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Caïssan squares: the magic of chess

by George P. H. Styan[1], McGill University

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(left panel) Frontispiece from *The Poetical Works of Sir William Jones* [86, vol. 1 (1810)]: online at *Chess Notes* by Edward Winter, (right panel) Saeed Jaffrey as Mir, Sanjeev Kumar as Mirza in "Shatranj Ke Khilari" (The Chess Players) [167].

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KEY WORDS AND PHRASES

alternate-couplets property, William Roxby Beverley (c. 1814–1889), Pavle Bidev (1912–1988), Caïssan magic squares, "Cavendish", EP magic matrices, William A. Firth (c. 1815–1890), Firth–Zukertort "magic chess board", 4-ply magic squares, Andrew Hollingworth Frost (1819–1907), involution-associated magic squares, involutory matrix, Henry Jones (1831–1899), Sir William Jones (1746–1794), Henry James Kesson (b. c. 1844), keyed magic matrices, knight-Nasik, magic key, most-perfect magic squares, Nasik squares, pan-diagonal magic square, philatelic items, Charles Planck (1856–1935), postage stamps, rhomboid, rhomboidal magic matrices, Robert Shortrede (1800–1868), "Ursus", Johannes Hermann Zukertort (1842–1888).

ABSTRACT

We study various properties of $n \times n$ Caïssan magic squares. A magic square is Caïssan whenever it is pandiagonal and knight-Nasik so that all paths of length n by a chess bishop are magic (pandiagonal) and by a (regular) chess knight are magic (CSP2-magic). Following the seminal 1881 article [7] by "Ursus" in *The Queen,* we show that 4-ply magic matrices, or equivalently magic matrices with the "alternate-couplets" property, have rank at most equal to 3. We also show that an $n \times n$ magic matrix $\mathbf{M}$ with rank 3 and index 1 is EP if and only if $\mathbf{M}^2$ is symmetric. We identify and study 46080 Caïssan beauties, which are pandiagonal and both CSP2- and CSP3-magic; a CSP3-path is by a special knight that leaps over 3 instead of 2 squares. We find that just 192 of these Caïssan beauties are EP. An extensive annotated and illustrated bibliography of over 300 items, many with hyperlinks, ends our report. We give special attention to items by (or connected with) "Ursus": Henry James Kesson (b. c. 1844), Andrew Hollingworth Frost (1819–1907), Charles Planck (1856–1935), and Pavle Bidev (1912–1988). We have tried to illustrate our findings as much as possible, and whenever feasible with images of postage stamps or other philatelic items.

REFERENCES

In Sections 9.1–9.4 we identify some publications by or connected with, respectively, "Ursus": Henry James Kesson (b. c. 1844), Andrew Hollingworth Frost (1819–1907), Charles Planck (1856–1935), and Pavle Bidev (1912–1988). Some other publications about Caïssan magic squares and related topics are listed in Section 9.5 with an associated portrait gallery in Section 9.6. Within Sections 9.1–9.5 references are listed chronologically. We end the report by listing some philatelic resources (Section 9.8) and some bio-bibliographic resources (Section 9.9).

ACKNOWLEDGEMENTS

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1. Caïssan magic squares

In this report we study various properties of $n \times n$ Caïssan magic squares. A magic square is Caïssan whenever it is pandiagonal and knight-Nasik so that all paths of length n by a chess bishop are magic (pandiagonal) and by a (regular) chess knight are magic (CSP2-magic). We give special emphasis to such squares with $n = 8$. Following the seminal 1881 article [7] by "Ursus" in *The Queen,* we look at involution-associated magic matrices and show that 4-ply magic matrices, or equivalently magic matrices with the "alternate-couplets" property, have rank at most equal to 3. We identify and study 46080 Caïssan beauties, which are pandiagonal and both CSP2- and CSP3-magic; a CSP3-path is by a special knight that leaps over 3 instead of 2 squares. We study the n-queens problem (1848), the Firth–Zukertort "magic chess board" (1887), and the Beverley "magic knight's tour" (1848). An extensive illustrated bibliography of over 250 items, many with hyperlinks, ends our report. We give special attention to items by (or connected with) "Ursus": Henry James Kesson (b. c. 1844), Andrew Hollingworth Frost (1819–1907), Charles Planck (1856–1935), and Pavle Bidev (1912–1988). We have tried to illustrate our findings as much as possible, and whenever feasible with images of postage stamps or other philatelic items.

1.1. **Classic magic squares.** An $n \times n$ "magic square" or "fully-magic square" is an array of numbers, usually integers, such that the numbers in all the rows, columns and two main diagonals add up to the same number, the magic sum $m \neq 0$. When only the the numbers in all the rows and columns all up to the magic sum $m \neq 0$ then we have a "semi-magic square". The associated matrix $\mathbf{M}$, say, will be called a "magic matrix"[2] and we will assume in this paper that $\text{rank}(\mathbf{M}) \geq 2$. An $n \times n$ magic square is said to be "classic" whenever it comprises n^2 consecutive integers, usually $1, 2, \ldots, n^2$ (or $0, 1, \ldots, n^2 - 1$) each precisely once; the magic sum is then $m = n(n^2 + 1)/2$ (or $n(n^2 - 1)/2$), and so when $n = 8$, the magic sum $m = 260$ (or 252). Other names for a classic magic square include natural, normal or pure.

1.2. **Pandiagonal magic squares and magic paths.** We define an $n \times n$ magic square to be "pandiagonal" whenever all $2n$ diagonal "paths" with wrap-around parallel to (and including) the two main diagonals are magic. A "path" here is continuous of length n (with wrap-around) and with consecutive entries a chess-bishop's (chess-piece's) move apart. Our paths all proceed in the same direction. A path is said to be magic whenever its n elements add up to the magic sum m. In our $n \times n$ pandiagonal magic square, therefore, all its rook's and bishop's (and queen's, king's and pawn's) paths are magic and so it has (at least) $4n$ magic paths. In a semi-magic square there are $2n$ magic paths for the rooks (and kings).

Other names for a pandiagonal magic square include continuous, diabolic, Indian, Jaina, or Nasik. The usage here of the word "Nasik" stems from a series of papers by Andrew H. Frost ([20, (1865)], [24, 25, 26, (1877)], [29, (1896)]) in which a magic square is defined to be "Nasik"[3]

[2] The term magic matrix seems to have been introduced by Charles Fox in 1956 in an article entitled "Magic matrices" in *The Mathematical Gazette* (vol. 40, pp. 209–211) and in a problem (untitled) in *The American Mathematical Monthly* (vol. 63, p. 584), both published in October 1956.

[3] Andrew Hollingworth Frost (1819–1907) was a British missionary with the Church Missionary Society living in Nasik, India, in 1853–1869 [**?**]. Nasik is in the northwest of Maharashtra state (180 km from Mumbai and 220 km from Pune), India and India Security Press in Nasik is where a wide variety of items like postage stamps, passports, visas, and non-postal adhesives are printed.

whenever it is pandiagonal[4] and satisfies "several other conditions"[5]. More recent usage indicates that a magic square is Nasik whenever it is (just) pandiagonal. The term "Nasik square" was apparently first defined by Andrew's older brother, the mathematician Percival Frost (1817–1898)[6] in his "introduction" of the paper by A. H. Frost [20, (1865)]. In that paper [20, p. 94] an $n \times n$ Nasik square is said to satisfy $4n$ (not completely specified) conditions as does a pandiagonal magic square.

Andrew H. Frost [29, (1896)] gives a method by which Nasik squares of the nth order can be formed for all values of n; a Nasik square being defined to be "A square containing n cells in each side, in which are placed the natural numbers from 1 to n in such an order that a constant sum $\frac{1}{2}n(n^2 + 1)$ is obtained by adding the numbers on n of the cells, these cells lying in a variety of directions denned by certain laws."

1.3. **Caïssa: the "patron goddess" of chess players.**

The "patron goddess" of chess players was named Caïssa by Sir William Jones (1746–1794), the English philologist and scholar of ancient India, in a poem entitled "Caïssa" [79] published in 1763.

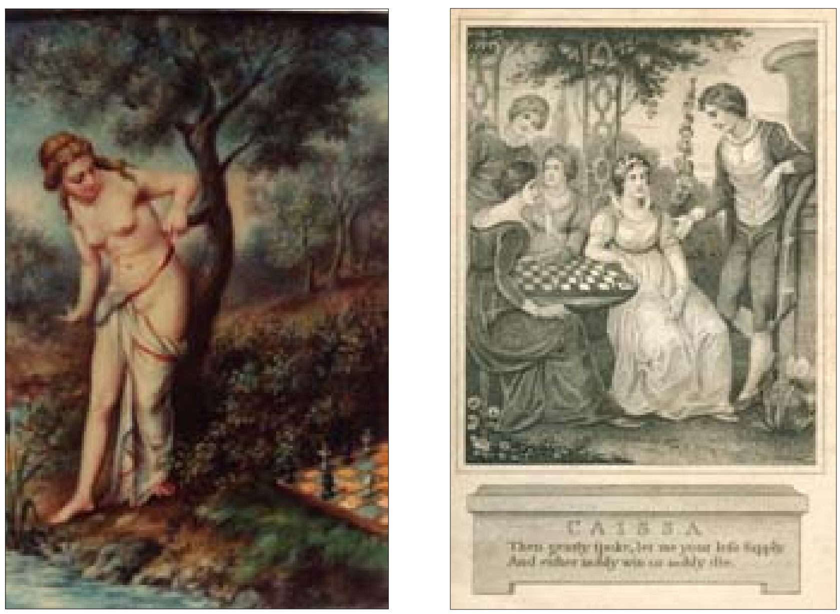


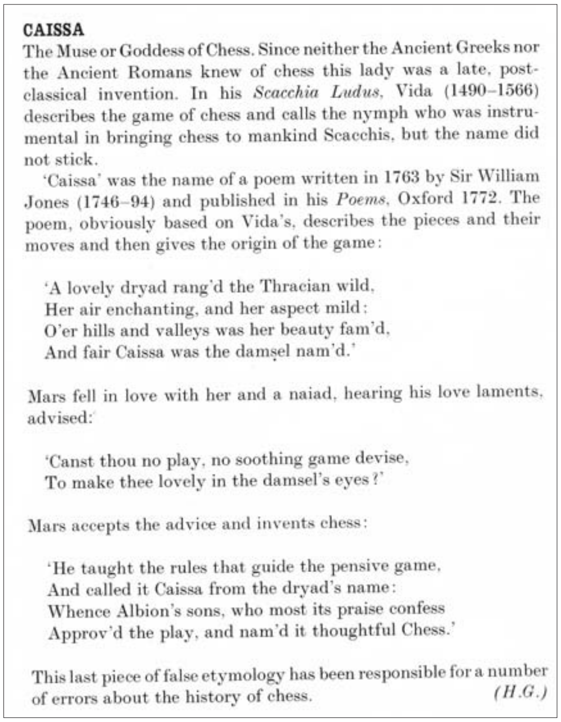
FIGURE 1.3.1: (left panel) Illustration of Caïssa, apparently by Domenico Maria Fratto (1669–1763)[7] [324]; (right panel) Frontispiece from *The Poetical Works of Sir William Jones,* volume 1 (London, 1810): online at *Chess Notes* by Edward Winter.

[4]Percival Frost in his introduction to [20, (1865)] by his younger brother Andrew H. Frost , says that "Mr. A. Frost has investigated a very elegant method of constructing squares, in which not only do the rows and columns form a constant sum, but also the same constant sum is obtained by the *same number of summations in the directions of the diagonals*— — I shall call them Nasik Squares".

[5]Andrew H. Frost [29, (1896)] defines a Nasik square to be "A square containing n cells in each side, in which are placed the natural numbers from 1 to n^2 in such an order that a constant sum $\frac{1}{2}n(n^2+1)$ (here called W) is obtained by adding the numbers on n of the cells, these cells *lying in a variety of directions defined by certain laws.*"

[6]For an obituary of Percival Frost (1817–1898) see [30].

[7]It seems the artist died in 1763, the year the poem was published.



CAISSA

The Muse or Goddess of Chess. Since neither the Ancient Greeks nor the Ancient Romans knew of chess this lady was a late, post-classical invention. In his *Scacchia Ludus*, Vida (1490–1566) describes the game of chess and calls the nymph who was instrumental in bringing chess to mankind Scacchis, but the name did not stick.

'Caissa' was the name of a poem written in 1763 by Sir William Jones (1746–94) and published in his *Poems*, Oxford 1772. The poem, obviously based on Vida's, describes the pieces and their moves and then gives the origin of the game:

'A lovely dryad rang'd the Thracian wild,
Her air enchanting, and her aspect mild:
O'er hills and valleys was her beauty fam'd,
And fair Caissa was the damsel nam'd.'

Mars fell in love with her and a naiad, hearing his love laments, advised:

'Canst thou no play, no soothing game devise,
To make thee lovely in the damsel's eyes?'

Mars accepts the advice and invents chess:

'He taught the rules that guide the pensive game,
And called it Caissa from the dryad's name:
Whence Albion's sons, who most its praise confess
Approv'd the play, and nam'd it thoughtful Chess.'

This last piece of false etymology has been responsible for a number of errors about the history of chess. *(H.G.)*

FIGURE 1.3.2: (left panel) "Caissa: The Muse or Goddess of Chess" from [168, p. 54]; (right panel) Women chess players and magic chessboard: Rwanda 2005, *Scott* TBC.

The first Caïssan magic square that we have found seems to be that presented in 1881 by "Ursus"[8] (1881), who introduces his seminal article in this way:

> ONCE UPON A TIME, when Orpheus was a little boy, long before the world was blessed with the "Eastern Question", there dwelt in the Balkan forests a charming nymph by name Caïssa. The sweet Caïssa roamed from tree to tree in Dryad meditation fancy free. As for trees, she was, no doubt, most partial to the box[9] and the ebony[10] See Figure 1.3.3.
>
> Mars read of her charms in a "society" paper, saw her photograph, and started by the first express for Thracia. He came, he saw, but he conquered not—in fact, was ingloriously repulsed. As he was wandering by the Danube blurting out, to give vent to his chagrin, many unparliamentary expressions, a Naiad advised him to go to Euphron, who ruled over a sort of Lowtherian Arcadia. Euphron good-naturedly produced from his stores a mimic war game, warranted to combine amusement with instruction.

[8]According to Irving Finkel [10], the adjective Caïssan was suggested to "Ursus" (Henry James Kesson) by "Cavendish" (Henry Jones), who originated it. Morever Iltyd Nicholl suggests that "Kesson" itself was a nom-de plume, deriving from the site called "Nassek" (Nasik, Nashik) where a contention-producing magic-square had been earlier discovered over a gateway. We do not know of any such magic square.

[9]*Buxus* is a genus of about 70 species in the family *Buxaceae*. Common names include box (majority of English-speaking countries) or boxwood (North America). See also Figure 1.3.3. [324]

[10]Ebony is a general name for very dense black wood. In the strictest sense it is yielded by several species in the genus *Diospyros,* but other heavy, black (or dark coloured) woods (from completely unrelated trees) are sometimes also called ebony. Some well-known species of ebony include *Diospyros ebenum* (Ceylon ebony), native to southern India and Sri Lanka, *Diospyros crassiflora* (Gaboon ebony), native to western Africa, and *Diospyros celebica* (Makassar ebony), native to Indonesia and prized for its luxuriant, multi-coloured wood grain. [324]

The warrior—in mufti—took the game to Caïssa, taught her how to play, called it after her name, and, throwing off his disguise, proposed. The delighted Caïssa consented to become Mrs Mars. He spent the honeymoon in felling box trees and ebony ditto, she in fashioning the wood to make tesselated boards and quaint chessmen to send all over the uncivilised world.

Such is our profane version of the story, told so prettily in Ovidian verse by Sir William Jones, the Orientalist. Such is Caïssa, as devoutly believed in by Phillidorians as though she were in Lemprière's *Biographical Dictionary.* Our Caïssa, however, shall herself be a chessman—we beg her pardon—chesswoman. Like the "queen" in the Russian game, she shall have the power of every man, moving as a king, rook, bishop, or pawn (like the queen in our game), and also as a knight. Caïssa then is complete mistress of the martial board.

FIGURE 1.3.3: (left panel) White pieces made of boxwood, black piece is ebonized, not ebony; (right) Elephant carvings from Ceylon (Sri Lanka), from ebony, likely Gaboon ebony (*Diospyros dendro*). [324]

And Ursus [7, (1881)] ends his article in this way:

> The idea of the foregoing Caïssan magic squares was suggested partly by the Brahminical squares, that from time immemorial have been used by the Hindoos as talismans; partly by the article [24] by the Rev. A. H. Frost, M.A., of St. John's College, Camb., on "Nasik Squares," in No 57 of *The Quarterly Journal of Pure and Applied Mathematics*" (1877). Our squares are, however, mostly original, as are the methods of construction, though one or two, notably that for the fifteen-square, may by Girtonians and Newnhamites readily be translated into the (mathematically) purer language adopted in Mr. Frost's able paper.
>
> The epithet "Caïssan" described the distinguishing characteristics of the squares, as the paths include all possible continuous chess moves. We commenced with Caïssa's mythical history; let us conclude with a few facts less apocryphal. We have alluded to the Brahminical squares. Of these the favorites are the four-square and the eight-square. We have shown that the latter is, from our point of view, the first perfect one; in fact as the Policeman of Penzance would say, "taking one consideration with another"[11], it may be pronounced the best of the lot. Now *chaturanga*, the great-grandfather of chess, is of unknown antiquity and Duncan Forbes given translations of venerable Sanskrit manuscripts, which describe the pieces, moves, and mode of playing the game. *Chaturanga* means four arms, i.e., the four parts of an army game–elephants, camels, boats (for fighting on the rivers, Sanskrit *roka*, a boat, hence rook), and foot-soldiers, the whole commanded by the kaisar or king.

[11]From *The Pirates of Penzance,* a comic opera in two acts, with music by Arthur Sullivan and libretto by W. S. Gilbert. The opera's official première was at the Fifth Avenue Theatre in New York City on 31 December 1879. [324]

> At first there were four armies, two as allies against two others; afterwards the allies united, one of the kings becoming prime minister with almost unlimited powers, and in the chivalrous west being denominated Queen. We have already noticed that the Russians, who are inveterate chess players, and who probably got their national game direct from the fountainhead, endow their queen with the powers of all the other pieces. So then in the west we have king, queen, bishop (ex-elephant), knight (ex-camel), castle or rook, and pawn, the line soldier on whom, after all, military success depends. Sir William Jones, who wrote "Caïssa" when he was a boy, thinking in Greek, but then little versed in Sanskrit, turned the corrupt Saxon "chess" into the pseudo-classic Caïssa. Strange coincidence, for much later, his admirer, Duncan Forbes, traces "chess", by such links as the French *échecs*, English *check* and *check mate*, German *schach matt*, to the Persian *Shah Met*, Shah being their rendering of the Sanskrit or Aryan *Kaisar*, still retained by Germans, and by Russians in *Czar*.
>
> Sir William's Caïssa would pass very well for the feminine of Kaisar; but this is not the coincidence we wish to accentuate. Both "chess" and "Caïssan" squares, under whatever name the reader pleases, have been known in India–the nursery of civilisation–from the remotest antiquity; hence there would seem to be a close connection between them. India is now the greatest possession of the British crown, and our Queen, as everyone knows, bears the masculine title of Kaisar-in-Hind[12].

The poem "Caïssa" (see Figure 1.3.2 above) by Sir William Jones[13] was based on the poem "Scacchi, Ludus" published in 1527 by the Italian humanist, bishop and poet Marco Girolamo [Marcus Hieronymus] Vida (c. 1485–1566), giving a mythical origin of chess that has become well known in the chess world. [324]. As observed by Murray [131, p. 793]:

> [26] Vida's description of the moves and rules, and the game (a Queen's Gambit), contain nothing of material importance. The name *Scacchis*, which Vida bestowed upon the nymph who was the means of teaching chess to mankind, has not commended itself to players, and *Caïssa*, the creation of Sir William Jones (1763), has supplanted it entirely.

[12]The Kaisar-i-Hind (sometimes misspelt as Kaiser-i-Hind or Kaisar-in-Hind) was a medal awarded by the British monarch between 1900 and 1947, to civilians of any nationality who rendered distinguished service in the advancement of the interests of the British Raj. The medal name literally means "Emperor of India" in the vernacular of the Hindi and Urdu languages. The world kaisar, meaning "emperor" is derived from the Latin "Caesar", derived from the name of the first Roman emperor, Julius Caesar. It is a cognate with the German title Kaiser, which was borrowed from the Latin at an earlier date. [324].

[13]The father of Sir William Jones (1746–1794) was William Jones, FRS (1675–1749), a Welsh mathematician, whose most noted contribution was his proposal for the use of the Greek letter π as the symbol to represent the ratio of the circumference of a circle to its diameter. [324].

FIGURE 1.3.4: (left panel) Marco Girolamo [Marcus Hieronymus] Vida (c. 1485–1566) [324]; (right panel) Sir William Jones (1746–1794), 250th birth anniversary: India 1997, *Scott*[14] 1626.

FIGURE 1.3.5: (left panel) Portable chess board used in postal chess contest: Brazil 1980, *Scott* 1723 [308]; and (right panel) on cover issued for the 15th anniversary of the "Associação Brasileira de Filatelistas Temâticas (Abrafite) "Caissa", Patrona do Xadrez, São Paulo, January 4, 1985. [296].

[14]*Scott* numbers are as published in the *Scott Standard Postage Stamp Catalogue* [308].

1.4. **"Ursus": Henry James Kesson (b. c. 1844).** It seems that the first person to explicitly connect Caïssa with magic squares was "Ursus" in a three-part article entitled "Caïssan magic squares" published in *The Queen: The Lady's Newspaper & Court Chronicle*[15] 1881. We believe that this "Ursus" was the *nom de plume* of Henry James Kesson (b. c. 1844), about whom we know very little. We do know that "Ursus" was a regular contributor to *The Queen* with a five-part article [5] on "Magic squares and circles" published two years earlier in 1879 and a six-part article [6] on "Trees in rows" in 1880, as well as many double acrostics (puzzles and solutions). Falkener (1892, p. 337) cites work on magic squares by "H. J. Kesson (Ursus)" in *The Queen,* 1879–1881. Whyld (1978), quoting Bidev (1977), says that the "pioneering work [on Caïssan magic squares] was done about a century ago by a London mathematician named Kesson, who under the pen-name Ursus, wrote a series of articles in *The Queen*".

