are sometimes called *diabolic, diabolical, Nasik,* or *panmagic*.

Ollerenshaw & Brée (1998, pp. 21–22) show that all most-perfect magic matrices are pandiagonal. We have recently shown (Chu *et al.* 2010a, 2010b) the stronger result that all $\mathbf{H}$-associated magic matrices with $n = 2h$ are pandiagonal.

All 4×4 pandiagonal matrices are most-perfect, and hence $\mathbf{H}$-associated (Heinz & Hendricks, *Magic Square Lexicon*, p. 97).

When the order $n = 4h$ with $h \geq 2$, however, a pandiagonal matrix need not be $\mathbf{H}$-associated and hence not most-perfect. For example,

$$\begin{pmatrix} 29 & 54 & 42 & 35 & 6 & 22 & 46 & 26 \\ 31 & 7 & 64 & 10 & 20 & 52 & 62 & 14 \\ 17 & 50 & 47 & 2 & 28 & 44 & 60 & 12 \\ 1 & 32 & 38 & 61 & 24 & 40 & 8 & 56 \\ 59 & 58 & 4 & 39 & 36 & 11 & 23 & 30 \\ 45 & 13 & 3 & 51 & 34 & 43 & 16 & 55 \\ 37 & 21 & 5 & 53 & 63 & 15 & 18 & 48 \\ 41 & 25 & 57 & 9 & 49 & 33 & 27 & 19 \end{pmatrix}$$

is pandiagonal but not $\mathbf{H}$-associated, and so not most-perfect.

Both the adjusted Daniels–Zlobec magic matrix

$$\mathbf{Z}_1 = \begin{pmatrix} 23 & \underline{10} & 21 & 16 \\ 20 & 17 & 22 & 11 \\ 14 & 19 & 12 & \underline{25} \\ 13 & 24 & 15 & 18 \end{pmatrix}$$

and its Moore–Penrose inverse $\mathbf{Z}_1^+ =$

$$\frac{1}{7280} \begin{pmatrix} 243 & \underline{33} & -9 & -163 \\ -373 & 201 & -121 & 397 \\ 61 & 215 & -191 & \underline{19} \\ 173 & -345 & 425 & -149 \end{pmatrix}$$

are pandiagonal, **H**-associated, most-perfect and *keyed*.

Let $\mathbf{M}$ denote a 4×4 magic matrix with magic sum m. When its characteristic polynomial

$$\det(\lambda \mathbf{I} - \mathbf{M}) = \lambda(\lambda - m)(\lambda^2 - \kappa); \quad \kappa \neq 0,$$

then we say that $\mathbf{M}$ is *keyed* with *magic key* $\kappa = \frac{1}{2}\left(\mathrm{tr}\mathbf{M}^2 - m^2\right) \neq 0$, and then

$$\mathbf{M}^{2p+1} = \kappa^p \mathbf{M} + m(m^{2p} - \kappa^p)\,\bar{\mathbf{E}}, \quad p = 0, 1, \ldots$$

and so the odd powers $\mathbf{M}^{2p+1}$ are all magic and inherit all the patterns present in the parent matrix $\mathbf{M}$.

We find that $\mathbf{Z}_1$ is keyed with magic key $\kappa = -48$ and hence for $p = 0, 1, \ldots,$

$$\mathbf{Z}_1^{2p+1} = (-48)^p \mathbf{Z}_1 + 70\left(70^{2p} - (-48)^p\right)\bar{\mathbf{E}}.$$

The $n \times n$ matrix $\mathbf{A}$ has *index* equal to 1 whenever $\mathrm{rank}(\mathbf{A}^2) = \mathrm{rank}(\mathbf{A})$. The *group inverse* $\mathbf{A}^{\#}$ of a matrix $\mathbf{A}$ with index 1 is the unique matrix $\mathbf{A}^{\#}$ which satisfies

$$\mathbf{A}\mathbf{A}^{\#}\mathbf{A} = \mathbf{A}, \quad \mathbf{A}^{\#}\mathbf{A}\mathbf{A}^{\#} = \mathbf{A}^{\#}, \quad \mathbf{A}\mathbf{A}^{\#} = \mathbf{A}^{\#}\mathbf{A}.$$

We find that the adjusted Daniels–Zlobec magic matrix $\mathbf{Z}_1$ has index 1 with group inverse

$$\mathbf{Z}_1^{\#} = \frac{1}{\kappa}\mathbf{Z}_1 + \Big(\frac{1}{m} - \frac{m}{\kappa}\Big)\bar{\mathbf{E}} = \frac{1}{3360}\begin{pmatrix} \underline{\overline{-373}} & 537 & -233 & 117 \\ -163 & 47 & -303 & 467 \\ 257 & -93 & \underline{397} & -513 \\ 327 & -443 & 187 & -23 \end{pmatrix},$$

which is both pandiagonal and $\mathbf{H}$-associated.

