Magic generalized inverses, with special emphasis on involution-associated magic matrices[1]

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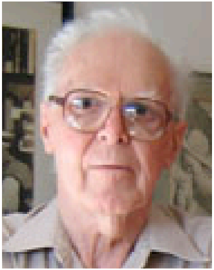


This talk is dedicated to the memory of Harold Dean Matheson (1933–2010) and Martin Gardner (1914–2010), who both died on Saturday, 22 May 2010.

My wife Evelyn's brother, Harold, was devoted to his family and community and his interests included gardening, golfing, fund raising—no task for him was ever impossible. He was the first member of Evelyn's family that George met in 1967.

Mathematics and science writer, Martin Gardner, specialized in recreational mathematics, with interests in micromagic, literature (especially the writings of Lewis Carroll), and philosophy. He wrote the Mathematical Games column in the *Scientific American* from 1956–1981.

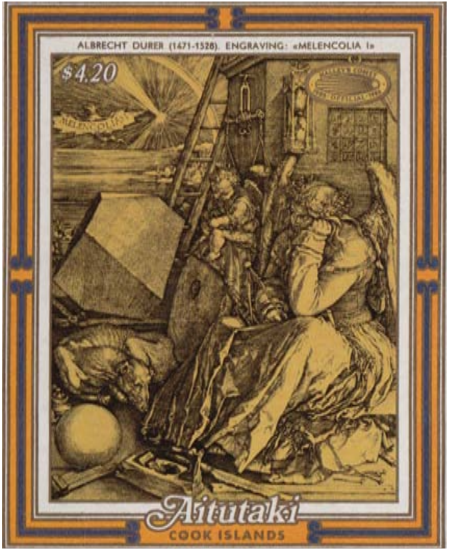
In *A Lifetime of Puzzles: A Collection of Puzzles in Honor of Martin Gardner's 90th Birthday* (AK Peters 2008) we find the warning

> *Martin Gardner has turned hundreds of mathematicians into magicians and hundreds of magicians into mathematicians!*

from Persi Diaconis, while from David Singmaster we learn that the magic square defined by the *magic matrix*

$$\mathbf{M} = \begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix}$$

is given by the Italian mathematician and Franciscan friar Luca Pacioli (c. 1446–1517) in his *De viribus quantitatis.* This magic square is in the top-right corner of *Melencolia I,* the 1514 copper-plate engraving by the German painter Albrecht Dürer (1471–1528) shown in this 1986 stamp from Aitutaki (Cook Islands).



The *magic matrix*

$$\mathbf{M} = \begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix}$$

has Moore–Penrose inverse

$$\mathbf{M}^{+} = \frac{1}{2720} \begin{pmatrix} 275 & -201 & -167 & 173 \\ 37 & -31 & -65 & 139 \\ -99 & 105 & 71 & 3 \\ -133 & 207 & 241 & -235 \end{pmatrix}.$$

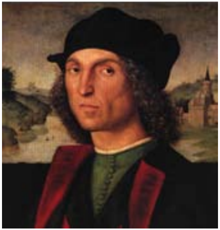
Both $\mathbf{M}$ and $\mathbf{M}^{+}$ define a magic square—the numbers in all the rows and columns and in both main diagonals all add up to the same *magic sum.*

Our main purpose in this talk is to identify magic matrices $\mathbf{A}$ with this property: that both $\mathbf{A}$ and $\mathbf{A}^{+}$ define a magic square.

Singmaster (*A Lifetime of Puzzles*, 2008), and Mackinnon (*The Mathematical Gazette*, 1993) note that the magic square defined by the matrix **M** also appears in Pacioli's *De viribus quantitatis*, written 1496–1508.

This well-known portrait of Pacioli with a "student" is attributed to Jacopo de' Barbari (c. 1440–c. 1515) and painted c. 1495.

Mackinnon (1993, p. 140) and Singmaster (2008, p. 83) suggest that the "student" may be Dürer (1471–1528), though R. Emmett Taylor: (*No Royal Road*, 1942, p. 203), says he is Guidobaldo da Montefeltro, Duke of Urbino (1472–1508).



The Dürer self-portrait on this stamp from France 1980 was painted c. 1491. The painting (on the right) is of Guidobaldo by Raffaello Sanzio (1483–1520).

The magic matrix

$$\mathbf{M} = \begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix},$$

which we now call the Pacioli–Dürer matrix, is *fully-magic*—the numbers in all the rows and columns and in both main diagonals all add up to the same *magic sum*, here $m = 34$.

Moreover $\mathbf{M}$ has rank 3, and so is singular, and its Moore-Penrose inverse

$$\mathbf{M}^{+} = \frac{1}{2720} \begin{pmatrix} 275 & -201 & -167 & 173 \\ 37 & -31 & -65 & 139 \\ -99 & 105 & 71 & 3 \\ -133 & 207 & 241 & -235 \end{pmatrix}$$

is also fully-magic, with magic sum $\frac{1}{34} = \frac{1}{m}$.