The 1894 book on magic squares by "Cavendish"[16] is dedicated to Henry James Kesson "by his sincere friend, the Author" and it seems that our "Ursus" is the H. J. Kesson, who wrote the words for five "operetta-cantatas for young people" (with music by Benjamin John W. Hancock, 1894a, 1894b, 1894c, 1901a, 1901b) and a booklet on the legend of the Lincoln Imp[17] (1904).

In addition, we believe that this Henry James Kesson is the one listed in the 1851 British Census as age 7, "born in St Pancras", with father John Kesson, age 40, "attendant at British Museum"[18]. The 1851 Census apparently also stated that the family then (1851) lived at 40 Chichester Place, Grays Inn Lane, St Pancras. As well as Henry James Kesson and his father John, the family then comprised Maria Kesson [John's wife], then also age 40, born in Finsbury, and children Maria Jane, then age 13, born Islington, Lucy Emma, age 9 and Arnold age 3, both born in St Pancras.

Horn [17, Document F] mentions a "[School]master: Henry James Kesson, Trained Two Years; Certificated First Class. Emily Kesson, Sewing Mistress and General Assistant. ..." in 1890 in Austrey, a village at the northern extremity of the county of Warwickshire, near Newton Regis and No Man's Heath, and close to the Leicestershire villages of Appleby Magna, Norton-juxta-Twycross and Orton on the Hill.

Following [7], we define a magic square to be "Caïssan" whenever it is pandiagonal and all the "knight's paths" are magic. And so in an $n \times n$ Caïssan magic square, all the $8n$ paths by a chess piece (rook, bishop, knight, queen, king) are magic. The knights we consider here are the regular chess-knights, and we may call such Caïssan squares "regular Caïssan squares".

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[15]Published from 1864–1922, *The Queen: The Lady's Newspaper & Court Chronicle* was started by Samuel Orchart Beeton (1831–1877) and Isabella Mary Beeton *née* Mayson (1836–1865), and Frederick Greenwood (1830–1909). Isabella Beeton is universally known as "Mrs Beeton", the principal author of *Mrs Beeton's Book of Household Management,* first published in 1861 and still in print.

[16]"Cavendish" was the *nom de plume* of Henry Jones (1831–1899), an English author well-known as a writer and authority on card games and who in 1877 founded the "The Championships, Wimbledon, or simply Wimbledon, the oldest tennis tournament in the world and considered the most prestigious" [324].

[17]According to a 14th-century legend two mischievous creatures called imps were sent by Satan to do evil work on Earth. After causing mayhem in Northern England, the two imps headed to Lincoln Cathedral where they smashed tables and chairs and tripped up the Bishop. When an angel came out of a book of hymns and told them to stop, one of the imps was brave and started throwing rocks at the angel but the other imp cowered under the broken tables and chairs. The angel turned the first imp to stone giving the second imp a chance to escape.

[18]The book, *The Cross and the Dragon* [3] is by John Kesson "of the British Museum". This is probably the John Kesson (d. 1876) who translated *Travels in Scotland* by J. G. Kohl from German into English [1] and *The Childhood of King Erik Menved* by B. S. Ingemann from Danish into English [2]. It is just possible that the Scottish novelist, playwright and radio producer Jessie Kesson (1916–1994), born as Jessie Grant McDonald, may be a descendant (in law) since in 1934 she married Johnnie Kesson, a cattleman, living in Abriachan (near Inverness) and then Rothienorman (near Aberdeen).

"Caïssa's special path" is defined by [7, p. 142 (our part 1/3, page 4/8)] as a special knight's path where the "special knight" (Caïssa) can move 3 steps instead of 2 (e.g., down 1 and over 3 or up 3 and over 1). We will call such a path Caïssa's special path of type 3 (CSP3). A "special knight" with paths of type CSP3 is called a "jumping rukh" by [186]: "Most important of all is the fact that the 'jumping rukh' which accesses a third square vertically or horizontally, as depicted in the theory of [140] appears here in an astonishing manner despite starting from any square." The regular knight's path, therefore, is of type 2 (CSP2).

In his study of a 15×15 Caïssan magic square [7, p. 391 (our part 3/3, page 4/9, Fig. R)], which we examine in Section 7 below, Caïssan special paths of types 4, 5, 6, and 7 (CSP4, CSP5, CSP6, CSP7) are (also) considered.

1.5. **"Caïssan beauties" and "knight-Nasik" magic squares.** The term "knight-Nasik" has been used by (at least) Planck [40, 44, 45], Woodruff [134, 137], Andrews [135], Foster [138], and Marder [147] to mean either a pandiagonal magic square with all $4n$ regular-knight's paths of type CSP2 magic, or just a magic square, not necessarily pandiagonal, with all $4n$ regular-knight's paths of type CSP2 magic. Ursus (1881) defined a Caïssan magic square to be pandiagonal with all $4n$ regular-knight's paths of type CSP2 magic.

The first use of the term "knight-Nasik" that we have found is in Planck[40, p. 17, footnote] where "the well-known" [8×8 magic square]

$$\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix}_8 \tag{1.5.1}$$

is said to be "knight-Nasik". We have not yet been able to identify the magic matrix defined by (1.5.1) but we expect it to be pandiagonal, with all $4n$ regular-knight's paths of type CSP2 magic.

We will use the following terminology:

DEFINITION 1.5.1. We define an $n \times n$ magic matrix, usually classic and 8×8, to be

(1) "CSP2-magic" when all $4n$ regular-knight's paths of type CSP2 and each of length n are magic,

(2) "CSP3-magic" when all $2n$ special-knight's paths of type CSP3 and each of length n are magic,

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The bishop's predecessor in "shatranj" (medieval chess) was the "alfil" (meaning "elephant" in Arabic, "éléphant" in French, "olifant" in Dutch), which could leap two squares along any diagonal, and could jump over an intervening piece. As a consequence, each alfil was restricted to eight squares, and no alfil could attack another. Even today, the word for the bishop in chess is "alfil" in Spanish and "alfiere" in Italian. Whyld [64] mentions the original alfil, and discusses several properties of the 8×8 Ursus magic matrix **U** (Figure 1.5.1).

Quotes & Queries

by **K. Whyld**, Moorland House, Kelsey Road, Caistor, Lincs. LN7 6SF.

No.3942 - Arising from Professor Bidev's article in July *BCM* a request has been made for an example of the magic squares to which he refers, and more background. In a paper on the subject, published in *Bonus Socius*, The Hague 1977, Bidev says that the pioneering work was done about a century ago by a London mathematician named Kesson who, under the pen-name Ursus, wrote a series of articles 'Magic Squares and Caissan Magic Squares' in *The Queen*, and he adds that Kesson did not reach the same conclusion because he did not know the original moves of the pieces. The German problemist and chess historian, Johannes Kohtz, investigated around 1918 the connection between chess and magic squares at the dawn of the game, but his work remains unpublished.

1	58	3	60	8	63	6	61
16	55	14	53	9	50	11	52
17	42	19	44	24	47	22	45
32	39	30	37	25	34	27	36
57	2	59	4	64	7	62	5
56	15	54	13	49	10	51	12
41	18	43	20	48	23	46	21
40	31	38	29	33	26	35	28

The book *From Magic Squares to Chess*, by N.M.Rudin, was published in 1969, and Pavle Bidev's important *Sah Simbol Kosmosa* appeared in 1972. Unfortunately these are both difficult for us because the first is in Russian and the second in Serbo-Croatian, but Bidev provides a good summary in German and a rather shorter one in English. The specimen square comes from Bidev's book.

The usual properties of a magic square are that each column and file and the two long diagonals total the same number, which on a 8×8 board is 260. This example has many other features. The two squares occupied by the white rooks plus the two occupied by their pawns total the half-constant of 130. The same applies for Black, and again for the corresponding knight, bishop and king plus queen sets of squares. The eight squares available to knights before pawns are moved total 260. The total of all squares covered or occupied by knights in the initial position is 520. The bishop's ancestor, the alfil, could move only to the next but one square diagonally, and thus could reach only eight squares on the board, and no two alfils could meet. Each of these four sets of eight squares totals 260. The four sets of four squares occupied by rooks, knights, bishops, and king plus queen, equal the half-constant. If the king moves through an octagon: Ke1-f2-f3-e4-d4-c3-c2-d1, or, Ke1-f1-g2-g3-f4-e4-d3-d2, the squares total 260, and the same for Black. This short outline barely scratches the surface. Professor Bidev concludes that the moves of the pieces were determined by the properties of the Nasik magic squares.

Figure 1.5.1: Quotes & Queries No. 3942 by K. Whyld [64, (1978)].

DEFINITION 1.5.2. We define an $n \times n$ (fully) magic matrix, usually classic and 8×8, to be a

(1) "Caïssan magic matrix" when it is pandiagonal ($2n$ magic paths) and CSP2-magic ($4n$ magic paths) and so there are (at least) $8n$ magic paths each of length n in all,

(2) "special-Caïssan magic matrix" when it is pandiagonal ($2n$ magic paths) and CSP3-magic ($2n$ magic paths) and so there are (at least) $6n$ magic paths each of length n in all,

(3) a "Caïssan beauty" (CB) when it is pandiagonal ($2n$ magic paths), and both CSP2- and CSP3-magic ($6n$ magic paths)and so there are (at least) $10n$ magic paths each of length n in all,

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"Cavendish" (1894) defines a Caïssan magic square as one being (just) pandiagonal. Planck (1900), however, citing the "late Henry Jones (Cavendish)", and we assume referring to the book on magic squares by Cavendish (1894), says that this "nomenclature is of doubtful propriety" and defines a Caïssan magic square as being magic in all regular chess-move paths, with wrap-around. And so such Caïssan magic squares are both pandiagonal with all CSP2 regular-knight's paths being magic.

Possibly the first person, however, to consider Caïssan magic squares was Simon de la Loubère (1642–1729), a French diplomat, writer, mathematician and poet. From his Siamese travels, he brought to France a very simple method for creating n-odd magic squares, now known as the "Siamese method" or the "de la Loubère method". This method apparently was initially brought from Surat, India, by a *médecin provençal* by the name of M. Vincent [77, TBC]. According to Marder [147, p. 5 (1940)] (see also Andrews [135, p. 165 (1917)]), the following words[19] come from the pen of La Loubère:

> In these [8×8] squares it is necessary not merely that the summation of the rows, columns and diagonals should be alike, but that the sum of any eight numbers in one direction as in the moves of a bishop or a knight should also be alike.

Andrews [135, p. 165, Fig. 262 (1917)] presents as "an example of one of these squares" the Ursus matrix **U** given by Ursus [7, p. 142, Fig. D (1881)], see our Figure 1.6.1 below.

[19] We have not found any mention of chess in the discussion of magic squares by Simon de La Loubère [77, vol. 2, pp. 227–247 (1693)].

1.6. **The** 8×8 **Ursus Caïssan magic matrix U.** The first so-called "Caïssan magic square" (CMS) that we have found is that presented by [7, p. 142, Fig. D] and displayed in our Figure 1.6.1. A magic regular-knight's path (CSP2) is marked with red circles (left panel) and a magic special-knight's path of type CSP3 is marked with red boxes (right panel).

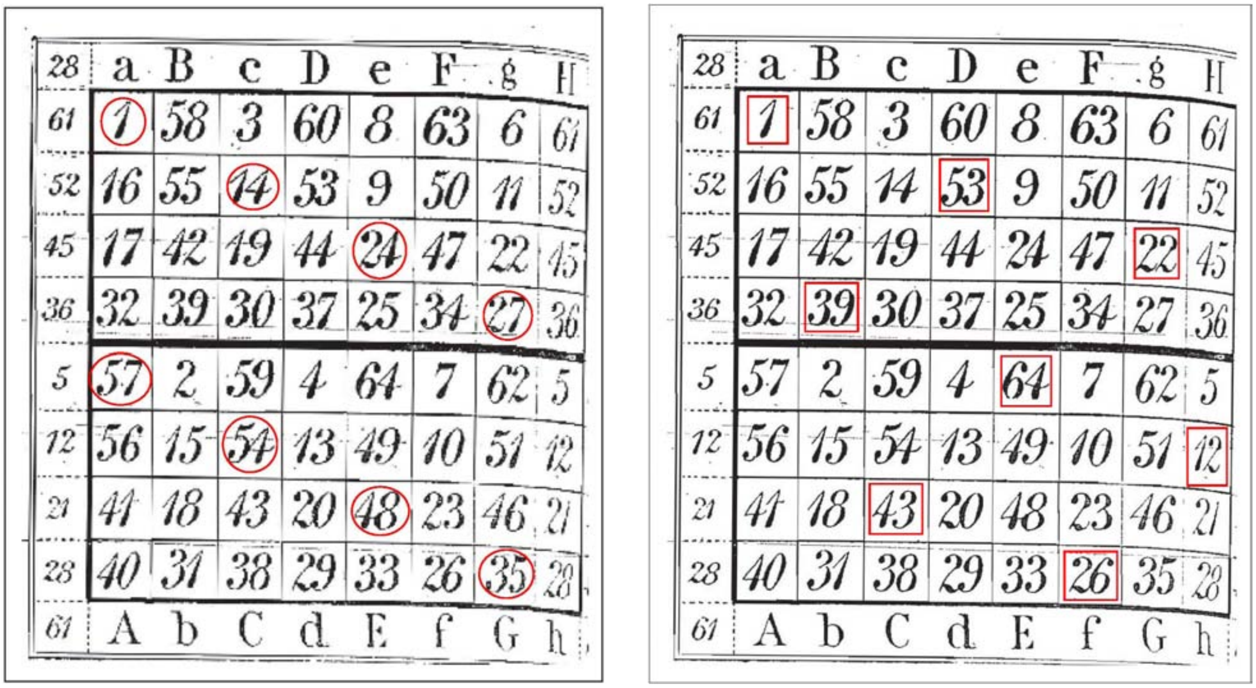

FIGURE 1.6.1: The 8×8 Caïssan magic square given by [7, p. 142, Fig. D], matrix **U** with a knight's path (CSP2) marked with red circles (left panel) and a special-knight's path (CSP3) marked with red boxes (right panel).

We will denote the "Ursus magic square" in Figure 1.6.1 by the "Ursus magic matrix"

$$\mathbf{U} = \begin{pmatrix} 1 & 58 & 3 & 60 & 8 & 63 & 6 & 61 \\ 16 & 55 & 14 & 53 & 9 & 50 & 11 & 52 \\ 17 & 42 & 19 & 44 & 24 & 47 & 22 & 45 \\ 32 & 39 & 30 & 37 & 25 & 34 & 27 & 36 \\ 57 & 2 & 59 & 4 & 64 & 7 & 62 & 5 \\ 56 & 15 & 54 & 13 & 49 & 10 & 51 & 12 \\ 41 & 18 & 43 & 20 & 48 & 23 & 46 & 21 \\ 40 & 31 & 38 & 29 & 33 & 26 & 35 & 28 \end{pmatrix}. \tag{1.6.1}$$

We note that the "Ursus magic matrix" $\mathbf{U}$ is a "Caïssan beauty" (CB), i.e., pandiagonal ($2n$ magic paths), and both CSP2- and CSP3-magic ($6n$ magic paths)and so there are (at least) $10n$ magic paths each of length n in all. Moreover, $\mathbf{U}$ has several other properties, as we will elaborate on below:

(1) rank 3 and index 1,

(2) keyed with magic key $\kappa = 2688$,

(3) all odd powers $\mathbf{U}^{2p+1}$ are linear in $\mathbf{U}$,

(4) the group inverse $\mathbf{U}^{\#}$ is linear in $\mathbf{U}$,

(5) $\mathbf{U}$ is $\mathbf{H}$-associated, i.e., $\mathbf{U}+\mathbf{HUH} = 2m\bar{\mathbf{E}}$, where m is the magic sum 260, $\bar{\mathbf{E}}$ has all elements equal to 1/8, and $\mathbf{H} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{I}_4$,

(6) the Moore–Penrose inverse $\mathbf{U}^{+}$ is $\mathbf{V}$-associated, and hence fully-magic,

(7) 4-pac (4-ply and with the alternate-couplets property),

(8) EP, and hence $\mathbf{U}^2$ is symmetric and $\mathbf{U}^{\#} = \mathbf{U}^{+}$.

Falkener (1892), citing Frost [20, 21, 24, 28, (1865, 1866, 1877, 1882)] and Ursus [7, (1881)] finds the term "Indian magic square" (see also Pickover [208, pp. 221–222]—"Gwalior square?) to be more appropriate than "Caïssan magic square" (with Nasik being in India and Sir William Jones's poem Caïssa set in eastern Europe) and presents the Ursus magic square and gives many properties in addition to it being pandiagonal and CSP2-magic. McClintock [117, pp. 111, 113 (1897)], who apparently does not mention Caïssa, presents two 8×8 magic squares, which we find to be pandiagonal and CSP2-magic.

1.7. **Open questions.** OPEN QUESTION 1.7.1. Does there exist an 8×8 classic magic matrix with all CSP2 and CSP3 knight's paths magic but which is *not* pandiagonal ?

OPEN QUESTION 1.7.2. Does there exist an 8×8 classic magic square which is (regular) knight-magic (CSP2) but *is not* pandiagonal ?

2. Some magic matrix properties

A key purpose in this report is to identify various matrix-theoretic properties of Caïssan magic squares.

2.1. "V-associated" magic matrices: "H-associated", "F-associated".

An important property of $n \times n$ magic matrices involves an $n \times n$ involutory matrix $\mathbf{V}$ that is symmetric, centrosymmetric, and has all row totals equal to 1 and defines an involution in that $\mathbf{V}^2 = \mathbf{I}_n$, the $n \times n$ identity matrix. The matrix $\mathbf{A}$ is centrosymmetric whenever $\mathbf{A} = \mathbf{FAF}$ where $\mathbf{F} = \mathbf{F}_n$ is the $n \times n$ flip matrix:

$$\mathbf{F} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{2.1.1}$$

DEFINITION 2.1.1. We define an $n \times n$ magic matrix $\mathbf{M}$ with magic sum m to be $\mathbf{V}$-associated whenever

$$\mathbf{M} + \mathbf{VMV} = 2m\bar{\mathbf{E}}, \tag{2.1.2}$$

where all the elements of $\bar{\mathbf{E}}$ are equal to $1/n$. Here the involutory matrix $\mathbf{V}$ is symmetric, centrosymmetric, and has all row totals equal to 1 and defines an involution in that $\mathbf{V}^2 = \mathbf{I}_n$, the $n \times n$ identity matrix.

THEOREM 2.1.1 [255, p. 21]. The Moore–Penrose inverse $\mathbf{M}^+$ of the $\mathbf{V}$-associated magic matrix $\mathbf{M}$ is also $\mathbf{V}$-associated.

The $n \times n$ Ursus magic matrix $\mathbf{U}$ with $n = 8$

$$\mathbf{U} = \begin{pmatrix} 1 & 58 & 3 & 60 & 8 & 63 & 6 & 61 \\ 16 & 55 & 14 & 53 & 9 & 50 & 11 & 52 \\ 17 & 42 & 19 & 44 & 24 & 47 & 22 & 45 \\ 32 & 39 & 30 & 37 & 25 & 34 & 27 & 36 \\ 57 & 2 & 59 & 4 & 64 & 7 & 62 & 5 \\ 56 & 15 & 54 & 13 & 49 & 10 & 51 & 12 \\ 41 & 18 & 43 & 20 & 48 & 23 & 46 & 21 \\ 40 & 31 & 38 & 29 & 33 & 26 & 35 & 28 \end{pmatrix} \tag{2.1.3}$$

is $\mathbf{V}$-associated with $\mathbf{V}$ equal to the $n \times n$ centrosymmetric involutory matrix with $n = 2h$ even

$$\mathbf{H} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_h \\ \mathbf{I}_h & \mathbf{0} \end{pmatrix} = \mathbf{F}_2 \otimes \mathbf{I}_h \,, \quad h = n/2 \,, \tag{2.1.4}$$

where

$$\mathbf{F}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.1.5}$$

is the 2×2 "flip matrix". When the order n is stressed we write $\mathbf{H}_n$. The Ursus matrix $\mathbf{U}$ is, therefore, $\mathbf{H}_8$-associated or just $\mathbf{H}$-associated.

In an $n \times n$ $\mathbf{H}$-associated magic matrix with magic sum m and $n = 2h$ even, the pairs of entries $h = n/2$ apart along the diagonals all add to $m/4$. And such a matrix is necessarily pandiagonal (YYT-TBC). The converse holds for $n = 4$ but not for $n \geq 6$.

THEOREM 2.1.2. An $\mathbf{H}$-associated $n \times n$ magic matrix with n even is pandiagonal. When $n = 4$ then a pandiagonal magic matrix is $\mathbf{H}$-associated.

THEOREM 2.1.3. An $\mathbf{H}$-associated 8×8 magic matrix is both special-knight (CSP3) magic and alfil-magic.

The $n \times n$ magic matrix $\mathbf{M}$ with magic sum m is $\mathbf{F}$-associated, whenever

$$\mathbf{M} + \mathbf{FMF} = 2m\bar{\mathbf{E}}, \tag{2.1.6}$$

where $\mathbf{F} = \mathbf{F}_n$ is the $n \times n$ flip matrix:

$$\mathbf{F} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{2.1.7}$$

In an $n \times n$ $\mathbf{F}$-associated magic matrix with magic sum m the sums of pairs of entries diametrically equidistant from the centre are all equal to $2m/n$. In the literature an $\mathbf{F}$-associated magic square is often called (just) "associated" (with no qualifier to the word "associated") or "regular" or "symmetrical" (e.g., Heinz & Hendricks [199, pp. 8, 166]).

THEOREM 2.1.4 [214, p. TBC]. Suppose that the magic matrix $\mathbf{M}$ is $\mathbf{F}$-associated. Then $\mathbf{M}^2$ is centrosymmetric.

Proof. By definition $\mathbf{M}+\mathbf{FMF} = 2m\bar{\mathbf{E}}$, which yields, by pre- and post-multiplication by $\mathbf{M}$ that $\mathbf{MFMF} = 2m^2\bar{\mathbf{E}} - \mathbf{M}^2 = \mathbf{FMFM}$. Hence $\mathbf{FM}^2\mathbf{F} = 2m^2\bar{\mathbf{E}} - \mathbf{FMFM} = 2m^2\bar{\mathbf{E}} - (2m^2\bar{\mathbf{E}} - \mathbf{M}^2) = \mathbf{M}^2$.