The group inverse $\mathbf{Z}_1^{\#}$ does not, however, coincide with the Moore-Penrose inverse

$$\mathbf{Z}_1^{+} = \frac{1}{7280}\begin{pmatrix} \underline{243} & 33 & -9 & -163 \\ -373 & 201 & -121 & 397 \\ 61 & 215 & \underline{\overline{-191}} & 19 \\ 173 & -345 & 425 & -149 \end{pmatrix},$$

even though the Moore–Penrose inverse $\mathbf{Z}_1^{+}$ is also both pandiagonal and $\mathbf{H}$-associated.

Following a suggestion by Peter D. Loly (2008), we introduce a *philatelic magic square* (PMS) as a square arrangement of images of postage stamps so that the associated nominal values form a magic square.

William L. Schaaf in his 1978 book *Mathematics and Science: An Adventure in Postage Stamps* found the rich and fascinating world of postage stamps to be "a mirror of civilization".

Our first PMS is based on the adjusted Daniels–Zlobec matrix $\mathbf{Z}_1$ with magic sum 70

$$\mathbf{Z}_1 = \begin{pmatrix} 23 & 10 & 21 & 16 \\ 20 & 17 & 22 & 11 \\ 14 & 19 & 12 & 25 \\ 13 & 24 & 15 & 18 \end{pmatrix}.$$

Accordingly, we have chosen stamps with some mathematical connection for this celebration.

Featured are

(row 1) Halley's comet, al-Biruni, Halley's comet (again, with telescopes from Newton, Cassegrain, Ritchey & Galileo), Pythagoras;

(row 2) Nunes, Kepler, Babbage, Franklin;

(row 3) Karamata, Napoleon Bonaparte, Terradas, Chagall's "Magician of Paris"; and

(bottom row) Dürer, Tesla, Banach, and Poincaré.

The stamps come from

(top row) Panama, Afghanistan, Nicaragua, Macedonia;

(row 2) Portugal, Czech Republic, Great Britain (2 stamps);

(row 3) Yugoslavia, Czech Republic, Spain, Bhutan, and

(bottom row) Bulgaria, Macedonia, Poland, and France.

Our birthday present for Sanjo:
a 3×3 *Croatian-diagonal* PMS
and a 4×4 *Farkas–Tesla–Napoleon* PMS:

The Croatian-diagonal PMS features

(1,1) *Arithmetika Horvatszka*, the first arithmetic book in Croatian, published in 1758; stamp issued by Croatia 2008 (*Scott* 674) to celebrate the 250th publication anniversary.

(2,2) Ruđer Josip Bošković [Rudjer Joseph Boscovich] (1711–1787), issued by Croatia 1943 (*Scott* 59).

(3,3) Nikola Tesla (1856–1943), stamp issued by Croatia 2006 (*Scott* 626) based on the 1901 photograph “Nikola Tesla, with Rudjer Boscovich's book *Theoria Philosophiae Naturalis*, in front of the spiral coil of his high-frequency transformer at East Houston Street, New York” by George Grantham Bain (1865–1944).

[*Hvala* to Sanjo for getting this stamp for me.]

The Croatian-diagonal magic matrix

$$\mathbf{S}_1 = \begin{pmatrix} 3\frac{1}{2} & 5\frac{1}{2} & 1\frac{1}{2} \\ 1\frac{1}{2} & 3\frac{1}{2} & 5\frac{1}{2} \\ 5\frac{1}{2} & 1\frac{1}{2} & 3\frac{1}{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 7 & 11 & 3 \\ 3 & 7 & 11 \\ 11 & 3 & 7 \end{pmatrix}$$

is nonsingular with magic sum $m = 10\frac{1}{2} = \frac{21}{2}$.

Its inverse

$$\mathbf{S}_1^{-1} = \frac{1}{126} \begin{pmatrix} 4 & -17 & 25 \\ 25 & 4 & -17 \\ -17 & 25 & 4 \end{pmatrix}$$

is magic with magic sum $\frac{1}{m} = \frac{2}{21}$.

Both $\mathbf{S}_1$ and $\mathbf{S}_1^{-1}$ define Latin squares
and both are **F**-associated:
all 3×3 magic matrices are **F**-associated!