Our major interest in this research is in identifying fully-magic matrices, like $\mathbf{M}$, where the Moore-Penrose inverse is <u>also</u> fully-magic.

We are also interested in illustrating our results with postage stamps: This 1994 stamp from Sri Lanka was issued in celebration of the 500th anniversary of double-entry bookkeeping: Luca Pacioli is (also) considered the "father of accountancy".

We define the $n \times n$ flip matrix

$$\mathbf{F} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

and the $n \times n$ matrix

$$\bar{\mathbf{E}} = \frac{1}{n} \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix}.$$

The Pacioli–Dürer matrix

$$\mathbf{M} = \begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix}$$

and its Moore–Penrose inverse $\mathbf{M}^+$ satisfy

$$\mathbf{FM} + \mathbf{MF} = \bar{\mathbf{E}}$$

or equivalently

$$\mathbf{M} + \mathbf{FMF} = \bar{\mathbf{E}},$$

and we will say that such matrices are **F**-associated (associative or regular are often used in the literature).

The $n \times n$ matrices with n even:

$$\mathbf{F} = \begin{pmatrix} \mathbf{0} & \mathbf{F}_{n/2} \\ \mathbf{F}_{n/2} & \mathbf{0} \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} \mathbf{F}_{n/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_{n/2} \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_{n/2} \\ \mathbf{I}_{n/2} & \mathbf{0} \end{pmatrix}; \quad n = 2h,$$

are *involutory matrices* in that $\mathbf{F}^2 = \mathbf{G}^2 = \mathbf{H}^2 = \mathbf{I}$; an involutory matrix defines an *involution*. Moreover $\mathbf{F}, \mathbf{G}$ and $\mathbf{H}$ are symmetric and have row totals all equal to 1.

When $n = 2h + 1$ is odd, then we define $\mathbf{F}, \mathbf{G}, \mathbf{H}$ similarly, but with an extra row and column in the middle with all elements 0 except for 1 in the centre.

We define $\mathbf{V}$ to be a symmetric involutory matrix with all row totals equal to 1. Then the $n \times n$ *semi-magic* matrix $\mathbf{A}$ with magic sum m is $\mathbf{V}$-*associated*, or *involution-associated*, whenever

$$\mathbf{AV} + \mathbf{VA} = 2m\bar{\mathbf{E}} \quad \text{or equivalently} \quad \mathbf{A} + \mathbf{VAV} = 2m\bar{\mathbf{E}}.$$

We note that $\mathbf{V}$ is semi-magic with magic sum 1.

We will say that the $n \times n$ matrix $\mathbf{A}$ is *semi-magic* whenever all the rows and columns of $\mathbf{A}$ add up to the same magic sum m, and *trace-magic* when $\mathbf{A}$ is semi-magic and, in addition, the trace $\mathrm{tr}\mathbf{A} = m$.

The trace-magic matrix $\mathbf{A}$ is *fully-magic* when, in addition, the trace $\mathrm{tr}\mathbf{AF} = m$.

The trace-magic matrix $\mathbf{A}$ is $\mathbf{V}$-*magic* when, in addition, $\mathrm{tr}\mathbf{AV} = m$, and so an $\mathbf{F}$-*magic* matrix is fully-magic.

The $\mathbf{V}$-associated trace-magic matrix $\mathbf{A}$ is $\mathbf{V}$-magic.

Whenever the involutory matrix $\mathbf{V}$ commutes with the flip matrix $\mathbf{F}$ or equivalently $\mathbf{V}$ is *centrosymmetric*, i.e.,

$$\mathbf{V} = \mathbf{FVF},$$

then the $\mathbf{V}$-associated trace-magic matrix $\mathbf{A}$ is fully-magic ($\mathbf{F}$-magic).

Our key result is that the Moore–Penrose inverse $\mathbf{A}^+$ of a $\mathbf{V}$-associated semi-magic matrix $\mathbf{A}$ is also $\mathbf{V}$-associated, and if, in addition, $\mathbf{V}$ is *centrosymmetric*, then both $\mathbf{A}$ and $\mathbf{A}^+$ are fully-magic.

For example, the Pacioli–Dürer matrix $\mathbf{M}$, and its Moore-Penrose inverse $\mathbf{M}^+$,

$$\mathbf{M} = \begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix}, \quad \mathbf{M}^+ = \frac{1}{2720} \begin{pmatrix} 275 & -201 & -167 & 173 \\ 37 & -31 & -65 & 139 \\ -99 & 105 & 71 & 3 \\ -133 & 207 & 241 & -235 \end{pmatrix}$$

are both $\mathbf{F}$-associated and fully-magic.