THEOREM 2.1.5 (new?). Suppose that the $n\times n$ magic matrix $\mathbf{M}$ with magic sum m is $\mathbf{H}$-associated with $n = 2h$ even. Then $\mathbf{M}^2$ and $\mathbf{MHM}$ are block-Latin, i.e.,

$$\mathbf{M}^2 = \begin{pmatrix} \mathbf{K}_1 & \mathbf{L}_1 \\ \mathbf{L}_1 & \mathbf{K}_1 \end{pmatrix}, \quad \mathbf{MHM} = \begin{pmatrix} \mathbf{K}_2 & \mathbf{L}_2 \\ \mathbf{L}_2 & \mathbf{K}_2 \end{pmatrix}, \tag{2.1.8}$$

for some $h\times h$ matrices $\mathbf{K}_1, \mathbf{K}_2, \mathbf{L}_1, \mathbf{L}_2$. Moreover,

$$\mathbf{K}_1 + \mathbf{L}_2 = \mathbf{K}_2 + \mathbf{L}_1 = m^2\bar{\mathbf{E}}_h, \tag{2.1.9}$$

where $\bar{\mathbf{E}}_h$ is the $n\times h$ matrix with all entries equal to $1/h$, with $h = n/2$.

Proof. By definition, $\mathbf{M}+\mathbf{HMH} = 2m\bar{\mathbf{E}}_n$, which pre- and post-multiplied by $\mathbf{M}$, respectively, yields

$$\mathbf{MHMH} = \mathbf{HMHM} = 2m^2\bar{\mathbf{E}}_n - \mathbf{M}^2. \tag{2.1.10}$$

Let the $n\times h$ matrices $\mathbf{J}_1 = (\mathbf{I}_h, \mathbf{0})'$ and $\mathbf{J}_2 = (\mathbf{0}, \mathbf{I}_h)'$. Then from (2.1.10) we obtain, since $\mathbf{J}_2 = \mathbf{HJ}_1$, that

$$\mathbf{J}_1'\mathbf{MHMJ}_2 = \mathbf{J}_2'\mathbf{MHMJ}_1 = m^2\bar{\mathbf{E}}_h - \mathbf{J}_1'\mathbf{M}^2\mathbf{J}_1 = m^2\bar{\mathbf{E}}_h - \mathbf{J}_2'\mathbf{M}^2\mathbf{J}_2, \tag{2.1.11}$$

$$\mathbf{J}_1'\mathbf{MHMJ}_1 = \mathbf{J}_2'\mathbf{MHMJ}_2 = m^2\bar{\mathbf{E}}_h - \mathbf{J}_1'\mathbf{M}^2\mathbf{J}_2 = m^2\bar{\mathbf{E}}_h - \mathbf{J}_2'\mathbf{M}^2\mathbf{J}_1, \tag{2.1.12}$$

and our proof is complete.

THEOREM 2.1.6 [203, 249]. Suppose that the $2p\times 2q$ block-Latin matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{K} & \mathbf{L} \\ \mathbf{L} & \mathbf{K} \end{pmatrix}, \tag{2.1.13}$$

where $\mathbf{K}$ and $\mathbf{L}$ are both $p\times q$.

$$\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{K}+\mathbf{L}) + \operatorname{rank}(\mathbf{K}-\mathbf{L}). \tag{2.1.14}$$

When the magic matrix $\mathbf{M}$ in Theorem 2.1.5 has rank 3 and index 1 then $\mathbf{M}^2$ has rank 3 and from (2.1.8) it follows that $\mathbf{K}_1+\mathbf{L}_1$ and $\mathbf{K}_1-\mathbf{L}_1$ each have rank at most 3, with $\operatorname{rank}(\mathbf{K}_1+\mathbf{L}_1)=3$ if and only if $\mathbf{L}_1=\mathbf{K}_1$, and then $\operatorname{rank}(\mathbf{K}_1)=3$. When, however, $\mathbf{M}$ defines a Caïssan beauty with rank 3 and index 1, our findings are that $\mathbf{K}_1+\mathbf{L}_1$ has rank 2 and $\mathbf{K}_1-\mathbf{L}_1$ has rank 1, and when the Caïssan beauty has rank 3 and index 3, we find that $\mathbf{M}^2$ has rank 2, and that both $\mathbf{K}_1+\mathbf{L}_1$ and $\mathbf{K}_1-\mathbf{L}_1$ have rank 1. TBC

Motivated by an observation of A. C. Thompson [187, (1994)], see also the "A–D method" used by Planck [46, (1919)], we note that reversing the first $h=n/2$ rows and the first $h=n/2$ columns of an $n\times n$ $\mathbf{H}$-associated magic matrix with $n=2h$ (even) makes it $\mathbf{F}$-associated and vice versa, since $\mathbf{HT}=\mathbf{TF}$ and $\mathbf{FT}=\mathbf{TH}$, where the $n\times n$ involutory "Thompson matrix"

$$\mathbf{T}=\begin{pmatrix}\mathbf{F}_h & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_h\end{pmatrix},\ h=n/2. \tag{2.1.15}$$

We will refer to this as "Thompson's trick" and applying it to the Ursus matrix $\mathbf{U}$ yields the $\mathbf{F}$-associated "Ursus–Thompson matrix"

$$\mathbf{U}^*=\begin{pmatrix}37&30&39&32&25&34&27&36\\44&19&42&17&24&47&22&45\\53&14&55&16&9&50&11&52\\60&3&58&1&8&63&6&61\\4&59&2&57&64&7&62&5\\13&54&15&56&49&10&51&12\\20&43&18&41&48&23&46&21\\29&38&31&40&33&26&35&28\end{pmatrix} \tag{2.1.16}$$

which is $\mathbf{F}$-associated but neither pandiagonal nor CSP2-magic.

OPEN QUESTION 2.1.1. Does there exist an 8×8 magic square that is both CSP2-magic and $\mathbf{F}$-associated?

2.2. **"4-ply" and the "alternate couplets" property: "4-pac" magic matrices.** McClintock (1897) considered $n \times n$ magic squares with $n = 4k$ doubly-even that have an "alternate couplets" property.

DEFINITION 2.2.1 (McClintock [117, (1897)]). The $n \times n$ magic matrix $\mathbf{M}$, with $n = 4k$ doubly-even, and magic sum m, has the "alternate couplets" property whenever

$$\mathbf{RM} = \mathbf{M}_2\mathbf{J}_3' \tag{2.2.1}$$

for some $n \times 2$ matrix $\mathbf{M}_2$ with row totals $\frac{1}{2}m$. The $2 \times n$ $(= 4k)$ matrix

$$\mathbf{J}_3' = \mathbf{e}_{2k}' \otimes \mathbf{I}_2 = (\mathbf{I}_2,\ \mathbf{I}_2,\ \ldots,\ \mathbf{I}_2), \tag{2.2.2}$$

with $\mathbf{e}_{2k}'$ the $1 \times 2k$ $(= n/2)$ vector with each entry equal to 1, and the $n \times n$ "couplets-summing" matrix

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{2.2.3}$$

THEOREM 2.2.1. Let the $n \times n$ magic matrix $\mathbf{M}$, with $n = 4k$ doubly-even, and magic sum m, have the "alternate couplets" property (2.2.1). Then the 2×2 matrix

$$\mathbf{J}_3'\mathbf{M}_2 = m\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} m & m \\ m & m \end{pmatrix}. \tag{2.2.4}$$

Proof. We note first that

$$\mathbf{J}_3'\mathbf{R} = (\mathbf{e}_{2k}' \otimes \mathbf{I}_2)\mathbf{R} = \mathbf{E}_{2,n}, \tag{2.2.5}$$

where $\mathbf{E}_{2,n}$ is the $2 \times n$ matrix with every entry equal to 1. and hence

$$\mathbf{J}_3'\mathbf{RMJ}_3 = \mathbf{E}_{2,n}\mathbf{MJ}_3 = m\mathbf{E}_{2,n}(\mathbf{e}_{2k} \otimes \mathbf{I}_2) = 4m\mathbf{E}_{2,2}\,. \tag{2.2.6}$$

On the other hand we have

$$\mathbf{J}_3'\mathbf{RMJ}_3 = \mathbf{J}_3'\mathbf{M}_2\mathbf{J}_3'\mathbf{J}_3 = 4\mathbf{J}_3'\mathbf{M}_2. \tag{2.2.7}$$

Equating (2.2.6) and (2.2.7) yields (2.2.4) at once.

Planck ([37, (1900)]) observed that the magic matrix formed from the Ursus matrix **U** by flipping (reversing) rows 2–8 is "4-ply".

DEFINITION 2.2.2 (Planck [37, (1900)]). The $n \times n$ magic square with $n = 4k$ doubly-even and magic sum m is "4-ply" whenever the four numbers in each of the n^2 subsets of order 2×2 of 4 contiguous numbers (with wrap-around) add up to the same sum $4m/n = m/k$, i.e.,

$$\mathbf{RMR}' = 4m\bar{\mathbf{E}} = \frac{m}{k}\mathbf{E}, \tag{2.2.8}$$

where all the entries of the $n \times n$ matrix $\mathbf{E}$ are equal to 1 and $\bar{\mathbf{E}} = \frac{1}{n}\mathbf{E}$.

THEOREM 2.2.2 (McClintock [117, (1897)]). Let $\mathbf{M}$ denote an $n \times n$ magic matrix with $n = 4k$ doubly-even. Then $\mathbf{M}$ is 4-ply (Definition 2.2.2) if and only if it has the alternate couplets property (Definition 2.2.1).

Proof. If $\mathbf{M}$ has the alternate couplets property then from (2.2.1)

$$\mathbf{RM} = \mathbf{M}_2\mathbf{J}_3' = \mathbf{M}_2(\mathbf{e}_{2k}' \otimes \mathbf{I}_2); \quad k = n/4\,. \tag{2.2.9}$$

Postmultiplying (2.2.9) by $\mathbf{R}'$ yields (2.2.8)

$$\mathbf{RMR}' = 4m\bar{\mathbf{E}} \tag{2.2.10}$$

at once since the numbers in the rows of $\mathbf{M}_2$ all add to $4m/n = m/k$, and so $\mathbf{M}$ is 4-pac.

To go the other way our proof is not quite so quick. With $n = 8$ we define

$$\mathbf{Q}_1 = \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 & 1 & -1 & 1 & -1 & -3 \\ -3 & 3 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & -3 & 3 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -3 & 3 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -3 & 3 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -3 & 3 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & -3 & 3 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -3 & 3 \end{pmatrix}, \quad \mathbf{Q}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}. \tag{2.2.11}$$

We postmultiply (2.2.10) by $\mathbf{Q}_1$. Then (2.2.9) follows since $\mathbf{R}'\mathbf{Q}_1 = \mathbf{Q}_2$ and $\mathbf{EQ}_1 = \mathbf{0}$.

DEFINITION 2.2.3. We will say that the $n \times n$ magic matrix $\mathbf{M}$ with $n = 4k$ doubly-even is "4-pac"[20] whenever (with wrap-around) it is 4-ply (Definition 2.2.2) or equivalently has the alternate couplets property (Definition 2.2.1).

From Theorem 2.2.2 it follows at once, therefore, that if a magic matrix $\mathbf{M}$ is 4-pac then so its transpose $\mathbf{M}'$. An example is the Ursus matrix $\mathbf{U}$ with

$$\mathbf{U} = \begin{pmatrix} 1 & 58 & 3 & 60 & 8 & 63 & 6 & 61 \\ 16 & 55 & 14 & 53 & 9 & 50 & 11 & 52 \\ 17 & 42 & 19 & 44 & 24 & 47 & 22 & 45 \\ 32 & 39 & 30 & 37 & 25 & 34 & 27 & 36 \\ 57 & 2 & 59 & 4 & 64 & 7 & 62 & 5 \\ 56 & 15 & 54 & 13 & 49 & 10 & 51 & 12 \\ 41 & 18 & 43 & 20 & 48 & 23 & 46 & 21 \\ 40 & 31 & 38 & 29 & 33 & 26 & 35 & 28 \end{pmatrix}, \quad \mathbf{U}_2 = \begin{pmatrix} 17 & 113 \\ 33 & 97 \\ 49 & 81 \\ 89 & 41 \\ 113 & 17 \\ 97 & 33 \\ 81 & 49 \\ 41 & 89 \end{pmatrix}, \quad (\mathbf{U}')_2 = \begin{pmatrix} 59 & 71 \\ 61 & 69 \\ 63 & 67 \\ 68 & 62 \\ 71 & 59 \\ 69 & 61 \\ 67 & 63 \\ 62 & 68 \end{pmatrix}. \tag{2.2.12}$$

The row totals of $\mathbf{U}_2$ and $(\mathbf{U}')_2$ are all equal to $m/2 = 130$ (but $\mathbf{U}_2 \neq (\mathbf{U}')_2$). The rows of $\mathbf{RU}$ and of $\mathbf{RU}'$ are the sums of successive pairs of rows of $\mathbf{U}$ and of $\mathbf{U}'$ (with wrap-around). These sums "alternate" and are given in "couplets" (pairs) in the corresponding rows of the $n \times 2$ matrices $\mathbf{U}_2$ and $(\mathbf{U}')_2$.

When the magic matrix $\mathbf{M}$ is 4-pac then from (2.2.1) we see that $\text{rank}(\mathbf{RM}) \leq 2$ with equality when $\mathbf{M}$ is classic. Moreover, from Sylvester's Law of Nullity we find that

$$\text{rank}(\mathbf{M}) \leq \text{rank}(\mathbf{RM}) - \text{rank}(\mathbf{P}) + n = \text{rank}(\mathbf{RM}) + 1 \leq 3 \tag{2.2.13}$$

since $\text{rank}(\mathbf{R}) = n-1$ $(n \geq 4)$. When $\mathbf{M}$ is classic, then from Drury [232] we know that $\text{rank}(\mathbf{M}) \geq 3$, and so we have proved

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THEOREM 2.2.3. Let $\mathbf{M}$ denote an $n \times n$ magic matrix with $n = 4k$ doubly-even. If $\mathbf{M}$ is 4-pac then $\text{rank}(\mathbf{M}) \leq 3$. When $\mathbf{M}$ is also classic then $\text{rank}(\mathbf{M}) = 3$.

The converse of Theorem 2.2.3 does not hold. For example, the classic magic matrix $\mathbf{M}_0$ (2.2.14) generated by Matlab has rank 3 but does not have the alternate couplets property and is not 4-ply, but is $\mathbf{F}$-associated.

[20] We choose the term "4-pac", in part, since "ac" are the initial letters of the two words "alternate couplets". According to *Wikipedia* [324] "A 'sixpack' is a set of six canned or bottled drinks, typically soft drink or beer, which are sold as a single unit." In Germany, "Yesterday I had dinner with seven courses!", "Wow, and what did you have?", "Oh, a sixpack and a hamburger!"—[277].

$$\mathbf{M}_0 = \begin{pmatrix} 64 & 2 & 3 & 61 & 60 & 6 & 7 & 57 \\ 9 & 55 & 54 & 12 & 13 & 51 & 50 & 16 \\ 17 & 47 & 46 & 20 & 21 & 43 & 42 & 24 \\ 40 & 26 & 27 & 37 & 36 & 30 & 31 & 33 \\ 32 & 34 & 35 & 29 & 28 & 38 & 39 & 25 \\ 41 & 23 & 22 & 44 & 45 & 19 & 18 & 48 \\ 49 & 15 & 14 & 52 & 53 & 11 & 10 & 56 \\ 8 & 58 & 59 & 5 & 4 & 62 & 63 & 1 \end{pmatrix}, \quad \mathbf{M}_0^* = \begin{pmatrix} 64 & 2 & 61 & 3 & 60 & 6 & 57 & 7 \\ 9 & 55 & 12 & 54 & 13 & 51 & 16 & 50 \\ 40 & 26 & 37 & 27 & 36 & 30 & 33 & 31 \\ 17 & 47 & 20 & 46 & 21 & 43 & 24 & 42 \\ 32 & 34 & 29 & 35 & 28 & 38 & 25 & 39 \\ 41 & 23 & 44 & 22 & 45 & 19 & 48 & 18 \\ 8 & 58 & 5 & 59 & 4 & 62 & 1 & 63 \\ 49 & 15 & 52 & 14 & 53 & 11 & 56 & 10 \end{pmatrix}, \quad (2.2.14)$$

If, however, we switch rows and columns 3 and 4, and rows and columns 7 and 8 then $\mathbf{M}_0$ becomes $\mathbf{M}_0^*$, which has the alternate couplets property and is 4-ply, is pandiagonal, and is $\mathbf{V}$-associated with $\mathbf{V} = \mathbf{F}_4 \otimes \mathbf{I}_2$, but $\mathbf{M}_0^*$ is not $\mathbf{H}$-associated. As we will show below (Theorem 2.2.4) a 4-pac magic matrix is necessarily pandiagonal.

DEFINITION 2.2.4 (McClintock [117, (1897), §16, pp. 110–111]). We define an $n \times n$ magic matrix $\mathbf{M}$ with $n = 4k$ doubly-even to be "most-perfect", or "complete" or "complete most-perfect" or "most-perfect pandiagonal" (MMPM) [196, 198, 202, 216, 275], whenever it is

(1) pandiagonal,

(2) $\mathbf{H}$-associated,

(3) 4-ply,

(4) and has the alternate-couplets property.

We have already shown that properties (3) and (4) in Definition 2.2.4 are equivalent and we introduced the term 4-pac for this. We have also shown that when n is even then condition (2) implies (1). In our next theorem we show that condition (3) implies (1). It follows, therefore, that an $n \times n$ magic matrix $\mathbf{M}$ with $n = 4k$ doubly-even is "most-perfect" whenever it is 4-pac and $\mathbf{H}$-associated. The 8×8 matrix $\mathbf{M}_0^*$ (2.2.14) is 4-pac but not $\mathbf{H}$-associated and the matrix

$$\mathbf{M}_0^{**} = \begin{pmatrix} 1 & 62 & 5 & 59 & 2 & 61 & 12 & 58 \\ 57 & 14 & 50 & 48 & 9 & 18 & 45 & 19 \\ 10 & 27 & 34 & 25 & 54 & 33 & 36 & 41 \\ 49 & 30 & 26 & 23 & 52 & 21 & 22 & 37 \\ 63 & 4 & 53 & 7 & 64 & 3 & 60 & 6 \\ 56 & 47 & 20 & 46 & 8 & 51 & 15 & 17 \\ 11 & 32 & 29 & 24 & 55 & 38 & 31 & 40 \\ 13 & 44 & 43 & 28 & 16 & 35 & 39 & 42 \end{pmatrix} \quad (2.2.15)$$

given by Setsuda [283], is $\mathbf{H}$-associated but not 4-pac, and so neither $\mathbf{M}_0^*$ nor $\mathbf{M}_0^{**}$ is "most-perfect".

THEOREM 2.2.4 (Pickover [208, p. 73]). A 4-ply magic matrix is necessarily pandiagonal.

To prove Theorem 2.2.4 we note first that the $n \times n$ magic matrix $\mathbf{M}$ with magic sum m is pandiagonal whenever

$$\mathrm{tr}\mathbf{S}^p\mathbf{M} = \mathrm{tr}\mathbf{S}^p\mathbf{F}\mathbf{M} = m = \mathrm{tr}\mathbf{M} = \mathrm{tr}\mathbf{F}\mathbf{M}, \quad p = 1, 2, \ldots, n-1. \tag{2.2.16}$$

where the "one-step forwards shift" matrix

$$\mathbf{S} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{2.2.17}$$

To prove Theorem 2.2.4, we now use the fact that $\mathbf{S} + \mathbf{I} = \mathbf{R}$, the "couplets-summing" matrix (2.2.3), and Theorem 2.2.1.

2.3. "Keyed" magic matrices and the "magic key". When at most 3 eigenvalues of the magic matrix are nonzero then there are two eigenvalues in addition to the magic-sum eigenvalue, which add to 0. This observation leads to

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DEFINITION 2.3.1 [233, 234]. We define the $n \times n$ magic matrix $\mathbf{M}$ with magic sum m to be "keyed" whenever its characteristic polynomial is of the form

$$\det(\lambda \mathbf{I} - \mathbf{M}) = \lambda^{n-3}(\lambda - m)(\lambda^2 - \kappa), \tag{2.3.1}$$

where the "magic key"

$$\kappa = \tfrac{1}{2}\big(\mathrm{tr}\mathbf{M}^2 - m^2\big) \tag{2.3.2}$$

may be positive, negative or zero.

DEFINITION 2.3.2. The $n \times n$ matrix $\mathbf{A}$ has "index 1" whenever $\mathrm{rank}(\mathbf{A}^2) = \mathrm{rank}(\mathbf{A})$.

THEREM 2.3.1 [233, 234]. Suppose that the $n \times n$ keyed magic matrix $\mathbf{M}$ has index 1 with magic sum $m \neq 0$ and magic key κ. Then

$$\kappa \neq 0 \;\Leftrightarrow\; \mathrm{rank}(\mathbf{M}) = 3. \tag{2.3.3}$$

DEFINITION 2.3.3. The $n \times n$ index-1 matrix $\mathbf{A}$ has a "group inverse" $\mathbf{A}^{\#}$ which satisfies the three conditions

$$\mathbf{A}\mathbf{A}^{\#}\mathbf{A} = \mathbf{A}, \quad \mathbf{A}^{\#}\mathbf{A}\mathbf{A}^{\#} = \mathbf{A}^{\#}, \quad \mathbf{A}\mathbf{A}^{\#} = \mathbf{A}^{\#}\mathbf{A}. \tag{2.3.4}$$

THEOREM 2.3.2 [206, Ex 11, p. 58]. When $\mathbf{A}$ has index 1 then the group inverse

$$\mathbf{A}^{\#} = \mathbf{A}(\mathbf{A}^3)^{+}\mathbf{A}, \tag{2.3.5}$$

where $(\mathbf{A}^3)^{+}$ is the Moore–Penrose inverse of $\mathbf{A}^3$.