Featured on the off-diagonal are

(1,2) Halley's comet, named after the astronomer Edmond Halley (1656–1742), and the Giotto space probe, named after the painter Giotto di Bondone (c. 1267–1337).

(1,3) Silk-thread writings by the Muslim polymath Abū Rayḥ ān Muḥammad ibn Aḥmad Bīrūnī [Al-Biruni] (973–1048).

(2,1) Emanuel Lasker (1868–1941), German mathematician who was World Chess Champion for 27 years (1894–1921).

(2,3) Niels Henrik Abel (1802–1829), Norwegian mathematician after whom the Abel Prize is named. The Abel Prize is presented annually by the King of Norway to one or more outstanding mathematicians.

(3,1) Francisco Ruiz Lozano (1607–1677), Peruvian astronomer and mathematician.

(3, 2) Ernest Rutherford, 1st Baron Rutherford of Nelson (1871–1937), who held the chair of physics at McGill University from 1898–1907. His work at McGill gained him the Nobel Prize in 1908 in chemistry, which is ironic since had had claimed that "All science is either physics or stamp collecting" [*Rutherford in Manchester* (J. B. Birks, ed.), Heywood, London, 1962, p. 108].

The Farkas–Tesla–Napoleon PMS has 70 in the top-left corner and 40 in the top-right. (I have not been able to build a PMS which also includes a stamp with nominal value 19.)

These two stamps in the top row were issued for Farkas Bolyai (1775–1856), while the stamp in position $(2, 3)$ depicts his son János Bolyai (1802–1860).

Both Bolyai *père et fils* were Hungarian mathematicians, as was Julius Farkas (1847–1930) for whom we have not found a stamp.

To remind myself about the (well-known?) Farkas Lemma, I Googled "Zlobec Farkas" and found the answer in the first hit ...

The Farkas lemma [Zlobec's *Stable Parametric Programming*, Kluwer 2001, Cor. 3.8, p. 32] :

> *Let* $\mathbf{A}$ *be an* $m \times n$ *matrix and* $\mathbf{b}$ *an* $m \times 1$ *vector. Then there is an* $n \times 1$ *vector* $\mathbf{x} \geq \mathbf{0}$ *such that* $\mathbf{Ax} = \mathbf{b}$ *if and only if* $\mathbf{A}'\mathbf{y} \geq \mathbf{0}$ *implies* $\mathbf{b}'\mathbf{y} \geq \mathbf{0}$.

The Farkas Lemma is named after Julius Farkas (1847–1930) for whom we have not found a stamp and who it seems is not related to Farkas Bolyai (1775–1856).

The stamps in positions (2,4) and (3,1) depict Nikola Tesla (1856–1943), inventor and electrical engineer. He was an important contributor to the birth of commercial electricity, and is best known for his many revolutionary developments in the field of electromagnetism.

Tesla's patents and theoretical work formed the basis of modern alternating current (AC) electric power systems.

He was born in the village of Smiljan, now in Croatia (Smiljan is about 6 km northeast of Gospić, and 15 km from the Zagreb–Split highway). The Tesla memorial complex, including a museum inside his restored childhood home, was opened in Smiljan in 2006 to celebrate Tesla's 150th birthyear.

Many thanks to Sanjo for introducing me to the work of Tesla.

The stamp in position (2,2) depicts Longwood House, the residence of Napoleon Bonaparte during his exile on the island of Saint Helena, 1815–1821. The stamp in position (3,2) is based on the painting "Napoléon sur son lit de mort d'après Rouget"; Inti Zlobec *et al.* (2007) recently found the cause of Napoleon's death "was strongly suggestive of stomach cancer".

Napoleon is best known as a military genius and Emperor of France but it seems he was also an outstanding mathematics student. His favourite topic apparently was geometry.

From Coxeter & Greitzer's *Geometry Revisited* [MAA, 1967, Th. 3.36, p. 63] we find Napoleon's Theorem:

> *If equilateral triangles are erected externally on the sides of any triangle, their centres form an equilateral triangle.*

The Farkas–Tesla–Napoleon magic matrix

$$\mathbf{Z}_2 =$$

$$\begin{pmatrix} 70 & 65 & 15 & 40 \\ 45 & 10 & 60 & 75 \\ 20 & 35 & 85 & 50 \\ 55 & 80 & 30 & 25 \end{pmatrix} = 5 \begin{pmatrix} 14 & 13 & 3 & 8 \\ 9 & 2 & 12 & 15 \\ 4 & 7 & 17 & 10 \\ 11 & 16 & 6 & 5 \end{pmatrix}$$

is **F**-associated and so its Moore–Penrose inverse is magic and **F**-associated.