We have found no fully-magic matrix $\mathbf{A}$ with a fully-magic Moore–Penrose inverse that is not $\mathbf{V}$-associated for some involutory matrix $\mathbf{V}$ that is both symmetric and centrosymmetric (*bisymmetric*).

We recall that we write $\mathbf{V}$ for a symmetric involutory matrix with all row totals equal to 1.

And that the $n \times n$ *semi-magic* matrix $\mathbf{A}$ with magic sum m is $\mathbf{V}$-*associated*, or *involution-associated*, whenever

$$\mathbf{AV} + \mathbf{VA} = 2m\bar{\mathbf{E}} \tag{1}$$

or equivalently

$$\mathbf{A} + \mathbf{VAV} = 2m\bar{\mathbf{E}}, \tag{2}$$

with $\mathrm{tr}\mathbf{A} = \mathrm{tr}\mathbf{AV} = m$ and so $\mathbf{A}$ is trace-magic and $\mathbf{V}$-magic.

If $\mathbf{V}$ is centrosymmetric then in addition $\mathrm{tr}\mathbf{AF} = m$ and $\mathbf{A}$ is fully-magic.

It follows at once from (1) that when $\mathbf{A}$ is $\mathbf{V}$-associated then

$$\mathbf{A}^2\mathbf{V} = \mathbf{VA}^2,$$

and from (1) and (2) that then $\mathbf{A}^3$ is also $\mathbf{V}$-associated, i.e.,

$$\mathbf{A}^3\mathbf{V} + \mathbf{VA}^3 = 2m^3\bar{\mathbf{E}}$$

or equivalently

$$\mathbf{A}^3 + \mathbf{VA}^3\mathbf{V} = 2m^3\bar{\mathbf{E}}.$$

Higher order even and odd powers of $\mathbf{A}$ behave similarly.

The French mathematician Bernard Frénicle de Bessy (c. 1605–1675) identified the 880 (or $7040 = 8 \times 880$ to include reflections and rotations) *classic* 4×4 fully-magic squares and the English mathematician Henry Ernest Dudeney (1857–1930) classified them into 12 types.

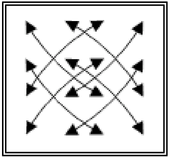
An $n \times n$ *classic* fully-magic square contains n^2 consecutive integers, usually $1, 2, \ldots, n^2$, as with the Pacioli-Dürer magic matrix $\mathbf{M}$, which is Dudeney Type III, and so now we call $\mathbf{M}_3$.

We define

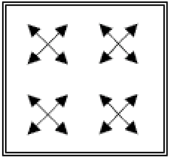
$$\mathbf{M}_1 = \begin{pmatrix} 8 & 11 & 14 & 1 \\ 13 & 2 & 7 & 12 \\ 3 & 16 & 9 & 6 \\ 10 & 5 & 4 & 15 \end{pmatrix}, \quad \mathbf{M}_2 = \begin{pmatrix} 12 & 3 & 6 & 13 \\ 14 & 5 & 4 & 11 \\ 7 & 16 & 9 & 2 \\ 1 & 10 & 15 & 8 \end{pmatrix}, \quad \mathbf{M}_3 = \begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix}.$$

The $\mathbf{H}$-associated matrix $\mathbf{M}_1$ of Dudeney Type I is contained in the *Rasa'il: Encyclopedia of the Brethren of Purity* first published in the late 10th century.

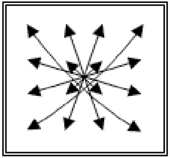
The $\mathbf{G}$-associated matrix $\mathbf{M}_2$ of Dudeney Type II is given by Ṭhakkura Pherū (fl. 1291–1323) in *Gaṇitasārakaumudī: The Moonlight of the Essence of Mathematics*, first published in the early 14th century in Sanskrit and recently reset and translated into English.



Type I



Type II



Type III

$$\mathbf{M}_1 = \begin{pmatrix} 8 & 11 & 14 & 1 \\ 13 & 2 & 7 & 12 \\ 3 & 16 & 9 & 6 \\ 10 & 5 & 4 & 15 \end{pmatrix}, \quad \mathbf{M}_2 = \begin{pmatrix} 12 & 3 & 6 & 13 \\ 14 & 5 & 4 & 11 \\ 7 & 16 & 9 & 2 \\ 1 & 10 & 15 & 8 \end{pmatrix}, \quad \mathbf{M}_3 = \begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix},$$

Rasa'il *Pherū* *Pacioli–Dürer*

$$\mathbf{H} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Out of 880 classic 4×4 fully-magic matrices enumerated by Frénicle, the

- 48 of Dudeney Type I are all **H**-associated,
- 48 of Dudeney Type II are all **G**-associated,
- 48 of Dudeney Type III are all **F**-associated.