THEOREM 2.3.3 [233, 234]. Let the magic matrix $\mathbf{M}$ with magic sum $m \neq 0$ be keyed with magic key $\kappa \neq 0$ and index 1. Then $\mathbf{M}$ has rank 3 and all odd powers are "linear in the parent" in that

$$\mathbf{M}^{2p+1} = \kappa^p \mathbf{M} + m(m^{2p} - \kappa^p)\bar{\mathbf{E}}; \quad p = 1, 2, 3, \ldots; \tag{2.3.6}$$

here each element of the $n \times n$ matrix $\bar{\mathbf{E}}$ is equal to $1/n$.

Moreover, the group inverse

$$\mathbf{M}^{\#} = \frac{1}{\kappa}\mathbf{M} + m\left(\frac{1}{m^2} - \frac{1}{\kappa}\right)\bar{\mathbf{E}}. \tag{2.3.7}$$

is also "linear in the parent". The right-hand side of (2.3.7 is the right-hand side of (2.3.6) with $p = -1$.

2.4. **EP matrices.** We now consider EP matrices. We believe that the term EP was introduced by Schwerdtfeger[21] [151, p. 130 (1950)]:

> An n-matrix $\mathbf{A}$ may be called an EP_r-matrix if it is a P_r-matrix and the linear relations existing among its rows are the same as those among the columns". The $n \times n$ matrix $\mathbf{A}$ is said to be a "P_r" matrix [151, Th. 18.1, p. 130] whenever there is an r-rowed principal submatrix $\mathbf{B}_r$ of rank r.

Ben-Israel & Greville [206, p. 157] call a complex EP matrix "range-Hermitian" and cite Schwerdtfeger [151, (1950)] and Pearl [155, (1959)]. Campbell & Meyer [170, p. 74 (1979)] define a square matrix $\mathbf{A}$ to be EP whenever

$$\mathbf{A}\mathbf{A}^{+} = \mathbf{A}^{+}\mathbf{A}, \tag{2.4.1}$$

i.e., whenever $\mathbf{A}$ commutes with its Moore–Penrose inverse [206, Ex. 16, p. 159]. Oskar Maria Baksalary [282, (2011)] notes that the property (2.4.1) may be called "Equal Projectors", which has the initial letters EP. See also Baksalary, Styan & Trenkler [246, (2009)].

DEFINITION 2.4.1 [170, p. 74]. We define a square matrix to be EP whenever the "Equal Projectors" property (2.4.1) holds.

THEOREM 2.4.1 [206, Th. 4, p. 157]. The $n \times n$ matrix $\mathbf{A}$ is EP if and only if the group and Moore–Penrose inverses coincide, i.e., $\mathbf{A}^{\#} = \mathbf{A}^{+}$.

THEOREM 2.4.2 [264, p. TBC]. The $n \times n$ matrix $\mathbf{A}$ with index 1 is EP if and only if $\mathbf{A}\mathbf{A}^{+}\mathbf{A}' = \mathbf{A}'$.

THEOREM 2.4.3 (new?). Suppose that the magic matrix $\mathbf{M}$ with magic sum $m \neq 0$ has rank 3 and index 1. Then $\mathbf{M}$ is EP if and only if $\mathbf{M}^2$ is symmetric.

Proof. From (2.3.7) we know that the group inverse $\mathbf{M}^{\#}$ is linear in the parent:

$$\mathbf{M}^{\#} = \frac{1}{\kappa}\mathbf{M} + m\left(\frac{1}{m^2} - \frac{1}{\kappa}\right)\bar{\mathbf{E}}, \tag{2.4.2}$$

where the matrix $\bar{\mathbf{E}}$ has every entry equal to $1/n$. The matrix $\mathbf{M}$ is EP if and only if $\mathbf{M}^{\#} = \mathbf{M}^{+}$ (Theorem 2.4.1) or equivalently if and only the projectors $\mathbf{M}\mathbf{M}^{\#}$ and $\mathbf{M}^{\#}\mathbf{M}$ are symmetric, and the result follows at once from (2.4.2).

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[21] Hans Wilhelm Eduard Schwerdtfeger (1902–1990) was Professor of Mathematics at McGill University from 1960–1983.

THEOREM 2.4.3X. Suppose that the $n \times n$ magic matrix $\mathbf{M}$ with magic sum $m \neq 0$ has rank 3 and that $\mathbf{M}^2$ is symmetric. Then $\mathbf{M}$ has index 1 and is EP (at least when $n = 4$).

Proof. If $\mathbf{M}^2$ is symmetric then

$$3 = \operatorname{rank}(\mathbf{M}) \geq \operatorname{rank}(\mathbf{M}^2) = \operatorname{rank}(\mathbf{M}^3) = \cdots \tag{2.4.3}$$

and so $\mathbf{M}^2$ has rank $1, 2$ or 3. Let $\mathbf{M}$ have magic key κ. Then $\mathbf{M}^2$ has 2 eigenvalues equal to κ and the magic eigenvalue $m^2 \neq 0$; the other $n - 3$ eigenvalues are 0. If $\kappa \neq 0$ then $\operatorname{rank}(\mathbf{M}^2) = 3$ and $\mathbf{M}$ has index 1 and is EP (Theorem 2.4.3) and our theorem is established.

If, however, $\kappa = 0$ then $\operatorname{rank}(\mathbf{M}^2) = 1$. This is impossible when $n = 4$ since from Sylvester's Law of Nullity

$$3 = \operatorname{rank}(\mathbf{M}) \geq \operatorname{rank}(\mathbf{M}^2) \geq 2 \operatorname{rank}(\mathbf{M}) - n = 6 - n = 2 \tag{2.4.4}$$

when $n = 4$. More generally for $n \geq 4$

$$\operatorname{rank}(\mathbf{M}^2) = 1 \Rightarrow \mathbf{M}^2 = m^2 \bar{\mathbf{E}}. \tag{2.4.5}$$

OPEN QUESTION 2.4.3X. Let the $n \times n$ magic matrix $\mathbf{M}$ have magic sum $m \neq 0$ and rank 3. Does there exist such a matrix $\mathbf{M}$ with magic key $\kappa = 0$ and $n \geq 5$ such that $\mathbf{M}^2$ satisfies (2.4.5)?

The Ursus matrix $\mathbf{U}$ and its square $\mathbf{U}^2$ are

$$\mathbf{U} = \begin{pmatrix} 1 & 58 & 3 & 60 & 8 & 63 & 6 & 61 \\ 16 & 55 & 14 & 53 & 9 & 50 & 11 & 52 \\ 17 & 42 & 19 & 44 & 24 & 47 & 22 & 45 \\ 32 & 39 & 30 & 37 & 25 & 34 & 27 & 36 \\ 57 & 2 & 59 & 4 & 64 & 7 & 62 & 5 \\ 56 & 15 & 54 & 13 & 49 & 10 & 51 & 12 \\ 41 & 18 & 43 & 20 & 48 & 23 & 46 & 21 \\ 40 & 31 & 38 & 29 & 33 & 26 & 35 & 28 \end{pmatrix}, \quad \mathbf{U}^2 = \begin{pmatrix} 9570 & 8674 & 9122 & 8226 & 8002 & 7554 & 8450 & 8002 \\ 8674 & 9186 & 8354 & 8866 & 7554 & 8386 & 7874 & 8706 \\ 9122 & 8354 & 8930 & 8162 & 8450 & 7874 & 8642 & 8066 \\ 8226 & 8866 & 8162 & 8802 & 8002 & 8706 & 8066 & 8770 \\ 8002 & 7554 & 8450 & 8002 & 9570 & 8674 & 9122 & 8226 \\ 7554 & 8386 & 7874 & 8706 & 8674 & 9186 & 8354 & 8866 \\ 8450 & 7874 & 8642 & 8066 & 9122 & 8354 & 8930 & 8162 \\ 8002 & 8706 & 8066 & 8770 & 8226 & 8866 & 8162 & 8802 \end{pmatrix} \tag{2.4.6}$$

and since $\mathbf{U}$ has rank 3 and index 1, and is keyed with magic key $\kappa = 2688 \neq 0$ it follows, using Theorem 2.4.3, that $\mathbf{U}$ is EP since $\mathbf{U}^2$ is symmetric. And we note that $\mathbf{U}^2$ is block-Latin, which also follows since $\mathbf{U}$ is $\mathbf{H}$-associated (Theorem 2.1.5).

Moreover, using Theorem 2.3.1 we find that the odd powers are all "linear in the parent":

$$\mathbf{U}^{2p+1} = 2688^p\mathbf{U} + 260(260^{2p} - 2688^p)\bar{\mathbf{E}}; \quad p = -1, +1, +2, \ldots \tag{2.4.7}$$

where the 8×8 matrix $\bar{\mathbf{E}}$ has every entry equal to 1/8. Since $\mathbf{U}$ is EP, we see (using Theorem 2.3.2) that the group inverse $\mathbf{U}^{\#}$ coincides with its Moore–Penrose inverse $\mathbf{U}^+$:

$$\mathbf{U}^{\#} = \mathbf{U}^+ = \frac{1}{2688}\mathbf{U} - \frac{4057}{43680}\bar{\mathbf{E}}, \tag{2.4.8}$$

which is (2.4.7) with $p = -1$. We note also that the Ursus matrix $\mathbf{U}$ is 4-pac and $\mathbf{H}$-associated.

THEOREM 2.4.4 (new?). Suppose that the magic matrix $\mathbf{M}$ is $\mathbf{F}$-associated and EP. Then the "row-flipped" matrix $\mathbf{FM}$ is $\mathbf{F}$-associated and EP if and only if the "column-flipped" matrix $\mathbf{MF}$ is $\mathbf{F}$-associated and EP, and $\mathbf{M}^2$ is bisymmetric, i.e., symmetric and centrosymmetric.

Proof. The row-flipped

$$\mathbf{FM} \text{ is EP} \Leftrightarrow \mathbf{FM}(\mathbf{FM})^+(\mathbf{FM})' = (\mathbf{FM})' \tag{2.4.9}$$

$$\Leftrightarrow \mathbf{FM}(\mathbf{FM})^+\mathbf{M}' = \mathbf{M}' \tag{2.4.10}$$

$$\Leftrightarrow (2m\bar{\mathbf{E}} - \mathbf{MF})\big((2/m)\bar{\mathbf{E}} - \mathbf{FM}^+\big)\mathbf{M}' = \mathbf{M}' \tag{2.4.11}$$

$$\Leftrightarrow \mathbf{MM}^+\mathbf{M}' = \mathbf{M}' \tag{2.4.12}$$

and $\mathbf{M}$ is EP, using Theorems 2.1.4 and 2.4.3. If follows at once from (2.4.1) that $\mathbf{M}$ is EP if and only if $\mathbf{FMF}$ is EP and the result then follows since $\mathbf{F}(\mathbf{FM})\mathbf{F} = \mathbf{MF}$.

Let $\mathbf{M}_{\mathrm{A}}$ denote the Agrippa "Mercury" magic matrix [185, p. 738][22]. Then $\mathbf{M}_{\mathrm{A}}$ and the column-flipped $\mathbf{M}_{\mathrm{A}}\mathbf{F}$ ([220, p. 49, Fig. 3]) are:

$$\mathbf{M}_{\mathrm{A}} = \begin{pmatrix} 8 & 58 & 59 & 5 & 4 & 62 & 63 & 1 \\ 49 & 15 & 14 & 52 & 53 & 11 & 10 & 56 \\ 41 & 23 & 22 & 44 & 45 & 19 & 18 & 48 \\ 32 & 34 & 35 & 29 & 28 & 38 & 39 & 25 \\ 40 & 26 & 27 & 37 & 36 & 30 & 31 & 33 \\ 17 & 47 & 46 & 20 & 21 & 43 & 42 & 24 \\ 9 & 55 & 54 & 12 & 13 & 51 & 50 & 16 \\ 64 & 2 & 3 & 61 & 60 & 6 & 7 & 57 \end{pmatrix}, \quad \mathbf{M}_{\mathrm{A}}\mathbf{F} = \begin{pmatrix} 1 & 63 & 62 & 4 & 5 & 59 & 58 & 8 \\ 56 & 10 & 11 & 53 & 52 & 14 & 15 & 49 \\ 48 & 18 & 19 & 45 & 44 & 22 & 23 & 41 \\ 25 & 39 & 38 & 28 & 29 & 35 & 34 & 32 \\ 33 & 31 & 30 & 36 & 37 & 27 & 26 & 40 \\ 24 & 42 & 43 & 21 & 20 & 46 & 47 & 17 \\ 16 & 50 & 51 & 13 & 12 & 54 & 55 & 9 \\ 57 & 7 & 6 & 60 & 61 & 3 & 2 & 64 \end{pmatrix} \tag{2.4.13}$$

are both $\mathbf{F}$-associated and EP since

$$\mathbf{M}_{\mathrm{A}}^2 = \begin{pmatrix} 7330 & 9346 & 9122 & 8002 & 8226 & 8450 & 8226 & 8898 \\ 9346 & 7714 & 7874 & 8866 & 8706 & 8354 & 8514 & 8226 \\ 9122 & 7874 & 7970 & 8834 & 8738 & 8258 & 8354 & 8450 \\ 8002 & 8866 & 8834 & 8098 & 8130 & 8738 & 8706 & 8226 \\ 8226 & 8706 & 8738 & 8130 & 8098 & 8834 & 8866 & 8002 \\ 8450 & 8354 & 8258 & 8738 & 8834 & 7970 & 7874 & 9122 \\ 8226 & 8514 & 8354 & 8706 & 8866 & 7874 & 7714 & 9346 \\ 8898 & 8226 & 8450 & 8226 & 8002 & 9122 & 9346 & 7330 \end{pmatrix}, \quad (\mathbf{M}_{\mathrm{A}}\mathbf{F})^2 = \begin{pmatrix} 9570 & 7554 & 7778 & 8898 & 8674 & 8450 & 8674 & 8002 \\ 7554 & 9186 & 9026 & 8034 & 8194 & 8546 & 8386 & 8674 \\ 7778 & 9026 & 8930 & 8066 & 8162 & 8642 & 8546 & 8450 \\ 8898 & 8034 & 8066 & 8802 & 8770 & 8162 & 8194 & 8674 \\ 8674 & 8194 & 8162 & 8770 & 8802 & 8066 & 8034 & 8898 \\ 8450 & 8546 & 8642 & 8162 & 8066 & 8930 & 9026 & 7778 \\ 8674 & 8386 & 8546 & 8194 & 8034 & 9026 & 9186 & 7554 \\ 8002 & 8674 & 8450 & 8674 & 8898 & 7778 & 7554 & 9570 \end{pmatrix} \tag{2.4.14}$$

are both bisymmetric (and, maybe surprisingly, are quite different).

In a pamphlet [91, (1845)] describing *A New Method of Ascertaining Interest and Discount*, Israel Newton includes *A Few Magic Squares of a Singular Quantity*. One of these magic squares is 16×16 and dated "September 28, 1844, in the 82nd year of his age" and "containing 4 squares of 8 and 16 squares of 4"[23]. We define this 16×16 magic square by the magic Newton matrix[24]

$$\mathbf{N} = \begin{pmatrix} 1 & 254 & 255 & 4 & 121 & 252 & 8 & 133 & 118 & 11 & 247 & 138 & 244 & 113 & 15 & 142 \\ 128 & 131 & 130 & 125 & 134 & 7 & 251 & 122 & 137 & 248 & 12 & 117 & 14 & 143 & 241 & 116 \\ 132 & 127 & 126 & 129 & 135 & 6 & 250 & 123 & 140 & 245 & 9 & 120 & 141 & 16 & 114 & 243 \\ 253 & 2 & 3 & 256 & 124 & 249 & 5 & 136 & 119 & 10 & 246 & 139 & 115 & 242 & 144 & 13 \\ 159 & 30 & 100 & 225 & 25 & 230 & 103 & 156 & 106 & 150 & 235 & 23 & 145 & 148 & 111 & 110 \\ 97 & 228 & 158 & 31 & 104 & 155 & 26 & 229 & 107 & 151 & 234 & 22 & 112 & 109 & 146 & 147 \\ 226 & 99 & 29 & 160 & 154 & 101 & 232 & 27 & 152 & 108 & 21 & 233 & 18 & 19 & 240 & 237 \\ 32 & 157 & 227 & 98 & 231 & 28 & 153 & 102 & 149 & 105 & 24 & 236 & 239 & 238 & 17 & 20 \\ 33 & 164 & 95 & 222 & 168 & 37 & 217 & 92 & 86 & 215 & 41 & 172 & 84 & 209 & 46 & 175 \\ 96 & 221 & 34 & 163 & 91 & 218 & 38 & 167 & 43 & 170 & 88 & 213 & 174 & 47 & 212 & 81 \\ 162 & 35 & 224 & 93 & 90 & 219 & 39 & 166 & 216 & 85 & 171 & 42 & 211 & 82 & 173 & 48 \\ 223 & 94 & 161 & 36 & 165 & 40 & 220 & 89 & 169 & 44 & 214 & 87 & 45 & 176 & 83 & 210 \\ 191 & 68 & 65 & 190 & 200 & 59 & 186 & 69 & 53 & 203 & 182 & 76 & 208 & 51 & 77 & 178 \\ 194 & 61 & 64 & 195 & 185 & 70 & 199 & 60 & 202 & 56 & 73 & 183 & 177 & 78 & 52 & 207 \\ 62 & 193 & 196 & 63 & 71 & 188 & 57 & 198 & 75 & 181 & 204 & 54 & 50 & 205 & 179 & 80 \\ 67 & 192 & 189 & 66 & 58 & 197 & 72 & 187 & 184 & 74 & 55 & 201 & 79 & 180 & 206 & 49 \end{pmatrix}. \tag{2.4.15}$$

[22]From Swetz [242, p.117], we note that the Agrippa "Mercury" magic square [185, p. 738], see also Paracelsus [75] TBC, is called the "Jupiter" magic square by Girolamo Cardano (1501–1576) [74, p. TBC].

[23]There are also 5 more magic squares: 3 are 8×8 and 2 are 4×4.

[24]We have corrected several typos in the original given by Newton [91].

Both the Newton matrix **N** (2.4.15) and its column-flipped partner **NF** are (surprisingly) EP but **N** is not **F**-associated (in fact $\mathbf{N}^+$ is not fully-magic). The 16×16 Newton matrix **N** has rank 13 and index 1 and each of the 4 magic 8×8 submatrices and its 4 column-flipped partners are EP (all with rank 7 and index 1). We will comment on the 16 magic 4×4 submatrices in Section TBC, below, but it seems that none are EP though 15 of them are **V**-associated.

Deacon Israel Newton (1763–1856) was [124, p. 228] "the inventor of the well-known medical preparations widely known as 'Newton's Bitters', 'Newton's Pills', etc, and sold extensively for many years throughout New England and New York[25]. Doctor Newton was a thoroughly educated physician, though not in general practice of his profession, and was much respected as a man and a citizen. Besides his medicines, which were valuable, he invented and built a church organ, which was placed in the old first church, and was there used for many years. He was gifted with rare mechanical skill, which he exhibited in many ways to the benefit of mankind. He was a soldier of the Revolution, and the last of those soldiers to die in Norwich, Vermont, at age 93.

Known variously as the Norwich Hotel, Curtis Hotel, The Union House, and the Newton Inn, the Norwich Inn was the first tavern in Vermont to entertain a Chief Executive of the United States. On July 22, 1817, President James Monroe (1758–1831) visited the Inn, and while there, he addressed the townspeople of Norwich and "partook of a dinner, prepared ... in handsome style". Built by Colonel Jasper Murdock in 1797, the Norwich Inn served as a stagecoach tavern and hostelry. Jasper Murdock's Alehouse is a Vermont microbrewery at the Norwich Inn with "Consistently good brews made right across the parking lot in their own brew house."

Figure 2.4.1: (left panel) The Newton Inn, Norwich, Vermont, c. 1850 [91, facing p. 51].
(right panel) The Norwich Inn, Norwich, Vermont, c. 2010: photograph online at
The Preservation Trust of Vermont, 142 Church Street, Burlington, Vermont.

[25]The Newton–Timmermann Pharmacy is located at 799 Lexington Avenue, New York City.

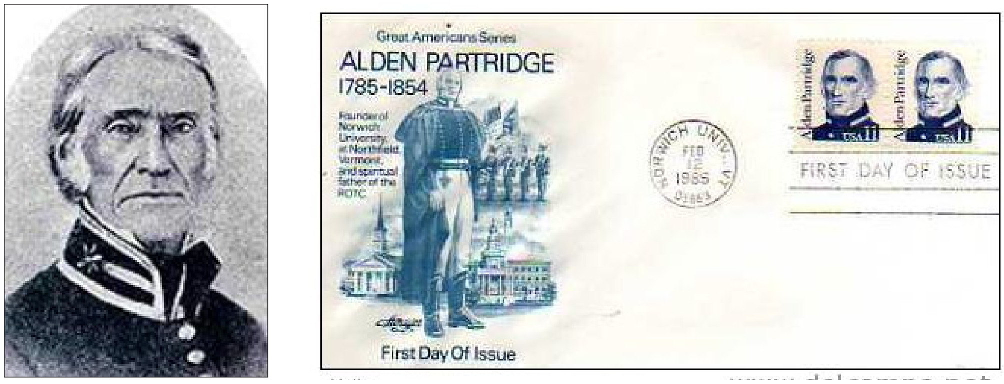

FIGURE 2.4.2: Alden Partridge (1785–1854), contemporary of Israel Newton (1763–1856).

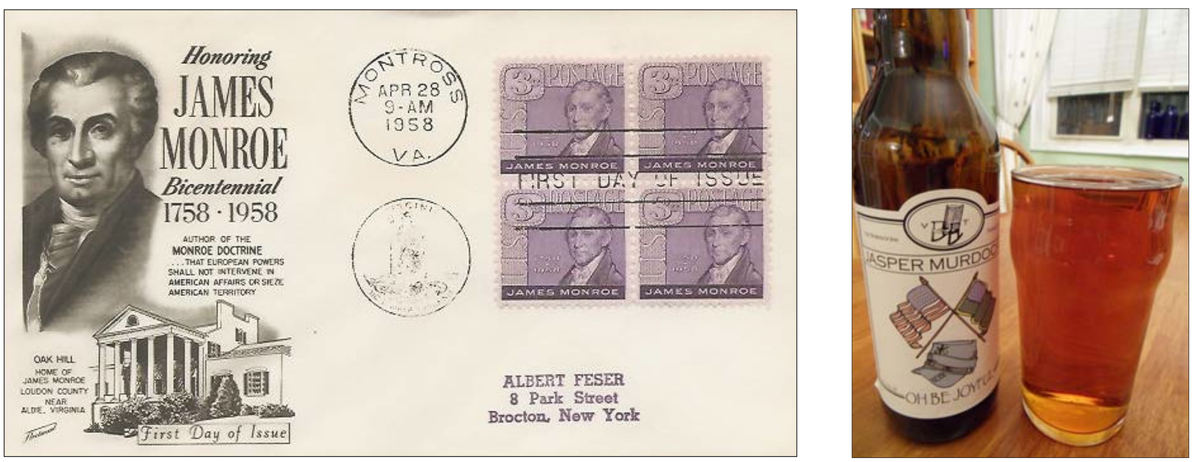

FIGURE 2.4.3: (left panel) James Monroe first-day cover, USA 1958, *Scott* 1105.
(right panel) Jasper Murdock's "Oh Be Joyful": photograph online at The Feisty Foodie.