Moreover, $\mathbf{Z}_2$ has index 1 and hence it has a group inverse $\mathbf{Z}^{\#}$. In addition the group inverse and Moore–Penrose inverse coincide:

$$\mathbf{Z}_2^{\#} = \mathbf{Z}_2^{+}$$

$$= \frac{1}{25840} \begin{pmatrix} 205 & 167 & -213 & -23 \\ 15 & -251 & 129 & 243 \\ -175 & -61 & 319 & 53 \\ 91 & 281 & -99 & -137 \end{pmatrix}.$$

Another goal is our research is to identify magic matrices with a magic Moore–Penrose inverse that are also EP.

The square matrix $\mathbf{A}$ with index 1 is said to be EP whenever

$$\text{rank}(\mathbf{A} : \mathbf{A}') = \text{rank}(\mathbf{A}),$$

i.e., whenever the column spaces of $\mathbf{A}$ and its transpose $\mathbf{A}'$ coincide.

The group inverse $\mathbf{A}^{\#}$ of the index 1 matrix $\mathbf{A}$ equals the Moore–Penrose inverse $\mathbf{A}^{+}$ if and only if $\mathbf{A}$ is EP [Ben-Israel & Greville (2003, p. 157)].

The Farkas–Tesla–Napoleon magic matrix $\mathbf{Z}_2$ is, therefore, EP, but the adjusted Daniels–Zlobec magic matrix $\mathbf{Z}_1$ is not.

We believe that the term EP was introduced in 1950 in the First Edition of Schwerdtfeger's *Introduction to Linear Algebra and the Theory of Matrices* [2nd edition (1961, pp. 130–131)].

Hans Schwerdtfeger (1902–1990) was Professor of Mathematics at McGill University from 1960–1983.

The $\mathbf{H}$-associated EP magic matrix

$$\mathbf{W} = \begin{pmatrix} 16 & 9 & 4 & 5 \\ 3 & 6 & 15 & 10 \\ 13 & 12 & 1 & 8 \\ 2 & 7 & 14 & 11 \end{pmatrix}$$

may be considered as one of the oldest EP magic matrices with a magic Moore–Penrose inverse. It corresponds to the magic square found in "an old temple in the hill fort of Gwalior" in India by "Captain Shortreede" in 1842 and "bears the date A.D. 1483".

Position (1,2): Benjamin Franklin (1706–1790), one of the Founding Fathers of the United States, also printer, satirist, political theorist, politician, postmaster, scientist, inventor, civic activist, statesman, and diplomat. Franklin also worked on magic squares:

> *I confess that in my younger days, having once some leisure which I still think I might have employed more usefully, I amused myself in making a kind of magic squares ...*

Position (1,3): a new local mail stamp from Sakhalin (Russia) for Ruđer Josip Bošković [Rudjer Joseph Boscovich] (1711–1787), mathematician, physicist, astronomer, philosopher, diplomat, poet, theologian, born in Ragusa (now Dubrovnik).

Positions (2,1) and (4,3): these are part of a set issued by several countries in the Middle East to celebrate Arabic achievements in various fields including specifically "algebra".

Position (3,3): the 39th Mersenne prime number

$$2^{13466917} - 1,$$

discovered in 2001 (stamp issued by Liechtenstein 2004).

The mathematician and physicist Baron Jurij Bartolomej Vega (1754–1802), born in Zagorica (now in Slovenia), is depicted in position (3,4), while a slide rule is shown in in postion (4,1) issued by Romania for the 2nd Congress of the Society of Engineers and Technicians, held in Bucharest, 29–31 May 1957.

Position (4,3): Jacob Bernoulli (1654–1705), born in Basel, Switzerland, who first proved the law of large numbers (illustrated on the stamp), while the last stamp shows the Polish astronomer and geodesist Tadeusz Banachiewicz (1882–1954), who invented "Cracovians"—a special kind of matrix algebra.

We end this talk with a 3×3 Tesla PMS defined by the magic matrix

$$\mathbf{Z}_3 = \begin{pmatrix} 20 & 12+33 & 10 \\ 15 & 25 & 2+33 \\ 2 \times 3\frac{1}{2} + 33 & 5 & 30 \end{pmatrix}$$

$$= \begin{pmatrix} 20 & 45 & 10 \\ 15 & 25 & 35 \\ 40 & 5 & 30 \end{pmatrix} = 5 \begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix},$$

which is **F**-associated and so its Moore–Penrose inverse is magic and **F**-associated.

...