And so these 144 ($= 3 \times 48$) all have fully-magic Moore–Penrose inverses.

Of the other $880 - 144 = 736$ classic fully-magic matrices

- none has a fully-magic or trace-magic Moore–Penrose inverse
- none is **V**-associated.

Closely related to every involutory matrix is an idempotent matrix.

Let the $n \times n$ symmetric matrix $\mathbf{V}$ be involutory, i.e., $\mathbf{V}^2 = \mathbf{I}$, and suppose that $\mathbf{Ve} = \mathbf{e}$. Suppose further that $\mathbf{V}$ is not necessarily centrosymmetric. Then the *complementary* involutory matrix

$$\mathbf{W} = 2\bar{\mathbf{E}} - \mathbf{V}$$

also satisfies $\mathbf{We} = \mathbf{e}$, and $\mathbf{W}$ is centrosymmetric if and if $\mathbf{V}$ is centrosymmetric.

The matrix

$$\mathbf{Z} = \tfrac{1}{2}(\mathbf{I} - \mathbf{V}) = \tfrac{1}{2}(\mathbf{I} + \mathbf{W} - 2\bar{\mathbf{E}}) = \mathbf{Z}^2$$

is idempotent and hence

$$\mathrm{rank}(\mathbf{Z}) = \mathrm{rank}(\mathbf{I} - \mathbf{V}) = \tfrac{1}{2}(n - \mathrm{tr}\mathbf{V}) = \tfrac{1}{2}(n + \mathrm{tr}\mathbf{W}) - 1,$$

and so $n + \mathrm{tr}\mathbf{V}$ and $n + \mathrm{tr}\mathbf{W}$ must both be even and hence

the three numbers n, $\mathrm{tr}\mathbf{V}$ and $\mathrm{tr}\mathbf{W}$ are all even or all odd.

Let the $n \times n$ magic matrix $\mathbf{A}$ be $\mathbf{V}$-associated (with $\mathbf{V}$ not necessarily centrosymmetric). Then

$$\mathbf{AV} + \mathbf{VA} = \mathbf{A} + \mathbf{VAV} = 2m\bar{\mathbf{E}}$$

and with $\mathbf{Z} = \frac{1}{2}(\mathbf{I} - \mathbf{V})$, it follows that $\mathbf{Ze} = \mathbf{0}$ and

$$\mathbf{ZAZ} = \tfrac{1}{4}(\mathbf{I} - \mathbf{V})\mathbf{A}(\mathbf{I} - \mathbf{V}) = \mathbf{0}.$$

Hence

$$\mathbf{A} = m\bar{\mathbf{E}} + \mathbf{AZ} + \mathbf{ZA}$$

$$\begin{aligned} \operatorname{rank}(\mathbf{A}) &= 1 + \operatorname{rank}(\mathbf{AZ}) + \operatorname{rank}(\mathbf{ZA}) \\ &\leq 1 + 2\operatorname{rank}(\mathbf{Z}), \end{aligned}$$

which leads to

$$\operatorname{rank}(\mathbf{A}) \leq n - |\operatorname{tr}\mathbf{V} - 1| \leq n,$$

where $|\cdot|$ denotes absolute value.

It follows at once that (with **V** not necessarily centrosymmetric)

$$\mathbf{A} = \mathbf{V}\text{-associated} \quad \& \quad \{\mathrm{tr}\mathbf{V} \le 0 \quad \text{or} \quad \mathrm{tr}\mathbf{V} \ge 2\} \quad \Rightarrow \quad \mathrm{rank}(\mathbf{A}) \le n-1,$$

$$\mathbf{A} = \mathbf{V}\text{-associated} \quad \& \quad \mathrm{rank}(\mathbf{A}) = n \quad \Rightarrow \quad \mathrm{tr}\mathbf{V} = 1 \quad \Rightarrow \quad n = 2h+1.$$

These implications do not go the other way—we have found a 7×7 **V**-associated fully-magic matrix with rank 6 and $\mathrm{tr}\mathbf{V} = 1$.

When **A** is **V**-associated and nonsingular then $\mathbf{V} + \mathbf{A}^{-1}\mathbf{V}\mathbf{A} = 2\bar{\mathbf{E}}$ and so

$$\mathrm{tr}\mathbf{V} = 1 \quad \text{and} \quad \mathrm{rank}(\mathbf{AZ}) = \mathrm{rank}(\mathbf{ZA}) = \mathrm{rank}(\mathbf{Z}) = \tfrac{1}{2}(n+1) - 1,$$

and hence n is odd as noted above. And so all **V**-associated trace-magic matrices of even order are singular.

The 4×4 matrix

$$\mathbf{L} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

defines a *Latin square* since the numbers $1, 2, 3, 4$ appear precisely once in each row and each column.