2.5. **Checking that an 8×8 magic matrix is CSP2-magic.** To check that an 8×8 magic matrix $\mathbf{M}$ is CSP2-magic (regular-knight-magic) we check that four 4×4 related matrices are $\mathbf{H}$-associated.

THEOREM 2.5.1. The 32 CSP2-paths in the 8×8 magic matrix $\mathbf{M}$ are all magic if and only if the four 4×4 matrices

$$\mathbf{J}_1'\mathbf{K}_2\mathbf{M}\mathbf{J},\ \mathbf{J}_2'\mathbf{K}_2\mathbf{M}\mathbf{J},\ \mathbf{J}_1'\mathbf{K}_2\mathbf{M}'\mathbf{J},\ \mathbf{J}_2'\mathbf{K}_2\mathbf{M}'\mathbf{J}, \tag{2.5.1}$$

are all $\mathbf{H}$-associated (and hence pandiagonal). Here the 8×4 matrices

$$\mathbf{J}_1 = \begin{pmatrix} \mathbf{I}_4 \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{J}_2 = \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_4 \end{pmatrix}, \quad \mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 = \begin{pmatrix} \mathbf{I}_4 \\ \mathbf{I}_4 \end{pmatrix}, \tag{2.5.2}$$

where $\mathbf{I}_4$ is the 4×4 identity matrix, and the "knight-selection matrix"

$$\mathbf{K}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.5.3}$$

The "knight-selection matrix" $\mathbf{K}_2$ is almost a regular-knight's move (CSP2) matrix but the "move" (with wrap-around) from row 4 to row 5 is that of a special-knight (CSP3) rather than that of the usual knight (CSP2) in chess.

We recall the Ursus matrix

$$\mathbf{U} = \begin{pmatrix} \mathbf{1} & 58 & 3 & 60 & \mathbf{8} & 63 & 6 & 61 \\ 16 & 55 & 14 & 53 & 9 & 50 & 11 & 52 \\ 17 & \mathbf{42} & 19 & 44 & 24 & \mathbf{47} & 22 & 45 \\ 32 & 39 & 30 & 37 & 25 & 34 & 27 & 36 \\ 57 & 2 & \mathbf{59} & 4 & 64 & 7 & \mathbf{62} & 5 \\ 56 & 15 & 54 & 13 & 49 & 10 & 51 & 12 \\ 41 & 18 & 43 & \mathbf{20} & 48 & 23 & 46 & \mathbf{21} \\ 40 & 31 & 38 & 29 & 33 & 26 & 35 & 28 \end{pmatrix}. \tag{2.5.4}$$

where we have identified a CSP2-magic path in red. We find that

$$\mathbf{K}_2\mathbf{U} = \begin{pmatrix} \mathbf{1} & 58 & 3 & 60 & \mathbf{8} & 63 & 6 & 61 \\ 17 & \mathbf{42} & 19 & 44 & 24 & \mathbf{47} & 22 & 45 \\ 57 & 2 & \mathbf{59} & 4 & 64 & 7 & \mathbf{62} & 5 \\ 41 & 18 & 43 & \mathbf{20} & 48 & 23 & 46 & \mathbf{21} \\ 16 & 55 & 14 & 53 & 9 & 50 & 11 & 52 \\ 32 & 39 & 30 & 37 & 25 & 34 & 27 & 36 \\ 56 & 15 & 54 & 13 & 49 & 10 & 51 & 12 \\ 40 & 31 & 38 & 29 & 33 & 26 & 35 & 28 \end{pmatrix}, \quad \mathbf{K}_2\mathbf{U}' = \begin{pmatrix} 1 & 16 & 17 & 32 & 57 & 56 & 41 & 40 \\ 3 & 14 & 19 & 30 & 59 & 54 & 43 & 38 \\ 8 & 9 & 24 & 25 & 64 & 49 & 48 & 33 \\ 6 & 11 & 22 & 27 & 62 & 51 & 46 & 35 \\ 58 & 55 & 42 & 39 & 2 & 15 & 18 & 31 \\ 60 & 53 & 44 & 37 & 4 & 13 & 20 & 29 \\ 63 & 50 & 47 & 34 & 7 & 10 & 23 & 26 \\ 61 & 52 & 45 & 36 & 5 & 12 & 21 & 28 \end{pmatrix}. \tag{2.5.5}$$

(2.5.6)

and hence

$$\mathbf{J}_1'\mathbf{K}_2\mathbf{U}\mathbf{J} = \begin{pmatrix} \mathbf{9} & 121 & 9 & 121 \\ 41 & \mathbf{89} & 41 & 89 \\ 121 & 9 & \mathbf{121} & 9 \\ 89 & 41 & 89 & \mathbf{41} \end{pmatrix}, \quad \mathbf{J}_2'\mathbf{K}_2\mathbf{U}\mathbf{J} = \begin{pmatrix} 25 & 105 & 25 & 105 \\ 57 & 73 & 57 & 73 \\ 105 & 25 & 105 & 25 \\ 73 & 57 & 73 & 57 \end{pmatrix}, \tag{2.5.7}$$

$$\mathbf{J}_1'\mathbf{K}_2\mathbf{U}'\mathbf{J} = \begin{pmatrix} 58 & 72 & 58 & 72 \\ 62 & 68 & 62 & 68 \\ 72 & 58 & 72 & 58 \\ 68 & 62 & 68 & 62 \end{pmatrix}, \quad \mathbf{J}_2'\mathbf{K}_2\mathbf{U}'\mathbf{J} = \begin{pmatrix} 60 & 70 & 60 & 70 \\ 64 & 66 & 64 & 66 \\ 70 & 60 & 70 & 60 \\ 66 & 64 & 66 & 64 \end{pmatrix} \tag{2.5.8}$$

are indeed all **H**-associated and pandiagonal and so all 32 regular-knight's (CSP2) paths are magic.

2.6. **Checking that an 8×8 magic matrix is CSP3-magic.** To check that an 8×8 magic matrix $\mathbf{M}$ is CSP3-magic (special-knight-magic) we compute $\mathbf{K}_3\mathbf{M}$, where the 8×8 CSP3 (special-knight) selection matrix

$$\mathbf{K}_3 = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0 \end{pmatrix}, \tag{2.6.1}$$

which we note is symmetric and magic, and commutes with $\mathbf{H}$. This leads to

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THEOREM 2.6.1. The 8×8 magic matrix $\mathbf{M}$ is CSP3-magic if and only if $\mathbf{K}_3\mathbf{M}$ is pandiagonal. If $\mathbf{M}$ is also $\mathbf{H}$-associated then so is $\mathbf{K}_3\mathbf{M}$ and hence $\mathbf{K}_3\mathbf{M}$ is pandiagonal and $\mathbf{M}$ is CSP3-magic.

For example, with the Ursus matrix $\mathbf{U}$ we find that

$$\mathbf{K}_3\mathbf{U} = \begin{pmatrix} 1&58&3&60&8&63&6&61\\ 32&39&30&37&25&34&27&36\\ 41&18&43&20&48&23&46&21\\ 16&55&14&53&9&50&11&52\\ 57&2&59&4&64&7&62&5\\ 40&31&38&29&33&26&35&28\\ 17&42&19&44&24&47&22&45\\ 56&15&54&13&49&10&51&12 \end{pmatrix} \tag{2.6.2}$$

is $\mathbf{H}$-associated and hence pandiagonal and so $\mathbf{U}$ is special-knight (CSP3) magic. Moreover, $\mathbf{U}$ is $\mathbf{H}$-associated implies that $\mathbf{K}_3\mathbf{U}$ is $\mathbf{H}$-associated and pandiagonal directly.

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3. Many 8×8 Caïssan beauties (CBs)

We now consider various properties of the 46080 Caïssan beauties (CBs) identified by Drury [272, (2011)] and present (Section 3.2) a generalization of an algorithm given by Cavendish [11, (1894)] for generating Caïssan beauties.

3.1. **Drury's 46080 Caïssan beauties (CBs).** We are most grateful to S. W. Drury [272] for identifying 46080 classic magic 8×8 matrices which are Caïssan beauties (CBs), i.e., pandiagonal and both CSP2- and CSP3-magic. Each of these 46080 CBs is classic with entries $0, 1, \ldots, 63$ and magic sum $m = 252$. The lead entry in the top left-hand corner or position $(1, 1)$ is 0. Since our magic paths all allow wrap-around we may shift row-blocks and/or column-blocks so that the resulting magic square with the entry $1, 2, \ldots, 63$ (as well as 0) in the top left-hand corner is also a CB. This "translation" property leads to

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CLAIM 3.1.1. We claim that there are precisely

$$46080 \times 64 = 2,949,120 \tag{3.1.1}$$

CBs in all.

The 46080 CBs include pairs which are transposes of each other, and pairs which are "double-flips" of each other, i.e., **M** and **FMF**. If we exclude these then there are just

$$\frac{46080}{4} = 11520 \tag{3.1.2}$$

8×8 CBs.

From Trump's Table [236] we find that the actual number (count) of classic magic pandiagonal 8×8 matrices is not known but exceeds the number H of **H**-associated 8×8 classic magic matrices. This number H is also not known exactly but from [236] we find that, with probability 99%, H lies in the interval

$$(2.5228 \pm 0.0014) \times 10^{27}. \tag{3.1.3}$$

CLAIM 3.1.2. We claim that the 46080 CBs [272] are all, in addition to being pandiagonal, CSP2-magic and CSP3-magic,

(1) **H**-associated,

(2) 4-pac, i.e., 4-ply and with the alternate couplets property, and

(3) all have rank 3 and index 1, and hence all are keyed with magic key $\kappa \neq 0$.

CLAIM 3.1.3. We claim that the 46080 CBs [272] have precisely 960 distinct top rows, with

$$48 = \frac{46080}{960} \tag{3.1.4}$$

CBs per top row.

In Claim 3.1.3 we claimed that there are 48 CBs with each distinct top row. We now claim that these 48 CBs may be generated from any particular starter-CB $\mathbf{M}_{\mathrm{A}}$ as follows. We assume that $\mathbf{M}_{\mathrm{A}}$ is $\mathbf{H}$-associated and from $\mathbf{M}_{\mathrm{A}}$ we generate $\mathbf{M}_{\mathrm{B}}$ and $\mathbf{M}_{\mathrm{C}}$ as follows:

$$\mathbf{M}_{\mathrm{A}} = \begin{pmatrix} R1a & R1b \\ R2a & R2b \\ R3a & R3b \\ R4a & R4b \\ \widehat{R1b} & \widehat{R1a} \\ \widehat{R2b} & \widehat{R2a} \\ \widehat{R3b} & \widehat{R3a} \\ \widehat{R4b} & \widehat{R4a} \end{pmatrix}, \quad \mathbf{M}_{\mathrm{B}} = \begin{pmatrix} R1a & R1b \\ R2a & R2b \\ R3b & R3a \\ R4b & R4a \\ \widehat{R1b} & \widehat{R1a} \\ \widehat{R2b} & \widehat{R2a} \\ \widehat{R3a} & \widehat{R3b} \\ \widehat{R4a} & \widehat{R4b} \end{pmatrix}, \quad \mathbf{M}_{\mathrm{C}} = \begin{pmatrix} R1a & R1b \\ R2b & R2a \\ R3b & R3a \\ R4a & R4b \\ \widehat{R1b} & \widehat{R1a} \\ \widehat{R2a} & \widehat{R2b} \\ \widehat{R3a} & \widehat{R3b} \\ \widehat{R4b} & \widehat{R4a} \end{pmatrix}, \tag{3.1.5}$$

where $R1a, \ldots, R4b$ are 4-tuples and $\widehat{xxxx}$ = complement of the 4-tuple $xxxx$ = the 4-tuple $63, 63, 63, 63$ minus the 4-tuple $xxxx$. Then (Claim 3.1.4)

CLAIM 3.1.4. We claim that the 48 CBs with each distinct top row may be generated from $\mathbf{M}_{\mathrm{A}}$ in (3.1.5) by forming $\mathbf{M}_{\mathrm{B}}$ and $\mathbf{M}_{\mathrm{C}}$ and then for each of the three matrices $\mathbf{M}_{\mathrm{A}}, \mathbf{M}_{\mathrm{B}}, \mathbf{M}_{\mathrm{C}}$ by

(1) switching rows 2 and 6,

(2) switching rows 3 and 7,

(3) switching rows 4 and 8,

(4) reversing rows 2, 3, …, 8.

We note that $3 \times 2^4 = 48$.

CLAIM 3.1.5. We claim that of the 46080 CBs [272] precisely 192 are EP and that precisely 96 of these have magic key $\kappa = 2688$ and precisely 96 have magic key $\kappa = 8736$.

We claim that of the 46080 CBs, precisely 672 are not EP but have magic key $\kappa = 2688$ and precisely 288 are not EP but have magic key $\kappa = 8736$. In all, therefore, of the 46080 CBs precisely 768 have magic key $\kappa = 2688$, and precisely 384 have magic key $\kappa = 8736$. We note that $768 = 2 \times 384$.

Moreover, we claim that none of these 192 remain EP when the columns are flipped (reversed) and that those CBs with $\kappa = 2688$ when flipped have magic keys

$$\kappa_{2688} = \pm 512,\ \pm 1024,\ \pm 1408,\ \pm 1664,\ \pm 2048,\ \pm 2432 \tag{3.1.6}$$

and that those CBs with $\kappa = 8736$ when flipped have magic keys

$$\kappa_{8736} = \pm 256,\ \pm 1024,\ \pm 4096,\ \pm 7648,\ \pm 7712,\ \pm 8672. \tag{3.1.7}$$

Each of these $2 \times 2 \times 6 = 24$ magic keys occurs 8 times ($8 \times 24 = 192$). Moreover the magic keys in (3.1.6) and (3.1.7) satisfy the inequalities

$$|\kappa_{2688}| < 2688, \quad |\kappa_{8736}| < 8736. \tag{3.1.8}$$

Of the absolute values of the 6 numbers in (3.1.6), we note that 3 are successive powers of 2:

$$512 = 2^9, \quad 1024 = 2^{10}, \quad 2048 = 2^{11}, \tag{3.1.9}$$

and 3 are (almost consecutive) prime multiples of 2^7:

$$1408 = 11 \times 2^7, \quad 1664 = 13 \times 2^7, \quad 2432 = 19 \times 2^7 \tag{3.1.10}$$

but, curiously, $2176 = 17 \times 2^7$ is absent! Of the absolute values of the 6 numbers in (3.1.7), we note that 3 are consecutive even powers of 2:

$$256 = 2^8, \quad 1024 = 2^{10}, \quad 4096 = 2^{12}, \tag{3.1.11}$$

and 3 are prime multiples of 2^5 (the first two consecutive prime multiples)

$$7648 = 239 \times 2^5, \quad 7712 = 241 \times 2^5, \quad 8672 = 271 \times 2^5. \tag{3.1.12}$$

CLAIM 3.1.6. We claim that the 46080 CBs [272] with the columns flipped are all, in addition to being pandiagonal, CSP2-magic and CSP3-magic,

(1) **H**-associated,

(2) 4-pac, i.e., 4-ply and with the alternate couplets property, and

(3) all have rank 3.

And that precisely 1920 (TBC) have index 3 and magic key $\kappa = 0$ and $46080 - 1920 = 44160$ have index 1 and magic key $\kappa \neq 0$ and that none of these are EP. We recall that *all* (unflipped) 46080 CBs with 0 in the top-left hand corner have index 1 and magic key $\kappa \neq 0$ and that precisely 192 of these are EP.

CLAIM 3.1.7. We claim that for each of the 45888 (= 46080 − 192) CBs **M** which are not EP that

$$\operatorname{rank}(\mathbf{M}^2 - (\mathbf{M}^2)') = \operatorname{rank}(\mathbf{M}^+ - \mathbf{M}^\#) = 2. \tag{3.1.13}$$

We recall that a magic matrix **M** with rank 3 and index 1 is EP if and only if $\mathbf{M}^2$ is symmetric (Theorem 2.4.3) and if and only if $\mathbf{M}^+ = \mathbf{M}^\#$ (Theorem TBC).

We recall that the Ursus-Caïssan beauty **U** is EP with magic key $\kappa = 2688$ and when flipped, i.e., **UF**, has index 1 and magic key $\kappa = 1408$ (< 2688) but is not EP. Moreover, the MATLAB magic(n) algorithm generates EP magic matrices for all $n = 4k$ doubly even with magic key

$$\kappa = \frac{n^3(n^2-1)}{12}, \tag{3.1.14}$$

as established by Kirkland & Neumann [189, (1995)]. The magic key (3.1.14) equals 2688 when $n = 8$. The EP magic matrix generated by magic(8) is $\mathbf{M}_0$ displayed in (2.2.14) above.

OPEN QUESTION 3.1.1. Concerning these 46080 8 × 8 Caïssan beauties we would like to know:

(1) Is it possible to "prove" that a Caïssan beauty MUST be 4-pac, **H**-associated, and have rank 3 and index 1.

(2) How many Caïssan beauties have the "Franklin" property?

(3) How many Caïssan beauties have the "rhomboid" property?

(4) Are 2688 and 8736 the most popular magic keys for a Caïssan beauty?

(5) What is special about the numbers 2688 and 8736? The MATLAB magic key (3.1.14) equals 2688 when $n = 8$.

3.2. **A generalized Cavendish (1894) algorithm for Caïssan beauties.** We now present a matrix representation of a generalized version of the algorithm for a (classic) pandiagonal magic square given for a special case by Cavendish [11, (1894)]. He defines a Caïssan magic square as (being just) pandiagonal but the algorithm he gives (for just a single special case) yields a Caïssan beauty that has all CSP2- and CSP3-paths magic as well as being pandiagonal. Our generalized Cavendish matrices $\mathbf{C}_{\mathbf{s},t,u}$, (3.2.5) below, are also Caïssan beauties, and like the Ursus matrix $\mathbf{U}$, are also 4-pac, $\mathbf{H}$-associated, and keyed with rank 3 and index 1. Moreover, the $\mathbf{C}_{\mathbf{s},t,u}$ are EP for all values of the "seed parameters" $\mathbf{s}, t, u$.

We define the 2×2 identity and flip matrices and the 1×4 unit (sum) vector

$$\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{F}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{e}_4' = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}', \tag{3.2.1}$$

and hence the 2×8 matrices

$$\mathbf{A}_1 = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix} = \mathbf{e}_4' \otimes (\mathbf{I}_2 - \mathbf{F}_2), \tag{3.2.2}$$

$$\mathbf{A}_2 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} = \mathbf{e}_4' \otimes \mathbf{F}_2. \tag{3.2.3}$$

Then for some 4×1 seed vector $\mathbf{s} = (a, b, c, d)'$ and some seed scalar t, we define the 8×8 matrix

$$\begin{aligned} \mathbf{B}_{\mathbf{s},t} &= \mathbf{A}_1 \otimes \mathbf{s} + \mathbf{A}_2 \otimes t\mathbf{e}_4 \\ &= \begin{pmatrix} a & t-a & a & t-a & a & t-a & a & t-a \\ b & t-b & b & t-b & b & t-b & b & t-b \\ c & t-c & c & t-c & c & t-c & c & t-c \\ d & t-d & d & t-d & d & t-d & d & t-d \\ t-a & a & t-a & a & t-a & a & t-a & a \\ t-b & b & t-b & b & t-b & b & t-b & b \\ t-c & c & t-c & c & t-c & c & t-c & c \\ t-d & d & t-d & d & t-d & d & t-d & d \end{pmatrix} \end{aligned} \tag{3.2.4}$$

and hence the 8×8 magic "Cavendish matrix"

$$\mathbf{C}_{\mathbf{s},t,u} = \mathbf{C}_{(a,b,c,d)',t,u} = \mathbf{B}_{s,t} + 8\mathbf{B}_{s,t}' - u\mathbf{E}_8 = \tag{3.2.5}$$

$$\begin{pmatrix} 9a-u & t-a+8b-u & a+8c-u & t-a+8d-u & -7a+8t-u & 9t-a-8b-u & a+8t-8c-u & 9t-a-8d-u \\ b+8t-8a-u & 9t-9b-u & b+8t-8c-u & 9t-b-8d-u & b+8a-u & t+7b-u & b+8c-u & t-b+8d-u \\ c+8a-u & t-c+8b-u & 9c-u & t-c+8d-u & c+8t-8a-u & 9t-c-8b-u & -7c+8t-u & 9t-c-8d-u \\ d+8t-8a-u & 9t-d-8b-u & d+8t-8c-u & 9t-9d-u & d+8a-u & t-d+8b-u & d+8c-u & t+7d-u \\ t+7a-u & a+8b-u & t-a+8c-u & a+8d-u & 9t-9a-u & a+8t-8b-u & 9t-a-8c-u & a+8t-8d-u \\ 9t-b-8a-u & -7b+8t-u & 9t-b-8c-u & b+8t-8d-u & t-b+8a-u & 9b-u & t-b+8c-u & b+8d-u \\ t-c+8a-u & c+8b-u & t+7c-u & c+8d-u & 9t-c-8a-u & c+8t-8b-u & 9t-9c-u & c+8t-8d-u \\ 9t-d-8a-u & d+8t-8b-u & 9t-d-8c-u & -7d+8t-u & t-d+8a-u & d+8b-u & t-d+8c-u & 9d-u \end{pmatrix}.$$

Here u is a second seed scalar and all the entries of the 8×8 matrix $\mathbf{E}_8$ are equal to 1. The Cavendish matrix $\mathbf{C}_{\mathbf{s},t,u}$ (3.2.5) is a Caïssan beauty (pandiagonal with all knight's paths of types CSP2 and CSP3 magic), though not necessarily classic, but with rank 3 and index 1, and $\mathbf{H}$-associated, 4-pac, and keyed for any choices of the seed parameters $\mathbf{s}=(a,b,c,d)', t, u$. The magic key

$$\kappa(\mathbf{C}_{\mathbf{s},t,u}) = 128\big(t^2 - t(a+b+c+d) + a^2+b^2+c^2+d^2\big) \tag{3.2.6}$$

is independent of the seed scalar u, while the magic sum $m = 36t - 8u$ is independent of the seed vector $\mathbf{s}=(a,b,c,d)'$. When $t=9$ and $u=8$ then $m=260$, the magic sum for a classic 8×8 magic matrix. And $\mathbf{C}_{\mathbf{s},t,u}$ (3.2.5) becomes the Cashmore beauty $\mathbf{M}^{(\mathrm{p})}_{2(\hat{b})} = (\mathbf{M}^{(\mathrm{p})}_{1(\hat{b})})'$, see (3.4) (3.4.16) below.