While all Latin squares define semi-magic matrices, the matrix $\mathbf{L}$ is fully-magic since the numbers in the two main diagonals also add up to the the magic sum $m = 10$.

In addition, the matrix $\mathbf{L}$ is $\mathbf{G}$-associated *and* $\mathbf{H}$-associated

$$\mathbf{L} + \mathbf{GLG} = 2m\bar{\mathbf{E}} = \mathbf{L} + \mathbf{HLH}.$$

The matrix $\mathbf{L}$ is in *mini-Sudoku form* in that the numbers $1, 2, 3, 4$ each appear once in the top left, top right, bottom left, and bottom right 2×2 corners.

Yates (*Biometrika*, 1939) says that such Latin squares have "balanced corners".

The involutory matrices $\mathbf{F}, \mathbf{G}$ and $\mathbf{H}$ satisfy

$$\mathbf{FG} = \mathbf{GF} = \mathbf{H},$$

$$\mathbf{FH} = \mathbf{HF} = \mathbf{G},$$

$$\mathbf{GH} = \mathbf{HG} = \mathbf{F}.$$

Since the matrix $\mathbf{L}$ is $\mathbf{G}$-associated and $\mathbf{H}$-associated, it follows that

$$\mathbf{GLG} = \mathbf{HLH} \quad \Rightarrow \quad \mathbf{L} = \mathbf{FLF},$$

i.e., $\mathbf{L}$ is centrosymmetric.

A classic fully-magic matrix cannot be centrosymmetric since $\mathbf{L} = \mathbf{FLF}$ implies that, e.g., the $(1,1)$ and $(4,4)$ elements be equal.

When $n = 4$ (and only when $n = 4$) then we have the "alphabet string"

$$\mathbf{E} = \mathbf{F} + \mathbf{G} + \mathbf{H} + \mathbf{I},$$

where $\mathbf{E}$ is the $n \times n$ matrix with every element equal to 1 and $\mathbf{I}$ is the $n \times n$ identity matrix.

The fully-magic Moore–Penrose inverse

$$\mathbf{L}^+ = \frac{1}{80}\begin{pmatrix} -13 & -3 & 7 & 17 \\ 7 & 17 & -13 & -3 \\ -3 & -13 & 17 & 7 \\ -17 & 7 & -3 & -13 \end{pmatrix}$$

is, like $\mathbf{L}$, both $\mathbf{G}$- and $\mathbf{H}$-associated, and centrosymmetric.

Moreover, $\mathbf{L}^+$ defines a Latin square since the 4 numbers

$$-\frac{13}{80}, \quad -\frac{3}{80}, \quad \frac{7}{80}, \quad \frac{17}{80}$$

each appear once in each row and in each column.

The fully-magic "Latin square" matrix $\mathbf{L}$, which we will now call $\mathbf{L}_1$

$$\mathbf{L}_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

is in *reduced form* since the numbers $1, 2, 3, 4$ appear in sequence in the top row.

There are 24 Latin square matrices of order 4 in reduced form and these are all semi-magic but only 4 of these are fully-magic: $\mathbf{L}_1$ and

$$\mathbf{L}_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \quad \mathbf{L}_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad \mathbf{L}_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

All four matrices $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3$ and $\mathbf{L}_4$ are singular with rank 3 and each has a fully-magic Moore–Penrose inverse. Moreover, $(\mathbf{L}_2)^2$ and $(\mathbf{L}_3)^2$, both with rank 2, also have fully-magic Moore–Penrose inverses.

The matrix $\mathbf{L}_2$ is $\mathbf{G}$-associated, the matrix $\mathbf{L}_3$ is $\mathbf{H}$-associated and, as already noted, $\mathbf{L}_1$ is both $\mathbf{G}$- and $\mathbf{H}$-associated.

The matrix $\mathbf{L}_4$, however, is neither $\mathbf{G}$- nor $\mathbf{H}$-associated and is also not $\mathbf{F}$-associated, but it is $\mathbf{V}$-associated with centrosymmetric

$$\mathbf{V} = \mathbf{I} - 2\mathbf{z}\mathbf{z}', \qquad \mathbf{z} = \frac{1}{\sqrt{2(9+x^2)}} \begin{pmatrix} 3 \\ x \\ -x \\ -3 \end{pmatrix} = -\mathbf{F}\mathbf{z}, \tag{3}$$

and

$$x = 1 + \sqrt{10} \quad \text{or} \quad x = 1 - \sqrt{10}.$$

The matrix $\mathbf{V}$ in (3) is called a *Householder transformation* and we say that $\mathbf{L}_4$ is *Householder-associated* or $\mathbf{z}$*-associated*.

We call the idempotent matrix $\mathbf{Z} = \mathbf{z}\mathbf{z}'$ the *Householder matrix*.