The singular values of the Cavendish matrix $\mathbf{C}_{\mathbf{s},t,u}$ (3.2.5) are, in addition to the magic sum m, 4ω and 32ω, while the eigenvalues are the magic sum m, and $\pm(8\sqrt{2})\omega$, where ω is the positive square root of

$$\omega^2 = t^2 - t(a+b+c+d) + a^2+b^2+c^2+d^2. \tag{3.2.7}$$

It follows that that the magic key κ is precisely 8 times the smallest (non-zero and non-magic) eigenvalue of $\mathbf{CC}'$ and that the largest (non-zero and non-magic) eigenvalue of $\mathbf{CC}'$ is precisely 8 times the magic key κ.

The Cavendish matrix $\mathbf{C}_{\mathbf{s},t,u}$ (3.2.5) is EP for any choices of the seed parameters $\mathbf{s}=(a,b,c,d)', t, u$. To establish this it suffices to show that $\mathbf{C}^2_{\mathbf{s},t,u}$ is symmetric (Theorem 2.4.3). An easy computation shows that, with $\mathbf{B}_{\mathbf{s},t}$ as defined in (3.2.4) above,

$$\mathbf{C}^2_{\mathbf{s},t,u} = 8(\mathbf{B}_{\mathbf{s},t}\mathbf{B}'_{\mathbf{s},t} + \mathbf{B}'_{\mathbf{s},t}\mathbf{B}_{\mathbf{s},t}) + (130t^2 - 72tu + 8u^2)\mathbf{E}_8, \tag{3.2.8}$$

which is symmetric. To establish (3.2.8) we used the fact that $\mathbf{B}^2 = 2t^2\mathbf{E}_8$.

If (a,b,c,d) is some permutation of $(1,2,3,4)$ then the magic key

$$\kappa = 128\big(t^2 - 10t + 30\big). \tag{3.2.9}$$

In our study of 46080 Caïssan beauties in Section 4 below we found that just 192 are EP: 96 with $\kappa = 2688$ and 96 with $\kappa = 8736$. The special key (3.2.9) equals 2688 if and only if $t=9$ or $t=1$. When $t=1$, however, the EP Caïssan beauty has magic sum $m=-28$ is not classic!

OPEN QUESTION 3.2.1. Are there "nice" values of the seed parameters for which the key (3.2.6) is equal to 8736? (The special key (3.2.9) equals 8736 only when $t = 5 \pm \frac{1}{2}\sqrt{253}$.)

For his special case, Cavendish (1894) used the seed parameters

$$\mathbf{s} = (1,2,3,4)', \quad t=9, \quad u=8, \tag{3.2.10}$$

and found that then

$$\mathbf{B}_{(1,2,3,4)',9} = \begin{pmatrix} 1 & 8 & 1 & 8 & 1 & 8 & 1 & 8 \\ 2 & 7 & 2 & 7 & 2 & 7 & 2 & 7 \\ 3 & 6 & 3 & 6 & 3 & 6 & 3 & 6 \\ 4 & 5 & 4 & 5 & 4 & 5 & 4 & 5 \\ 8 & 1 & 8 & 1 & 8 & 1 & 8 & 1 \\ 7 & 2 & 7 & 2 & 7 & 2 & 7 & 2 \\ 6 & 3 & 6 & 3 & 6 & 3 & 6 & 3 \\ 5 & 4 & 5 & 4 & 5 & 4 & 5 & 4 \end{pmatrix} \tag{3.2.11}$$

and hence

$$\mathbf{C}_{(1,2,3,4)',9,8} = \begin{pmatrix} 1 & 16 & 17 & 32 & 57 & 56 & 41 & 40 \\ 58 & 55 & 42 & 39 & 2 & 15 & 18 & 31 \\ 3 & 14 & 19 & 30 & 59 & 54 & 43 & 38 \\ 60 & 53 & 44 & 37 & 4 & 13 & 20 & 29 \\ 8 & 9 & 24 & 25 & 64 & 49 & 48 & 33 \\ 63 & 50 & 47 & 34 & 7 & 10 & 23 & 26 \\ 6 & 11 & 22 & 27 & 62 & 51 & 46 & 35 \\ 61 & 52 & 45 & 36 & 5 & 12 & 21 & 28 \end{pmatrix} = \mathbf{U}', \tag{3.2.12}$$

the transpose $\mathbf{C}'$ of the Ursus matrix $\mathbf{U}$.

3.3. **Cashmore beauties.** Cashmore [52, (1907)] discusses "chess magic squares" which he defines as "having constant summation along every chess path". We interpret this as synonymous to our Caïssan magic squares (pandiagonal and CSP2-magic). He presents two Caïssan magic squares which we define by the Caïssan magic matrices $\mathbf{C}_1$ and $\mathbf{C}_2$,

$$\mathbf{C}_1 = \begin{pmatrix} 19 & 41 & 20 & 47 & 22 & 48 & 21 & 42 \\ 55 & 14 & 56 & 13 & 50 & 11 & 49 & 12 \\ 37 & 26 & 35 & 25 & 36 & 31 & 38 & 32 \\ 1 & 60 & 7 & 62 & 8 & 61 & 2 & 59 \\ 46 & 24 & 45 & 18 & 43 & 17 & 44 & 23 \\ 10 & 51 & 9 & 52 & 15 & 54 & 16 & 53 \\ 28 & 39 & 30 & 40 & 29 & 34 & 27 & 33 \\ 64 & 5 & 58 & 3 & 57 & 4 & 63 & 6 \end{pmatrix}, \quad \mathbf{C}_2 = \begin{pmatrix} 19 & 6 & 27 & 54 & 43 & 62 & 35 & 14 \\ 55 & 42 & 63 & 34 & 15 & 18 & 7 & 26 \\ 37 & 12 & 21 & 4 & 29 & 52 & 45 & 60 \\ 1 & 32 & 49 & 48 & 57 & 40 & 9 & 24 \\ 46 & 59 & 38 & 11 & 22 & 3 & 30 & 51 \\ 10 & 23 & 2 & 31 & 50 & 47 & 58 & 39 \\ 28 & 53 & 44 & 61 & 36 & 13 & 20 & 5 \\ 64 & 33 & 16 & 17 & 8 & 25 & 56 & 41 \end{pmatrix},$$

which are pandiagonal and CSP2-magic, but only half CSP3-magic, in that only the n forwards but not the n backwards CSP3-paths are magic. Moreover, is $\mathbf{C}_i$ "semi-**H**-associated"

$$\mathbf{C}_i + \mathbf{H}\mathbf{C}_i = 65\mathbf{E}; \quad i = 1, 2. \tag{3.3.1}$$

DEFINITION 3.3.1. A magic matrix $\mathbf{M}$ with magic sum m is "semi-**H**-associated" whenever

$$\mathbf{M} + \mathbf{H}\mathbf{M} = 2m\bar{\mathbf{E}} \quad \text{or} \quad \mathbf{M} + \mathbf{M}\mathbf{H} = 2m\bar{\mathbf{E}}. \tag{3.3.2}$$

Suppose that the $n \times n$ magic matrix $\mathbf{M}$ with $n = 2k$, even, is partitioned with four $k \times k$ blocks as

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{pmatrix}. \tag{3.3.3}$$

Then $\mathbf{M}$ is semi-**H**-associated whenever either

$$\mathbf{M}_{11} + \mathbf{M}_{21} = \mathbf{M}_{12} + \mathbf{M}_{22} = 2m\bar{\mathbf{E}} \tag{3.3.4}$$

$$\text{or} \quad \mathbf{M}_{11} + \mathbf{M}_{12} = \mathbf{M}_{21} + \mathbf{M}_{22} = 2m\bar{\mathbf{E}}. \tag{3.3.5}$$

Moreover, $\mathbf{M}$ is **H**-associated whenever

$$\mathbf{M}_{11} + \mathbf{M}_{22} = \mathbf{M}_{12} + \mathbf{M}_{21} = 2m\bar{\mathbf{E}}. \tag{3.3.6}$$

It is easy to see that both $\mathbf{C}_1$ and $\mathbf{C}_2$ satisfy (3.3.4), and so both $\mathbf{C}_1$ and $\mathbf{C}_2$ are semi-**H**-associated.

THEOREM 3.3.1. Suppose that the magic matrix $\mathbf{M}$ is semi-$\mathbf{H}$-associated. Then all positive powers $\mathbf{M}^q$ are semi-$\mathbf{H}$-associated. When $\mathbf{M}$ has index 1 then $\mathbf{M}$ has a group-inverse $\mathbf{M}^\#$ and all positive powers $(\mathbf{M}^\#)^q$ are semi-$\mathbf{H}$-associated.

We find that both $\mathbf{C}_1$ and $\mathbf{C}_2$ have rank 5 and index 1 and so both have group inverses. From Theorem 3.3.1 we find that for $\mathbf{C} = \mathbf{C}_1$ or $\mathbf{C}_2$ all positive powers $\mathbf{C}^q$ and $(\mathbf{C}^\#)^q$ are semi-$\mathbf{H}$-associated.

Bidev [67, Fig. 7 & 8, (c. 1981)] observes that $\mathbf{C}_1$ and $\mathbf{C}_2$ *lassen sich natürlich aufrechtmachen und dann entstehen zwei Nasiks der Hauptklasse* (may be transformed into Caïssan beauties?). by transforming $\mathbf{C}_1$ and $\mathbf{C}_2$ into the *umpolarisierte* (repolarized) $\mathbf{C}_1^{(\mathrm{p})}$ and $\mathbf{C}_2^{(\mathrm{p})}$ by shifting the jth row of $\mathbf{C}_i$ to the left $j-1$ entries (with wrap-around), $i = 1, 2;\ j = 1, 2, \ldots, 8$. We observe that the first row of $\mathbf{C}_i$ and the first row of $\mathbf{C}_i^{(\mathrm{p})}$ coincide and the diagonal of $\mathbf{C}_i$ is the first column of $\mathbf{C}_i^{(\mathrm{p})}$, $i = 1, 2$. We will call $\mathbf{C}_i^{(\mathrm{p})}$ the "polarized partner" of $\mathbf{C}_i,\ i = 1, 2$. We have

$$\mathbf{C}_1 = \begin{pmatrix} 19 & 41 & 20 & 47 & 22 & 48 & 21 & 42 \\ 55 & 14 & 56 & 13 & 50 & 11 & 49 & 12 \\ 37 & 26 & 35 & 25 & 36 & 31 & 38 & 32 \\ 1 & 60 & 7 & 62 & 8 & 61 & 2 & 59 \\ 46 & 24 & 45 & 18 & 43 & 17 & 44 & 23 \\ 10 & 51 & 9 & 52 & 15 & 54 & 16 & 53 \\ 28 & 39 & 30 & 40 & 29 & 34 & 27 & 33 \\ 64 & 5 & 58 & 3 & 57 & 4 & 63 & 6 \end{pmatrix}, \quad \mathbf{C}_2 = \begin{pmatrix} 19 & 6 & 27 & 54 & 43 & 62 & 35 & 14 \\ 55 & 42 & 63 & 34 & 15 & 18 & 7 & 26 \\ 37 & 12 & 21 & 4 & 29 & 52 & 45 & 60 \\ 1 & 32 & 49 & 48 & 57 & 40 & 9 & 24 \\ 46 & 59 & 38 & 11 & 22 & 3 & 30 & 51 \\ 10 & 23 & 2 & 31 & 50 & 47 & 58 & 39 \\ 28 & 53 & 44 & 61 & 36 & 13 & 20 & 5 \\ 64 & 33 & 16 & 17 & 8 & 25 & 56 & 41 \end{pmatrix}, \tag{3.3.7}$$

$$\mathbf{C}_1^{(\mathrm{p})} = \begin{pmatrix} 19 & 41 & 20 & 47 & 22 & 48 & 21 & 42 \\ 14 & 56 & 13 & 50 & 11 & 49 & 12 & 55 \\ 35 & 25 & 36 & 31 & 38 & 32 & 37 & 26 \\ 62 & 8 & 61 & 2 & 59 & 1 & 60 & 7 \\ 43 & 17 & 44 & 23 & 46 & 24 & 45 & 18 \\ 54 & 16 & 53 & 10 & 51 & 9 & 52 & 15 \\ 27 & 33 & 28 & 39 & 30 & 40 & 29 & 34 \\ 6 & 64 & 5 & 58 & 3 & 57 & 4 & 63 \end{pmatrix}, \quad \mathbf{C}_2^{(\mathrm{p})} = \begin{pmatrix} 19 & 6 & 27 & 54 & 43 & 62 & 35 & 14 \\ 42 & 63 & 34 & 15 & 18 & 7 & 26 & 55 \\ 21 & 4 & 29 & 52 & 45 & 60 & 37 & 12 \\ 48 & 57 & 40 & 9 & 24 & 1 & 32 & 49 \\ 22 & 3 & 30 & 51 & 46 & 59 & 38 & 11 \\ 47 & 58 & 39 & 10 & 23 & 2 & 31 & 50 \\ 20 & 5 & 28 & 53 & 44 & 61 & 36 & 13 \\ 41 & 64 & 33 & 16 & 17 & 8 & 25 & 56 \end{pmatrix}.$$

We find that $\mathbf{C}_1^{(\mathrm{p})}$ and $\mathbf{C}_2^{(\mathrm{p})}$ are Caïssan beauties, i.e., pandiagonal, CSP2- and CSP3-magic, and in addition both $\mathbf{C}_1^{(\mathrm{p})}$ and $\mathbf{C}_2^{(\mathrm{p})}$ are

(1) **H**-associated,

(2) keyed, with magic key $\kappa = 2496$,

(3) 4-pac, and have

(4) rank 3, and index 1 but neither $\mathbf{C}_1^{(\mathrm{p})}$ nor $\mathbf{C}_2^{(\mathrm{p})}$ is EP.

The Cashmore magic matrices $\mathbf{C}_1$ and $\mathbf{C}_2$ both have rank 5 and index 1.

DEFINITION 3.3.2. A magic matrix $\mathbf{M}$ is a "Cashmore beauty" whenever $\mathbf{M}$ is

(1) a Caïssan magic matrix (pandiagonal and CSP2-magic),

(2) half CSP3-magic, i.e., the n forwards but not the n backwards CSP3-paths of $\mathbf{M}$ are magic, or the n backwards but not the n forwards CSP3-paths of $\mathbf{M}$ are magic, and

(3) semi-**H**-associated,

(4) the polarized partner $\mathbf{M}^{(\mathrm{p})}$ is a Caïssan beauty (pandiagonal, CSP2- and CSP3-magic).

CLAIM 3.3.1. We claim that all positive odd powers $\mathbf{C}_i^{2q+1}$ and $(\mathbf{C}_i^{\#})^{2q+1}$ are Cashmore beauties, $i = 1, 2;\ q = 0, 1, \ldots$

Bidev [67, Fig. 19] gives a magic square, our $\mathbf{C}_3$, which he finds to be *viermal so schachlich* (four times more chesslike) than $\mathbf{C}_1$ and $\mathbf{C}_2$. We find that $\mathbf{C}_3$ is actually *less* Caïssan in that its polarized partner $\mathbf{C}_3^{(\mathrm{p})}$ is not CSP2-magic! And so $\mathbf{C}_3$ is not a Cashmore beauty. We have

$$\mathbf{C}_3 = \begin{pmatrix} 50 & 1 & 29 & 46 & 15 & 64 & 36 & 19 \\ 30 & 47 & 16 & 60 & 35 & 18 & 49 & 5 \\ 12 & 59 & 34 & 17 & 53 & 6 & 31 & 48 \\ 33 & 21 & 54 & 7 & 32 & 44 & 11 & 58 \\ 55 & 8 & 28 & 43 & 10 & 57 & 37 & 22 \\ 27 & 42 & 9 & 61 & 38 & 23 & 56 & 4 \\ 13 & 62 & 39 & 24 & 52 & 3 & 26 & 41 \\ 40 & 20 & 51 & 2 & 25 & 45 & 14 & 63 \end{pmatrix}, \quad \mathbf{C}_3^{(\mathrm{p})} = \begin{pmatrix} 50 & 47 & 34 & 7 & 10 & 23 & 26 & 63 \\ 30 & 59 & 54 & 43 & 38 & 3 & 14 & 19 \\ 12 & 21 & 28 & 61 & 52 & 45 & 36 & 5 \\ 33 & 8 & 9 & 24 & 25 & 64 & 49 & 48 \\ 55 & 42 & 39 & 2 & 15 & 18 & 31 & 58 \\ 27 & 62 & 51 & 46 & 35 & 6 & 11 & 22 \\ 13 & 20 & 29 & 60 & 53 & 44 & 37 & 4 \\ 40 & 1 & 16 & 17 & 32 & 57 & 56 & 41 \end{pmatrix}. \tag{3.3.8}$$

Like $\mathbf{C}_1$ and $\mathbf{C}_2$, the Caïssan magic matrix $\mathbf{C}_3$ also has the n forwards but not the n backwards CSP3-paths magic and is semi-**H**-associated in that $\mathbf{C}_3$ satisfies (3.3.5).

To obtain $\mathbf{C}_3^{(\mathrm{p})}$ from $\mathbf{C}_3$ we transpose the procedure used to obtain $\mathbf{C}_i^{(\mathrm{p})}$, $i = 1, 2$. We shift the jth column of $\mathbf{C}_3$ up $j-1$ entries (with wrap-around), $i = 1, 2$; $j = 1, 2, \ldots, 8$. We observe that the first column of $\mathbf{C}_3$ and the first column of $\mathbf{C}_3^{(\mathrm{p})}$ coincide and the main diagonal of $\mathbf{C}_3$ is the first row of $\mathbf{C}_3^{(\mathrm{p})}$.

We find that $\mathbf{C}_3^{(\mathrm{p})}$ is not CSP2-magic, a surprise! It is, however, like $\mathbf{C}_1^{(\mathrm{p})}$ and $\mathbf{C}_2^{(\mathrm{p})}$, both pandiagonal and CSP3-magic, and **H**-associated and keyed. But $\mathbf{C}_3^{(\mathrm{p})}$

(1) has magic key $\kappa = 0$,

(2) is not 4-pac, and

(3) has rank 4, and index 3 (and so cannot be EP).

The Caïssan magic matrix $\mathbf{C}_3$ has rank 5 and index 1.