It is interesting that both

$$\mathbf{L}_2^2 = \begin{pmatrix} 27 & 23 & 27 & 23 \\ 23 & 27 & 23 & 27 \\ 23 & 27 & 23 & 27 \\ 27 & 23 & 27 & 23 \end{pmatrix}, \quad \mathbf{L}_3^2 = \begin{pmatrix} 27 & 27 & 23 & 23 \\ 23 & 23 & 27 & 27 \\ 23 & 23 & 27 & 27 \\ 27 & 27 & 23 & 23 \end{pmatrix},$$

are fully-magic, though neither $\mathbf{L}_1^2$ nor $\mathbf{L}_4^2$ is fully-magic. The Moore–Penrose inverses

$$(\mathbf{L}_2^2)^+ = \frac{1}{800}\begin{pmatrix} 27 & -23 & -23 & 27 \\ -23 & 27 & 27 & -23 \\ 27 & -23 & -23 & 27 \\ -23 & 27 & 27 & -23 \end{pmatrix}, \quad (\mathbf{L}_3^2)^+ = \frac{1}{800}\begin{pmatrix} 27 & -23 & -23 & 27 \\ 27 & -23 & -23 & 27 \\ -23 & 27 & 27 & -23 \\ -23 & 27 & 27 & -23 \end{pmatrix},$$

are both fully-magic.

The matrix $\mathbf{L}_2^2$ is both **F**- and **H**-associated but not **G**-associated, though $\mathbf{L}_2^2$ is **G**-magic and the fully-magic Latin square $\mathbf{L}_2$ is **G**-associated.

The matrix $\mathbf{L}_3^2$ is both **F**- and **G**-associated but not **H**-associated, though $\mathbf{L}_3^2$ is **H**-magic and the fully-magic Latin square $\mathbf{L}_3$ is **H**-associated.

matrix count	A type	A name	A n	A rank	A index	A V-assoc	A z-assoc	A^2 rank	A^2 z-assoc	A^2 V-assoc	A^2 V-magic
1	Latin	L2	4	3	3	G	1	2	1	FH	FGH
1	Latin	L3	4	3	3	H	1	2	1	FG	FGH
8	classic	Frenicle	4	3	3	H	1	2	1	0	H
8	classic	Frenicle	4	3	3	G	1	2	1	0	G
8	classic	Frenicle	4	3	3	F	1	2	1	0	F
26											

The matrices $\mathbf{L}_2$ and $\mathbf{L}_3$ have just one nonzero eigenvalue: the magic sum m, and both $\mathbf{L}_2$ and $\mathbf{L}_3$ have rank 3 and index 3. This property also holds for 24 of the Frénicle 880 classic fully-magic matrices: 8 each of Dudeney types I, II, III.

But for these 24 the squared matrix $\mathbf{A}^2$ is not **F**-, **G**- or **H**-associated though all 24 are **z**- or Householder-associated, with Householder matrix $\mathbf{Z} = \mathbf{zz}' = \mathbf{I} - \mathbf{A}^+\mathbf{A}$ or $\mathbf{I} - \mathbf{AA}^+$. The squares of the 8 matrices of Dudeney type I are, however, **H**-magic, type II **G**-magic, and type III **F**-magic.

Our major objective in this research has been to identify fully-magic matrices which have a fully-magic Moore–Penrose inverse.

Our major result is that a trace-magic matrix **A** that is **V**-associated:

$$\mathbf{A} + \mathbf{VAV} = \mathbf{AV} + \mathbf{VA} = 2m\bar{\mathbf{E}}$$

has a trace-magic Moore–Penrose inverse that is also **V**-associated.

When **V** is centrosymmetric: $\mathbf{V} = \mathbf{FVF}$, then both **A** and $\mathbf{A}^+$ are fully-magic.

We have not found a fully-magic matrix **A** with fully-magic Moore–Penrose inverse $\mathbf{A}^+$ that is not **V**-associated.

We have found, however, 63 fully-magic *philatelic Latin squares.* And each of these 63 is connected to a fully-magic matrix with a fully-magic Moore–Penrose inverse.

Postage stamps are occasionally issued in sheetlets (small sheets) of 4 different stamps printed in a 4×4 array with 4 copies of each of the 4 stamps.

Sometimes the sheetlet forms a philatelic Latin square (PLS): each of the 4 stamps appears exactly once in each row and exactly once in each column. When the Latin square is fully-magic we have a magic PLS:

$$\mathbf{L}_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \quad \mathbf{L}_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$

$$\mathbf{L}_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad \mathbf{L}_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

We have identified 172 4×4 PLS with 63 that are fully-magic:

54 type $\mathbf{L}_1$, 1 type $\mathbf{L}_2$, 8 type $\mathbf{L}_3$ & no type $\mathbf{L}_4$.

To illustrate we present this $\mathbf{L}_1$ PLS from Monaco 1991 depicting Hermann's tortoise (*Testudo hermanni*), a small- to medium-sized tortoise that we believe is found in Tuscany and makes an excellent pet.