3.4. **Generalization of the Cashmore (1907) algorithm.** Cashmore [52, (1907)] described a procedure (which we believe he used) to generate the magic matrices $\mathbf{C}_1$ and $\mathbf{C}_2$, see (3.3.7) above. We now generalize this method. Let

$$\mathbf{P}_1 = \begin{pmatrix} a & b & c & d \\ d & \hat{a} & \hat{b} & \hat{c} \\ \hat{c} & \hat{d} & a & b \\ b & c & d & \hat{a} \end{pmatrix}, \tag{3.4.1}$$

where $\hat{a} = 9 - a,\ \hat{b} = 9 - b,\ \hat{c} = 9 - c,\ \hat{d} = 9 - d$. To create $\mathbf{C}_1$ and $\mathbf{C}_2$, Cashmore [52] chose

$$a = 3,\ b = 1,\ c = 4,\ d = 7. \tag{3.4.2}$$

We will assume (at least implicitly) only that a, b, c, d are any positive integers between 1 and 8, inclusive. Let $\hat{\mathbf{P}}_1 = 9\mathbf{E} - \mathbf{P}_1$, where $\mathbf{E}$ here is the 4×4 matrix with each entry equal to 1, and

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}_1 & \hat{\mathbf{P}}_1 \\ \hat{\mathbf{P}}_1 & \mathbf{P}_1 \end{pmatrix}. \tag{3.4.3}$$

Let the 8×2 matrix

$$\mathbf{Q}_1 = \begin{pmatrix} \mathbf{q} & \hat{\mathbf{q}} \\ \hat{\mathbf{q}} & \mathbf{q} \end{pmatrix}, \quad \text{where} \quad \mathbf{q} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \quad \text{and} \quad \hat{\mathbf{q}} = 9\mathbf{e} - \mathbf{q}, \tag{3.4.4}$$

and where $\mathbf{e}$ here is the 4×1 vector with each entry equal to 1. To create $\mathbf{C}_1$ and $\mathbf{C}_2$, Cashmore [52] chose

$$x = 3,\ y = 7,\ z = 5,\ t = 1. \tag{3.4.5}$$

We will assume (at least implicitly) only that x, y, z, t are any positive integers between 1 and 8, inclusive. Let the 8×8 matrix

$$\mathbf{Q} = \big(\mathbf{Q}_1 \ :\ \mathbf{Q}_1 \ :\ \mathbf{Q}_1 \ :\ \mathbf{Q}_1\big). \tag{3.4.6}$$

Then, using an "Euler-type algorithm" [52], we find the 8×8 magic matrix

$\mathbf{M}_1 = 8(\mathbf{Q} - \mathbf{E}) + \mathbf{P} =$

$$\begin{pmatrix}
8x-8+a & 64-8x+b & 8x-8+c & 64-8x+d & 8x+1-a & 73-8x-b & 8x+1-c & 73-8x-d \\
8y-8+d & 73-8y-a & 8y+1-b & 73-8y-c & 8y+1-d & 64-8y+a & 8y-8+b & 64-8y+c \\
8z+1-c & 73-8z-d & 8z-8+a & 64-8z+b & 8z-8+c & 64-8z+d & 8z+1-a & 73-8z-b \\
8t-8+b & 64-8t+c & 8t-8+d & 73-8t-a & 8t+1-b & 73-8t-c & 8t+1-d & 64-8t+a \\
73-8x-a & 8x+1-b & 73-8x-c & 8x+1-d & 64-8x+a & 8x-8+b & 64-8x+c & 8x-8+d \\
73-8y-d & 8y-8+a & 64-8y+b & 8y-8+c & 64-8y+d & 8y+1-a & 73-8y-b & 8y+1-c \\
64-8z+c & 8z-8+d & 73-8z-a & 8z+1-b & 73-8z-c & 8z+1-d & 64-8z+a & 8z-8+b \\
73-8t-b & 8t+1-c & 73-8t-d & 8t-8+a & 64-8t+b & 8t-8+c & 64-8t+d & 8t+1-a
\end{pmatrix}, \tag{3.4.7}$$

and its polarized partner $\mathbf{M}_1^{(\mathrm{p})} =$

$$\begin{pmatrix}
8x-8+a & 64-8x+b & 8x-8+c & 64-8x+d & 8x+1-a & 73-8x-b & 8x+1-c & 73-8x-d \\
73-8y-a & 8y+1-b & 73-8y-c & 8y+1-d & 64-8y+a & 8y-8+b & 64-8y+c & 8y-8+d \\
8z-8+a & 64-8z+b & 8z-8+c & 64-8z+d & 8z+1-a & 73-8z-b & 8z+1-c & 73-8z-d \\
73-8t-a & 8t+1-b & 73-8t-c & 8t+1-d & 64-8t+a & 8t-8+b & 64-8t+c & 8t-8+d \\
64-8x+a & 8x-8+b & 64-8x+c & 8x-8+d & 73-8x-a & 8x+1-b & 73-8x-c & 8x+1-d \\
8y+1-a & 73-8y-b & 8y+1-c & 73-8y-d & 8y-8+a & 64-8y+b & 8y-8+c & 64-8y+d \\
64-8z+a & 8z-8+b & 64-8z+c & 8z-8+d & 73-8z-a & 8z+1-b & 73-8z-c & 8z+1-d \\
8t+1-a & 73-8t-b & 8t+1-c & 73-8t-d & 8t-8+a & 64-8t+b & 8t-8+c & 64-8t+d
\end{pmatrix} \tag{3.4.8}$$

both have magic sum $m = 260$ for all values of the parameters a, b, c, d, x, y, z, t.
To obtain $\mathbf{M}_1^{(\mathrm{p})}$ from $\mathbf{M}_1$ we shift the jth row of $\mathbf{M}_1$ to the left $j-1$ entries (with wrap-around). We observe that the first row of $\mathbf{M}_1$ and the first row of $\mathbf{M}_1^{(\mathrm{p})}$ coincide and the diagonal of $\mathbf{M}_1$ is the first column of $\mathbf{M}_1^{(\mathrm{p})}$.

Moreover, we find the 8×8 magic matrix

$\mathbf{M}_2 = 8(\mathbf{P} - \mathbf{E}) + \mathbf{Q} =$

$$\begin{pmatrix} 8a-8+x & 8b+1-x & 8c-8+x & 8d+1-x & 64-8a+x & 73-8b-x & 64-8c+x & 73-8d-x \\ 8d-8+y & 73-8a-y & 64-8b+y & 73-8c-y & 64-8d+y & 8a+1-y & 8b-8+y & 8c+1-y \\ 64-8c+z & 73-8d-z & 8a-8+z & 8b+1-z & 8c-8+z & 8d+1-z & 64-8a+z & 73-8b-z \\ 8b-8+t & 8c+1-t & 8d-8+t & 73-8a-t & 64-8b+t & 73-8c-t & 64-8d+t & 8a+1-t \\ 73-8a-x & 64-8b+x & 73-8c-x & 64-8d+x & 8a+1-x & 8b-8+x & 8c+1-x & 8d-8+x \\ 73-8d-y & 8a-8+y & 8b+1-y & 8c-8+y & 8d+1-y & 64-8a+y & 73-8b-y & 64-8c+y \\ 8c+1-z & 8d-8+z & 73-8a-z & 64-8b+z & 73-8c-z & 64-8d+z & 8a+1-z & 8b-8+z \\ 73-8b-t & 64-8c+t & 73-8d-t & 8a-8+t & 8b+1-t & 8c-8+t & 8d+1-t & 64-8a+t \end{pmatrix}, \quad (3.4.9)$$

and its polarized partner $\mathbf{M}_2^{(\mathrm{p})} =$

$$\begin{pmatrix} 8a-8+x & 8b+1-x & 8c-8+x & 8d+1-x & 64-8a+x & 73-8b-x & 64-8c+x & 73-8d-x \\ 73-8a-y & 64-8b+y & 73-8c-y & 64-8d+y & 8a+1-y & 8b-8+y & 8c+1-y & 8d-8+y \\ 8a-8+z & 8b+1-z & 8c-8+z & 8d+1-z & 64-8a+z & 73-8b-z & 64-8c+z & 73-8d-z \\ 73-8a-t & 64-8b+t & 73-8c-t & 64-8d+t & 8a+1-t & 8b-8+t & 8c+1-t & 8d-8+t \\ 8a+1-x & 8b-8+x & 8c+1-x & 8d-8+x & 73-8a-x & 64-8b+x & 73-8c-x & 64-8d+x \\ 64-8a+y & 73-8b-y & 64-8c+y & 73-8d-y & 8a-8+y & 8b+1-y & 8c-8+y & 8d+1-y \\ 8a+1-z & 8b-8+z & 8c+1-z & 8d-8+z & 73-8a-z & 64-8b+z & 73-8c-z & 64-8d+z \\ 64-8a+t & 73-8b-t & 64-8c+t & 73-8d-t & 8a-8+t & 8b+1-t & 8c-8+t & 8d+1-t \end{pmatrix}. \quad (3.4.10)$$

We find that for all choices of a, b, c, d and x, y, z, t, both $\mathbf{M}_1$ and $\mathbf{M}_2$ each have rank 5 and index 1 and are half-$\mathbf{H}$-associated, and both are Caïssan magic matrices (pandiagonal and CSP2-magic) and both are half-CSP3-magic.

The polarized partners $\mathbf{M}_1^{(\mathrm{p})}$ and $\mathbf{M}_2^{(\mathrm{p})}$, however, for all choices of a, b, c, d and x, y, z, t, each have rank 3 and index 1 and are $\mathbf{H}$-associated and 4-pac, and both are Caïssan beauties (pandiagonal, and CSP2- and CSP3-magic). And so both $\mathbf{M}_1$ and $\mathbf{M}_2$ are Cashmore beauties for all choices of a, b, c, d and x, y, z, t,. But, in general, neither $\mathbf{M}_1^{(\mathrm{p})}$ nor $\mathbf{M}_2^{(\mathrm{p})}$ is EP.

If, however, we choose

$$(\hat{a}): \qquad x = \hat{a} = 9 - a, \quad y = b, \quad z = \hat{c} = 9 - c, \quad t = d, \qquad (3.4.11)$$

then $\mathbf{M}_{1(\hat{a})}^{(\mathrm{p})} =$

$$\begin{pmatrix} 64-7a & 8a-8+b & 64-8a+c & 8a-8+d & 73-9a & 8a+1-b & 73-8a-c & 8a+1-d \\ 73-8b-a & 7b+1 & 73-8b-c & 8b+1-d & 64-8b+a & 9b-8 & 64-8b+c & 8b-8+d \\ 64-8c+a & 8c-8+b & 64-7c & 8c-8+d & 73-8c-a & 8c+1-b & 73-9c & 8c+1-d \\ 73-8d-a & 8d+1-b & 73-8d-c & 7d+1 & 64-8d+a & 8d-8+b & 64-8d+c & 9d-8 \\ 9a-8 & 64-8a+b & 8a-8+c & 64-8a+d & 7a+1 & 73-8a-b & 8a+1-c & 73-8a-d \\ 8b+1-a & 73-9b & 8b+1-c & 73-8b-d & 8b-8+a & 64-7b & 8b-8+c & 64-8b+d \\ 8c-8+a & 64-8c+b & 9c-8 & 64-8c+d & 8c+1-a & 73-8c-b & 7c+1 & 73-8c-d \\ 8d+1-a & 73-8d-b & 8d+1-c & 73-9d & 8d-8+a & 64-8d+b & 8d-8+c & 64-7d \end{pmatrix} \quad (3.4.12)$$

and $\mathbf{M}^{(\mathrm{p})}_{2(\hat{a})} = \mathbf{H}(\mathbf{M}^{(\mathrm{p})}_{1(\hat{a})})'\mathbf{H}$

$$\begin{pmatrix} 7a+1 & 8b-8+a & 8c+1-a & 8d-8+a & 73-9a & 64-8b+a & 73-8c-a & 64-8d+a \\ 73-8a-b & 64-7b & 73-8c-b & 64-8d+b & 8a+1-b & 9b-8 & 8c+1-b & 8d-8+b \\ 8a+1-c & 8b-8+c & 7c+1 & 8d-8+c & 73-8a-c & 64-8b+c & 73-9c & 64-8d+c \\ 73-8a-d & 64-8b+d & 73-8c-d & 64-7d & 8a+1-d & 8b-8+d & 8c+1-d & 9d-8 \\ 9a-8 & 8b+1-a & 8c-8+a & 8d+1-a & 64-7a & 73-8b-a & 64-8c+a & 73-8d-a \\ 64-8a+b & 73-9b & 64-8c+b & 73-8d-b & 8a-8+b & 7b+1 & 8c-8+b & 8d+1-b \\ 8a-8+c & 8b+1-c & 9c-8 & 8d+1-c & 64-8a+c & 73-8b-c & 64-7c & 73-8d-c \\ 64-8a+d & 73-8b-d & 64-8c+d & 73-9d & 8a-8+d & 8b+1-d & 8c-8+d & 7d+1 \end{pmatrix}. \quad (3.4.13)$$

Both $\mathbf{M}^{(\mathrm{p})}_{1(\hat{a})}$ and $\mathbf{M}^{(\mathrm{p})}_{2(\hat{a})}$ are EP for all choices of a, b, c, d. If we choose a, b, c, d so that $a+b+c+d = 10$ then both $\mathbf{M}^{(\mathrm{p})}_{1(\hat{a})}$ and $\mathbf{M}^{(\mathrm{p})}_{2(\hat{a})}$ have magic key $\kappa = -2688$.

If we choose

$$(\hat{b}): \qquad x = a, \quad y = \hat{b} = 9 - b, \quad z = c, \quad t = \hat{d} = 9 - d, \tag{3.4.14}$$

then $\mathbf{M}^{(\mathrm{p})}_{1(\hat{b})} =$

$$\begin{pmatrix} 9a-8 & 64-8a+b & 8a-8+c & 64-8a+d & 7a+1 & 73-8a-b & 8a+1-c & 73-8a-d \\ 8b+1-a & 73-9b & 8b+1-c & 73-8b-d & 8b-8+a & 64-7b & 8b-8+c & 64-8b+d \\ 8c-8+a & 64-8c+b & 9c-8 & 64-8c+d & 8c+1-a & 73-8c-b & 7c+1 & 73-8c-d \\ 8d+1-a & 73-8d-b & 8d+1-c & 73-9d & 8d-8+a & 64-8d+b & 8d-8+c & 64-7d \\ 64-7a & 8a-8+b & 64-8a+c & 8a-8+d & 73-9a & 8a+1-b & 73-8a-c & 8a+1-d \\ 73-8b-a & 7b+1 & 73-8b-c & 8b+1-d & 64-8b+a & 9b-8 & 64-8b+c & 8b-8+d \\ 64-8c+a & 8c-8+b & 64-7c & 8c-8+d & 73-8c-a & 8c+1-b & 73-9c & 8c+1-d \\ 73-8d-a & 8d+1-b & 73-8d-c & 7d+1 & 64-8d+a & 8d-8+b & 64-8d+c & 9d-8 \end{pmatrix} \quad (3.4.15)$$

and $\mathbf{M}^{(\mathrm{p})}_{2(\hat{b})} = (\mathbf{M}^{(\mathrm{p})}_{1(\hat{b})})' = \mathbf{C}_{\mathbf{s}=(a,b,c,d)',t=9,u=8} =$

$$\begin{pmatrix} 9a-8 & 8b+1-a & 8c-8+a & 8d+1-a & 64-7a & 73-8b-a & 64-8c+a & 73-8d-a \\ 64-8a+b & 73-9b & 64-8c+b & 73-8d-b & 8a-8+b & 7b+1 & 8c-8+b & 8d+1-b \\ 8a-8+c & 8b+1-c & 9c-8 & 8d+1-c & 64-8a+c & 73-8b-c & 64-7c & 73-8d-c \\ 64-8a+d & 73-8b-d & 64-8c+d & 73-9d & 8a-8+d & 8b+1-d & 8c-8+d & 7d+1 \\ 7a+1 & 8b-8+a & 8c+1-a & 8d-8+a & 73-9a & 64-8b+a & 73-8c-a & 64-8d+a \\ 73-8a-b & 64-7b & 73-8c-b & 64-8d+b & 8a+1-b & 9b-8 & 8c+1-b & 8d-8+b \\ 8a+1-c & 8b-8+c & 7c+1 & 8d-8+c & 73-8a-c & 64-8b+c & 73-9c & 64-8d+c \\ 73-8a-d & 64-8b+d & 73-8c-d & 64-7d & 8a+1-d & 8b-8+d & 8c+1-d & 9d-8 \end{pmatrix} \quad (3.4.16)$$

are both EP for all choices of a, b, c, d. Here $\mathbf{C}_{\mathbf{s}=(a,b,c,d)',t=9,u=8}$ is the Cavendish matrix (3.2.5) with parameters $\mathbf{s} = (a, b, c, d)', t = 9, u = 8$. The magic key

$$\kappa(\mathbf{M}^{(\mathrm{p})}_{1(\hat{b})}) = \kappa(\mathbf{M}^{(\mathrm{p})}_{2(\hat{b})}) = 128(81 - 9(a+b+c+d) + (a^2+b^2+c^2+d^2) = \kappa_1, \tag{3.4.17}$$

say. We recall that the Cavendish matrix $\mathbf{C}_{\mathbf{s},t,u}$ has magic key (3.2.6)

$$\kappa(\mathbf{C}_{\mathbf{s},t,u}) = 128\big(t^2 - t(a+b+c+d) + a^2+b^2+c^2+d^2\big) = \kappa_2, \tag{3.4.18}$$

say. And clearly $\kappa_1 = \kappa_2$ when $t = 9$.

We now choose $a = 1$ so that the lead (1,1) entry of $\mathbf{M}^{(\mathrm{p})}_{1(\hat{b})} = \mathbf{M}^{(\mathrm{p})}_{1(\hat{b}),a=1}$ equals 1 to match Drury's 46080 classic Caïssan beauties:

$$\mathbf{M}^{(\mathrm{p})}_{1(\hat{b}),a=1} = \begin{pmatrix} 1 & 56+b & c & 56+d & 8 & 65-b & 9-c & 65-d \\ 8b & 73-9b & 8b+1-c & 73-8b-d & 8b-7 & 64-7b & 8b-8+c & 64-8b+d \\ 8c-7 & 64-8c+b & 9c-8 & 64-8c+d & 8c & 73-8c-b & 7c+1 & 73-8c-d \\ 8d & 73-8d-b & 8d+1-c & 73-9d & 8d-7 & 64-8d+b & 8d-8+c & 64-7d \\ 57 & b & 56+c & d & 64 & 9-b & 65-c & 9-d \\ 72-8b & 7b+1 & 73-8b-c & 8b+1-d & 65-8b & 9b-8 & 64-8b+c & 8b-8+d \\ 65-8c & 8c-8+b & 64-7c & 8c-8+d & 72-8c & 8c+1-b & 73-9c & 8c+1-d \\ 72-8d & 8d+1-b & 73-8d-c & 7d+1 & 65-8d & 8d-8+b & 64-8d+c & 9d-8 \end{pmatrix}. \tag{3.4.19}$$

If we then choose, in addition to $a = 1$, the triple

$$\{b, c, d\} = \{2, 3, 4\},\ \{2, 3, 5\},\ \{2, 4, 6\},\ \{2, 5, 6\},\ \{3, 4, 7\},\ \{3, 5, 7\},\ \{4, 6, 7\}\ \text{ or }\ \{5, 6, 7\}, \tag{3.4.20}$$

then the magic key (3.4.17) $\kappa = 2688$. We found that precisely 192 of Drury's 46080 classic Caïssan beauties are EP, with 96 having magic key $\kappa = 2688$. These 96 may be generated from $\mathbf{M}^{(\mathrm{p})}_{1(\hat{b}),a=1}$ by choosing b, c, d equal to one of the 8 choices in (3.4.20), each of which has 6 permutations that leaves $\kappa = 2688$ unchanged. This yields $48 = 8 \times 6$, with the other 48 found by transposition.

When $a = 1,\ b = 2,\ c = 3,\ d = 4$ and $x = 1,\ y = 7,\ z = 3,\ t = 5$ (and so $(\hat{b})$: holds) then

$$\mathbf{M}^{(\mathrm{p})}_{1(\hat{b}),a=1,b=2,c=3,d=4} = \begin{pmatrix} 1 & 58 & 3 & 60 & 8 & 63 & 6 & 61 \\ 16 & 55 & 14 & 53 & 9 & 50 & 11 & 52 \\ 17 & 42 & 19 & 44 & 24 & 47 & 22 & 45 \\ 32 & 39 & 30 & 37 & 25 & 34 & 27 & 36 \\ 57 & 2 & 59 & 4 & 64 & 7 & 62 & 5 \\ 56 & 15 & 54 & 13 & 49 & 10 & 51 & 12 \\ 41 & 18 & 43 & 20 & 48 & 23 & 46 & 21 \\ 40 & 31 & 38 & 29 & 33 & 26 & 35 & 28 \end{pmatrix} = \mathbf{U}, \tag{3.4.21}$$

the Ursus matrix. Moreover,

$$\mathbf{U}^2 = \begin{pmatrix} 9570 & 8674 & 9122 & 8226 & 8002 & 7554 & 8450 & 8002 \\ 8674 & 9186 & 8354 & 8866 & 7554 & 8386 & 7874 & 8706 \\ 9122 & 8354 & 8930 & 8162 & 8450 & 7874 & 8642 & 8066 \\ 8226 & 8866 & 8162 & 8802 & 8002 & 8706 & 8066 & 8770 \\ 8002 & 7554 & 8450 & 8002 & 9570 & 8674 & 9122 & 8226 \\ 7554 & 8386 & 7874 & 8706 & 8674 & 9186 & 8354 & 8866 \\ 8450 & 7874 & 8642 & 8066 & 9122 & 8354 & 8930 & 8162 \\ 8002 & 8706 & 8066 & 8770 & 8226 & 8866 & 8162 & 8802 \end{pmatrix} \tag{3.4.22}$$

is symmetric and block-Latin.

CLAIM 3.4.1. The other 96 of the 192 EP Drury Caïssan beauties all have magic key $\kappa = 8736$ and we claim that none of these 96 can be generated by an EP Cashmore beauty. For example, consider the Caïssan beauty

$$\mathbf{X} = \begin{pmatrix} 1 & 62 & 11 & 56 & 43 & 24 & 33 & 30 \\ 63 & 4 & 53 & 10 & 21 & 42 & 31 & 36 \\ 6 & 57 & 16 & 51 & 48 & 19 & 38 & 25 \\ 60 & 7 & 50 & 13 & 18 & 45 & 28 & 39 \\ 22 & 41 & 32 & 35 & 64 & 3 & 54 & 9 \\ 44 & 23 & 34 & 29 & 2 & 61 & 12 & 55 \\ 17 & 46 & 27 & 40 & 59 & 8 & 49 & 14 \\ 47 & 20 & 37 & 26 & 5 & 58 & 15 & 52 \end{pmatrix}, \tag{3.4.23}$$

which is EP with magic key $\kappa = 8736$. We believe that there are no values of a, b, c, d, x, y, z, t so that $\mathbf{X}$ (3.4.23) can be formed from either $\mathbf{M}^{(\mathrm{p})}_1$ (3.4.8) or $\mathbf{M}^{(\mathrm{p})}_2$ (3.4.10). Moreover,

$$\mathbf{X}^2 = \begin{pmatrix} 11306 & 5762 & 10466 & 6266 & 7778 & 8954 & 8618 & 8450 \\ 5762 & 10986 & 6522 & 10530 & 8954 & 8098 & 8194 & 8554 \\ 10466 & 6522 & 10026 & 6786 & 8618 & 8194 & 9058 & 7930 \\ 6266 & 10530 & 6786 & 10218 & 8450 & 8554 & 7930 & 8866 \\ 7778 & 8954 & 8618 & 8450 & 11306 & 5762 & 10466 & 6266 \\ 8954 & 8098 & 8194 & 8554 & 5762 & 10986 & 6522 & 10530 \\ 8618 & 8194 & 9058 & 7930 & 10466 & 6522 & 10026 & 6786 \\ 8450 & 8554 & 7930 & 8866 & 6266 & 10530 & 6786 & 10218 \end{pmatrix}$$

is symmetric and block-Latin.

When

$$a = 3,\ b = 1,\ c = 4,\ d = 7 \text{ and } x = 3,\ y = 7,\ z = 5,\ t = 1$$

as given in (3.4.2) and (3.4.5), respectively, then we find

$$\mathbf{M}_1 = \mathbf{C}_1 = \begin{pmatrix} 19 & 41 & 20 & 47 & 22 & 48 & 21 & 42 \\ 55 & 14 & 56 & 13 & 50 & 11 & 49 & 12 \\ 37 & 26 & 35 & 25 & 36 & 31 & 38 & 32 \\ 1 & 60 & 7 & 62 & 8 & 61 & 2 & 59 \\ 46 & 24 & 45 & 18 & 43 & 17 & 44 & 23 \\ 10 & 51 & 9 & 52 & 15 & 54 & 16 & 53 \\ 28 & 39 & 30 & 40 & 29 & 34 & 27 & 33 \\ 64 & 5 & 58 & 3 & 57 & 4 & 63 & 6 \end{pmatrix}, \quad \mathbf{M}_2 = \mathbf{C}_2 = \begin{pmatrix} 19 & 6 & 27 & 54 & 43 & 62 & 35 & 14 \\ 55 & 42 & 63 & 34 & 15 & 18 & 7 & 26 \\ 37 & 12 & 21 & 4 & 29 & 52 & 45 & 60 \\ 1 & 32 & 49 & 48 & 57 & 40 & 9 & 24 \\ 46 & 59 & 38 & 11 & 22 & 3 & 30 & 51 \\ 10 & 23 & 2 & 31 & 50 & 47 & 58 & 39 \\ 28 & 53 & 44 & 61 & 36 & 13 & 20 & 5 \\ 64 & 33 & 16 & 17 & 8 & 25 & 56 & 41 \end{pmatrix},$$

as obtained by Cashmore [52, Fig. 1, 2 (1907)].