An(other) open question: Who was Hermann?

Another fully-magic square with an Italian connection is defined by the 9×9 matrix

$$\mathbf{A} =$$

$$\begin{pmatrix} 15 & 58 & 29 & 34 & 63 & 49 & 74 & 41 & 6 \\ 7 & 27 & 31 & 81 & 23 & 76 & 80 & 18 & 26 \\ 38 & 8 & 30 & 71 & 47 & 20 & 21 & 78 & 56 \\ 73 & 19 & 25 & 42 & 10 & 33 & 50 & 65 & 52 \\ 22 & 55 & 72 & 1 & 45 & 60 & 28 & 16 & 70 \\ 79 & 35 & 39 & 66 & 2 & 48 & 17 & 24 & 59 \\ 14 & 64 & 69 & 12 & 77 & 3 & 51 & 68 & 11 \\ 46 & 36 & 61 & 53 & 40 & 43 & 4 & 54 & 32 \\ 75 & 67 & 13 & 9 & 62 & 37 & 44 & 5 & 57 \end{pmatrix},$$

which is nonsingular; its inverse $\mathbf{A}^{-1}$ is not fully-magic, in fact its trace $\mathrm{tr}\mathbf{A}^{-1}$ is negative!

Called "Quadratus Maximus", this magic square is engraved in a marble plaque dated 1766 on a wall of the Villa Albani, near Rome. The plaque is in the wall opposite the foot of the stairs from the first floor apartments.

The Villa Albani was built in 1748 for Cardinal Alessandro Albani (1692–1779) to house his collection of antiquities, which were catalogued by the Cardinal's secretary, the first professional art historian, Johann Joachim Winckelmann (1717–1768).

The only magic matrix connected with Fibonacci, Leonardo Pisano seems to be the matrix

$$\mathbf{B} = \begin{pmatrix} 13 & 144 & 5 \\ 8 & 21 & 55 \\ 89 & 3 & 34 \end{pmatrix}$$

formed by replacing the entries in the 3×3 fully-magic *Luoshu* matrix

$$\mathbf{C} = \begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix}$$

with the corresponding Fibonacci numbers:

$$3, 5, 8, 13, 21, 34, 55, 89, 144.$$

The products of the 3 numbers in each of the 3 rows of **B** add up to the products of the 3 numbers in each of the 3 columns:

$$9078 + 9240 + 9360 = 27678 = 9256 + 9072 + 9350.$$

There is apparently no 3×3 fully-magic matrix with distinct Fibonacci numbers.

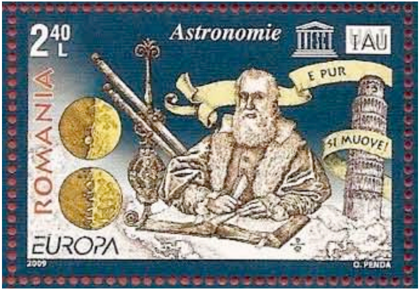
The Luoshu matrix **C** is nonsingular and its inverse $\mathbf{C}^{-1}$ is fully-magic. The *Fibonacci matrix* **B** is also nonsingular but its inverse

$$\mathbf{B}^{-1} = \frac{1}{221208} \begin{pmatrix} 183 & -1627 & 2605 \\ 1541 & -1 & -225 \\ -615 & 4259 & -293 \end{pmatrix},$$

however, does not seem to have any magical properties.

We have found no magic square connected to Galileo Galilei but there are many stamps in his honour. In particular this 1964 stamp from Czechoslovakia for his 400th birth anniversary.

In 2009 Romania issued a postage stamp featuring Galileo and the Tower of Pisa. After having to recant before the Inquisition in 1633 his belief that the Earth moves around the Sun, Galileo apprently said "E pur si muove" ("and yet it moves").



When the order n is even, then **H**-associated magic matrices are *pandiagonal*, i.e., the broken diagonals all sum to the magic number. We define the *one-step forward circulant* $n \times n$ *permutation matrix*

$$\mathbf{S} = \begin{pmatrix} \mathbf{0}_{n-1} & \mathbf{I}_{n-1} \\ 1 & \mathbf{0}'_{n-1} \end{pmatrix},$$

where $\mathbf{0}_{n-1}$ is the $(n-1) \times 1$ column vector with each element equal to zero. When $n = 2h$, then

$$\mathbf{S}^h = \begin{pmatrix} \mathbf{0} & \mathbf{I}_h \\ \mathbf{I}_h & \mathbf{0} \end{pmatrix} = \mathbf{H}; \quad n = 2h.$$

Then the $n \times n$ fully-magic matrix **A** with magic sum m is pandiagonal whenever

$$\mathrm{tr}\mathbf{AS} = \mathrm{tr}\mathbf{AS}^2 = \cdots = \mathrm{tr}\mathbf{AS}^{n-1} = m \tag{4}$$

$$= \mathrm{tr}\mathbf{FAS} = \mathrm{tr}\mathbf{FAS}^2 = \cdots = \mathrm{tr}\mathbf{FAS}^{n-1}. \tag{5}$$

The $n-1$ traces in (4) give the sums of the first $n-1$ broken-diagonals parallel to the forwards main-diagonal, while the $n-1$ traces in (5) give the sums of the first $n-1$ broken-diagonals parallel to the backwards main-diagonal.

When the $2h \times 2h$ matrix $\mathbf{A}$ is $\mathbf{H}$-associated, then

$$2m\bar{\mathbf{E}} = \mathbf{AH} + \mathbf{HA} = \mathbf{AS}^h + \mathbf{S}^h\mathbf{A} \tag{6}$$

and so $\mathrm{tr}(\mathbf{AS}^h) = m$. Premultiplying (6) by $\mathbf{S}^{-1}$ yields $\mathrm{tr}(\mathbf{AS}^{h-1}) = m$ and postmultiplying by $\mathbf{S}$ yields $\mathrm{tr}(\mathbf{AS}^{h+1}) = m$. Continuing like this establishes (1);

$$\mathrm{tr}\mathbf{AS} = \mathrm{tr}\mathbf{AS}^2 = \cdots = \mathrm{tr}\mathbf{AS}^{n-1} = m.$$

A similar argument leads to (2):

$$m = \mathrm{tr}\mathbf{FAS} = \mathrm{tr}\mathbf{FAS}^2 = \cdots \mathrm{tr}\mathbf{FAS}^{n-1}.$$

The only classic 4×4 fully-magic matrices that are pandiagonal are also $\mathbf{H}$-associated. Trump (2007) showed that the number of $\mathbf{H}$-associated classic 8×8 fully-magic matrices is in the interval $(2.5228 \pm .0014) \times 10^{27}$ with probability 99% and that the number of pandiagonal 8×8 classic fully-magic matrices is (strictly) larger than this—but apparently no estimate of this number is available.

It seems that there are "many" pandiagonal 8×8 classic fully-magic matrices that are not $\mathbf{H}$-associated, and possibly even "many" that are not even $\mathbf{V}$-associated.

As we will see below, there are no **H**-associated nonsingular fully-magic matrices of even order. The 8×8 fully-magic matrix

$$\mathbf{B} = \begin{pmatrix} 29 & 54 & 42 & 35 & 6 & 22 & 46 & 26 \\ 31 & 7 & 64 & 10 & 20 & 52 & 62 & 14 \\ 17 & 50 & 47 & 2 & 28 & 44 & 60 & 12 \\ 1 & 32 & 38 & 61 & 24 & 40 & 8 & 56 \\ 59 & 58 & 4 & 39 & 36 & 11 & 23 & 30 \\ 45 & 13 & 3 & 51 & 34 & 43 & 16 & 55 \\ 37 & 21 & 5 & 53 & 63 & 15 & 18 & 48 \\ 41 & 25 & 57 & 9 & 49 & 33 & 27 & 19 \end{pmatrix},$$

however, is pandiagonal, and nonsingular and hence is not **H**-associated.

Chater & Chater (1949) proved that all "pan-magic" fully-magic matrices of even order are singular, apparently intrerpreting "pan-magic" to mean both pandiagonal and **H**-associated.

Mattingly (2000) proved that all (classic) **F**-associated fully-magic matrices of even order are singular and Chater & Chater (1949) proved that all

Given a fully-magic $n \times n$ matrix $\mathbf{A}$ we wish to solve the equation

$$\mathbf{AV} + \mathbf{VA} = 2m\bar{\mathbf{E}} \tag{7}$$

for a symmetric involutory martrix $\mathbf{V}$, if one exists!

Using the vec-operator we may write (7) as

$$(\mathbf{I} \otimes \mathbf{A} + \mathbf{A}' \otimes \mathbf{I})\text{vec}\mathbf{V} = 2m\,\text{vec}\bar{\mathbf{E}} = 2m\,\mathbf{e}_{(n^2\times 1)}. \tag{8}$$

The eigenvalues of the matrix

$$\mathbf{K} = \mathbf{I} \otimes \mathbf{A} + \mathbf{A}' \otimes \mathbf{I} \tag{9}$$

are the n^2 pairs $\alpha_i + \alpha_j$ and so, under certain diagonability conditions, the matrix $\mathbf{K}$ has nullity ν equal to the number of these pairs that are equal to 0.

For $n = 3$ and $\mathbf{A}$ Luoshu

$$\begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix} \tag{10}$$

we find that this nullity $\nu = 2$ and the rank of the 9×9 matrix $\mathbf{K}$ is 7.