Furthermore,

$$\mathbf{M}_1^{(\mathrm{p})} = \mathbf{C}_1^{(\mathrm{p})} = \begin{pmatrix} 19 & 41 & 20 & 47 & 22 & 48 & 21 & 42 \\ 14 & 56 & 13 & 50 & 11 & 49 & 12 & 55 \\ 35 & 25 & 36 & 31 & 38 & 32 & 37 & 26 \\ 62 & 8 & 61 & 2 & 59 & 1 & 60 & 7 \\ 43 & 17 & 44 & 23 & 46 & 24 & 45 & 18 \\ 54 & 16 & 53 & 10 & 51 & 9 & 52 & 15 \\ 27 & 33 & 28 & 39 & 30 & 40 & 29 & 34 \\ 6 & 64 & 5 & 58 & 3 & 57 & 4 & 63 \end{pmatrix}, \quad \mathbf{M}_2^{(\mathrm{p})} = \mathbf{C}_2^{(\mathrm{p})} = \begin{pmatrix} 19 & 6 & 27 & 54 & 43 & 62 & 35 & 14 \\ 42 & 63 & 34 & 15 & 18 & 7 & 26 & 55 \\ 21 & 4 & 29 & 52 & 45 & 60 & 37 & 12 \\ 48 & 57 & 40 & 9 & 24 & 1 & 32 & 49 \\ 22 & 3 & 30 & 51 & 46 & 59 & 38 & 11 \\ 47 & 58 & 39 & 10 & 23 & 2 & 31 & 50 \\ 20 & 5 & 28 & 53 & 44 & 61 & 36 & 13 \\ 41 & 64 & 33 & 16 & 17 & 8 & 25 & 56 \end{pmatrix}.$$

and neither $\mathbf{M}_1^{(\mathrm{p})}$ nor $\mathbf{M}_2^{(\mathrm{p})}$ is EP and neither (3.4.11) nor (3.4.14) is satisfied.

OPEN QUESTION 3.4.1. Is there an EP $\mathbf{M}_1^{(\mathrm{p})}$ or $\mathbf{M}_2^{(\mathrm{p})}$ with neither $(\hat{a}) : (3.4.11)$ nor $(\hat{b}) : (3.4.14)$ satisfied? To answer this it seems we need only check for symmetry of $(\mathbf{M}_j^{(\mathrm{p})})^2,\ j = 1, 2$.

CLAIM 3.4.2. The magic matrix $\mathbf{C}_3$ given by Bidev [67, Fig. 19], our (3.3.8), is not a Cashmore beauty and so we claim that there are no values of $a, b, c, d;\ x, y, z, t$ such that

$$\mathbf{C}_3 = \mathbf{M}_1' \quad \text{or} \quad \mathbf{M}_2' \,.$$

3.5. **Cashmore** [52, (1907)]**.** "Chess magic squares: on magic squares constructed using chess moves", by M. Cashmore, *Report of the South African Association for the Advancement of Science, Cape Town,* vol. 3, pp. 83–90 (1907). Reprint (apparently by Pavle Bidev, c. 1975) entitled "Chess magic squares" (8 pp. unnumbered) at Koninklijke Bibliotheek, 's-Gravenhage [Royal Library, The Hague] Request number XSR 564: GHC G12/Cashmore.

Article (apparently) reproduced in [70, pp. 24, 46–48, 82–85]. JFM 36.0312.04 says "Chess magic squares", *Brit. Ass. Rep. South Africa,* 350 (1905). Abstract in *Nature,* vol. 72, no. 1878, p. 640 (October 26, 1905):

> Mr. M. Cashmore showed how chess magic squares, *i.e.* squares of numbers which add up to the same amount along every path across the square in the direction of a rook's, a bishop's, or a knight's move, can be constructed by superposing on each other two types of subsidiary squares, which can be formed by simple rules.

online at the Nature Publishing Group and online at Google Books. Abstract also in the *Report of the Seventy-Fifth Meeting of the British Association for the Advancement of Science: South Africa, August and September 1905:* Transactions of Section A.—Mathematical and Physical Science, Johannesburg (Friday, September 1, 1905), p. 350: full *Report* online at Google Books & in the *S2A3 Biographical Database of Southern African Science,* online at S$_2$A$_3$. On 1 September 1905 Cashmore read two papers at the joint meeting of the British and South African Associations for the Advancement of Science in Johannesburg. The first dealt with an aspect of recreational mathematics, namely "Chess magic squares", that is, magic squares having a constant sum along every chess path The method of construction was given, followed by an investigation into the number of possible chess magic squares, and an explanation of the theory of their construction. Both papers were published in Addresses and papers read at the joint meeting of the British and South African Associations for the Advancement of Science, South Africa, September 1905. In 1906, still a member of the South African Association for the Advancement of Science, Cashmore was living in London. There he showed himself to be not so well-informed about his subject, by publishing a monograph entitled *Fermat's Last Theorem: proofs by elementary algebra* [Third edition, pub. G. Bell & Sons, London, 1921 (First edition 1916, revised 1918).

4. 4 × 4 MAGIC MATRICES

We now discuss some properties of 4×4 magic matrices, which special emphasis on those that have rank 3 and index 1.

4.1. **Pandiagonal** 4×4 **magic matrices are 4-pac and H-associated.**

As we have seen (where? TBC) when $n = 8$ the three properties: (a) 4-pac, (b) **H**-associated and (c) pandiagonal are not equivalent but when $n = 4$ they are!

THEOREM 4.1.1. Let $\mathbf{M}$ be a 4×4 pandiagonal magic matrix. Then $\mathbf{M}$ is 4-pac and **H**-associated.

Proof of Theorem 4.1.1. Let m denote the magic sum of $\mathbf{M}$ and $\bar{\mathbf{E}}$ the 4×4 matrix with every entry equal to 1. Then $\mathbf{M}$ may be written, in general, as follows [156, 202, 275]

$$\mathbf{M} = m\bar{\mathbf{E}} + \begin{pmatrix} p+r & -p+s & p-r & -p-s \\ q-r & -q-s & q+r & -q+s \\ -p+r & p+s & -p-r & p-s \\ -q-r & q-s & -q+r & q+s \end{pmatrix} \tag{4.1.1}$$

$$= \begin{pmatrix} a+b+e & c+d+e & a+c & b+d \\ a+c+d & b & a+b+d+e & c+e \\ b+d+e & a+c+e & c+d & a+b \\ c & a+b+d & b+e & a+c+d+e \end{pmatrix}. \tag{4.1.2}$$

Theorem 4.1.1 then follows at once (by inspection) from (4.1.1) or (4.1.2).

The magic key κ of $\mathbf{M}$ as defined in in (4.1.1) and (4.1.2) is

$$\kappa = 8(pr + sq) = 4(ae + cd - bd). \tag{4.1.3}$$

When the magic key $\kappa \neq 0$ and the magic sum $m \neq 0$ then the 4×4 magic matrix $\mathbf{M}$ has rank 3 and index 1, and hence is EP if and only if $\mathbf{M}^2$ is symmetric (Theorem 2.4.3). We find that with $\mathbf{M}$ defined as in (4.1.1) and (4.1.2),

$$\mathbf{M}^2 - (\mathbf{M}^2)' = \sigma \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix} = \mathbf{T}, \tag{4.1.4}$$

say, where

$$\sigma = 4(ps - qr) = 2(ac - ab - de). \tag{4.1.5}$$

It follows at once that $\mathbf{M}$ is EP if and only if

$$ps = qr \;\Leftrightarrow\; ac = ab + de. \tag{4.1.6}$$

4.2. **16th century 4×4 magic squares with optional magic sum.** An early method for constructing 4×4 magic squares having any optional magic sum $m = 2a$ is given in *Smṛtitattva,* the 16th-century encyclopaedia on Hindu Law by the Bengali scholar Raghunandana Bhaṭṭācārya (fl. c. 1520/1570). The magic matrix

$$\mathbf{M}_{(a)} = \begin{pmatrix} 1 & 8 & a-7 & a-2 \\ a-5 & a-4 & 3 & 6 \\ 7 & 2 & a-1 & a-8 \\ a-3 & a-6 & 5 & 4 \end{pmatrix} \tag{4.2.1}$$

has magic sum $m = 2a$, magic key $\kappa = 16a - 136$, and is pandiagonal for all values of a, and hence 4-pac and $\mathbf{H}$-associated for all values of a (Theorem 4.1.1). When the magic sum $m \neq 0$ and the magic key $\kappa \neq 0$, i.e., $a \neq 0$ or $8\frac{1}{2}(= 136/16)$, then $\mathbf{M}_{(a)}$ has rank 3 and index 1 and is EP if and only if $\mathbf{M}^2_{(a)}$ is symmetric. We find that

$$\mathbf{M}^2_{(a)} - (\mathbf{M}^2_{(a)})' = \sigma \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix} = \mathbf{T}, \tag{4.2.2}$$

as in (4.1.4), but where now

$$\sigma = 4(a - 11). \tag{4.2.3}$$

It follows at once that $\mathbf{M}_{(a)}$ is EP if and only if $a = 11$. But when $a = 11$ the matrix $\mathbf{M}_{(a)}$ is not classic: in fact $\mathbf{M}_{(a)}$ is classic only when $a = 17$. [264]

4.3. **15th century 4×4 classic EP Shortrede–Gwalior magic matrix.** We have identified relatively few magic matrices that are EP. Of the 46080 Drury–Caïssan beauties only 192 are EP and of the 880 classic 4×4 magic matrices only 24 are EP. Only one of the 16th-century 4×4 magic squares $\mathbf{M}_{(a)}$ (4.2.1) is EP ($a = 11$) but it is not classic. All of these are $\mathbf{V}$-associated and so have a magic Moore–Penrose inverse, but only 8 of the classic 4×4 are pandiagonal (all 46080 Drury–Caïssan beauties are pandiagonal).

The oldest EP classic 4×4 magic matrix may be the "Shortrede–Gwalior magic square" defined by the "Shortrede–Gwalior magic matrix" $\mathbf{G}$ in (4.3.1) and discovered in 1841 [90, (1842)] by Robert Shortrede (1800–1866) but dated 1483:

$$\mathbf{G} = \begin{pmatrix} 16 & 9 & 4 & 5 \\ 3 & 6 & 15 & 10 \\ 13 & 12 & 1 & 8 \\ 2 & 7 & 14 & 11 \end{pmatrix}, \quad \mathbf{G}^2 = \begin{pmatrix} 345 & 281 & 273 & 257 \\ 281 & 313 & 257 & 305 \\ 273 & 257 & 345 & 281 \\ 257 & 305 & 281 & 313 \end{pmatrix}. \tag{4.3.1}$$

The Shortrede–Gwalior magic matrix $\mathbf{G}$ has rank 3 and index 1 and is EP since $\mathbf{G}^2$ is symmetric (Theorem 2.4.3). In addition the matrix $\mathbf{G}$ is 4-pac, $\mathbf{H}$-associated, and pandiagonal.

The 1842 article [90], which announced the discovery of the Shortrede–Gwalior magic square, is signed by "Captain Shortreede", who almost surely was Captain (later Major-General) Robert Shortrede (1800–1868), with the extra "e" in "Shortreede" here a typo. From his obituary [100] we find that Robert Shortrede was born on 19 July 1800 in Jedburgh (Scotland, about halfway between Edinburgh and Newcastle Upon Tyne). Having "early evinced unusual aptitude for mathematics ... thinking that India presented ample scope for his talents in that direction, he obtained an appointment to that country". The magic square defined by $\mathbf{G}$ was discovered in 1841 [90] in an old temple in Gwalior (Madhya Pradesh, about 120 km south of Agra).

***On an Ancient Magic Square, cut in a Temple at Gwalior. By Captain* SHORTREEDE.**

As every thing tending to throw any certain light on the antiquities of India has an interest, I send you the following inscription of a Magic Square, which I copied last year from an old temple in the hill fort of Gwalior. It bears the date समवत १५४० = A. D. 1483.

The temple is on the northern side of the hill, and at one time it has been a very magnificent edifice, though now it be sorely dilapidated.

There is another and larger ancient temple in the fort, of a peculiar form, which the Musalmans have converted into a Musjid.

If I remember rightly, the Magic Square is cut on the inner side of the northern wall, close to where the excavation has been made. I did not measure the dimensions; but the form is as follows:—

FIGURE 4.3.1: Comments by "Captain Shortreede" [90, (1842)] about the Shortrede–Gwalior magic square.

FIGURE 4.3.2: (left panel) The original Shortrede–Gwalior magic square with entries in Sanskrit; (right panel) The fort at Gwalior, India 1984, *Scott* 1065.

Shortrede went to India in 1822 and was appointed to the Deccan Survey. The Deccan Plateau extends over eight Indian states and encompasses a wide range of habitats, covering most of central and southern India. Shortrede was appointed to the Great Trigonometric Survey (GTS) in which he remained until 1845. The GTS was piloted in its initial stages by William Lambton (c. 1753–1823), and later by Sir George Everest (1790–1866). Among the many accomplishments of the GTS was the measurement of the height of the Himalayan giants: Everest, K2, and Kanchenjunga. In 1865, Mount Everest was named in Sir George Everest's honour, despite his objections. Pandit Nain Singh Rawat (c. 1826–1882; Figure 4.3.3, left stamp) was one of the first of the pundits who explored the Himalayas for the British; Radhanath Sikdar (1813–1870; Figure 4.3.3 right stamp) was an Indian mathematician who, among many other things, calculated the height of Peak XV in the Himalayas and showed it to be the tallest mountain above sea level; Peak XV was later named Mount Everest (Figure 4.3.4, left panel).

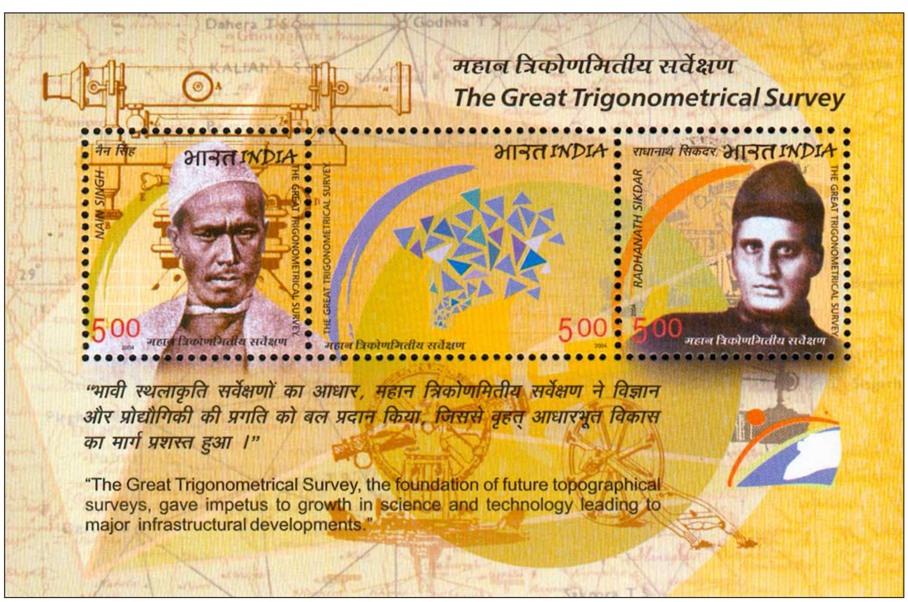

FIGURE 4.3.3: The Great Trigonometric Survey, India 2004, *Scott* 2067a.

The Scottish historical novelist and poet Sir Walter Scott, 1st Baronet (1771–1832), was an "old and intimate friend of the Shortrede family" and both Shortrede and Scott studied at the University of Edinburgh.

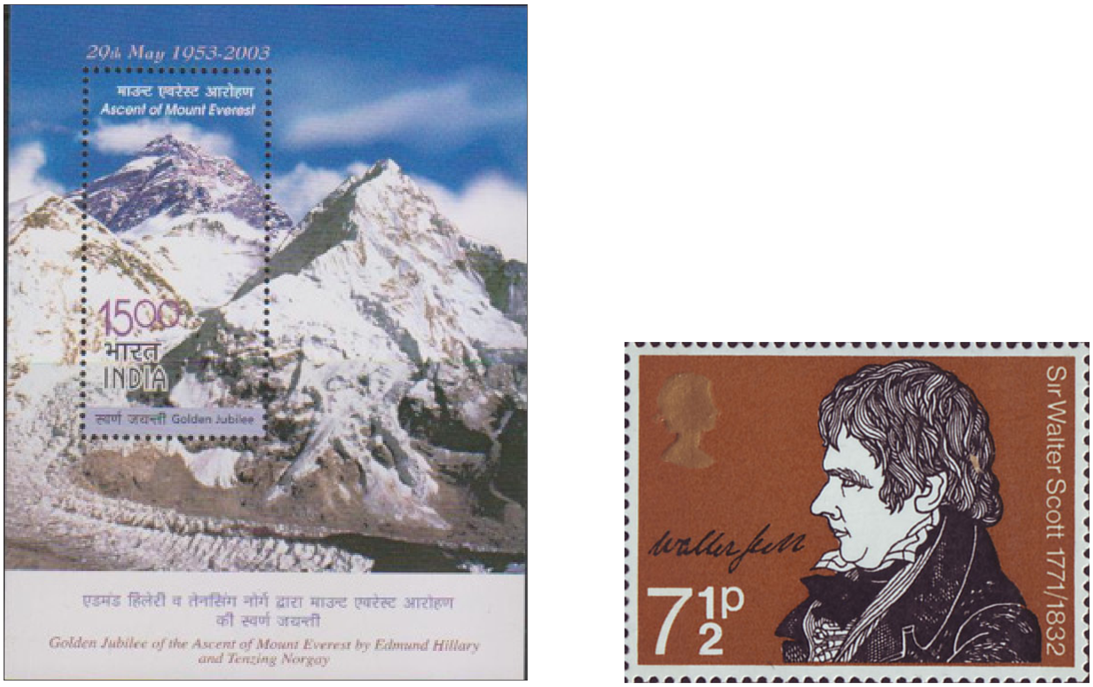

FIGURE 4.3.4 (left panel) Mount Everest: India 2003, *Scott* 2008a;
(right panel) Sir Walter Scott: Great Britain 1971, *Scott* 653.

FIGURE 4.3.5: TBC.

4.4. **Shortrede's "rhomboid" property: "rhomboidal" magic matrices.**

> **It will be observed, that the places of the numbers 1, 2, 3, 4, form a rhomboid, as do also 5, 6, 7, 8; 9, 10, 11, 12; 13, 14, 15, 16. It may be remarked also, that the sum of every two alternate numbers taken diagonally is 17: and that all these properties will hold good if the lines be transposed vertically or horizontally in the same order; that is, if the top line be brought to the bottom; or if the left hand vertical line be carried over to the right.**

FIGURE 4.4.1: Shortrede's comments [90, p. 292 (1842)] "that the places of the numbers $1, 2, 3, 4$ form a rhomboid". And "that the sum of every two alternate numbers taken diagonally is 17" (Dudeney Type I or our **H**-associated).

Shortrede [90, (1842)] observed (Figure 4.4.1) that the Gwalior magic square defined by

$$\mathbf{G} = \begin{pmatrix} 16 & 9 & \mathbf{4} & 5 \\ \mathbf{3} & 6 & 15 & 10 \\ 13 & 12 & \mathbf{1} & 8 \\ \mathbf{2} & 7 & 14 & 11 \end{pmatrix} \tag{4.4.1}$$

is **H**-associated and that it has a "rhomboid" property— the places of the numbers $1, 2, 3, 4$ (boldface red) in (4.4.1), form a "rhomboid", as do the numbers $5, 6, 7, 8$; $9, 10, 11, 12$ and $13, 14, 15, 16$. We will say that a magic matrix with such a rhomboid property is "rhomboidal".

In *Wikipedia* [324] we find that a "rhomboid" is a parallelogram in which adjacent sides are of unequal lengths and the angles are all oblique (not equal to 90°). A parallelogram with sides of equal length is a "rhombus"; a parallelogram with right-angled corners is a "rectangle" and a rectangle with all sides of equal length is a "square". A rhombus with acute angle 45° is sometimes called a "lozenge" [286] and a rhombus with acute angle

$$90^\circ - 2\tan^{-1}(\tfrac{1}{2}) = \tan^{-1}(2) - \tan^{-1}(\tfrac{1}{2}) \simeq 37.870^\circ \tag{4.4.2}$$

is sometimes called a "diamond" [279]. We will adopt the following definition:

DEFINITION 4.4.1. An $n \times n$ classic (magic) matrix with $n = 4h$ doubly-even is "rhomboidal" whenever the places of the numbers $1, 2, 3, 4$; $5, 6, 7, 8$; $9, 10, 11, 12$; $\ldots$; n^2-3, n^2-2, n^2-1, n^2 form n "rhomboids", where a "rhomboid" is defined in the extended sense to include a diamond, lozenge, rectangle, rhombus, or square.

The Shortrede–Gwalior magic matrix **G** (4.4.1) is rhomboidal (Definition 4.4.1). If we define the distance between any two numbers in **G** that are adjacent (either horizontally or vertically) as 1 unit, then the rhomboid defined by the red numbers and the other 3 (similar) rhomboids each have sides of lengths 2 and $\sqrt{5}$ units, and associated acute angle $\tan^{-1}(2) \simeq 63.435°$.

It seems that Euclid of Alexandria (fl. 300 BC) was the first to use the term "rhomboid" [153, Def. 22, pp. 188–189]:

> Of quadrilateral figures, ... an oblong that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled.

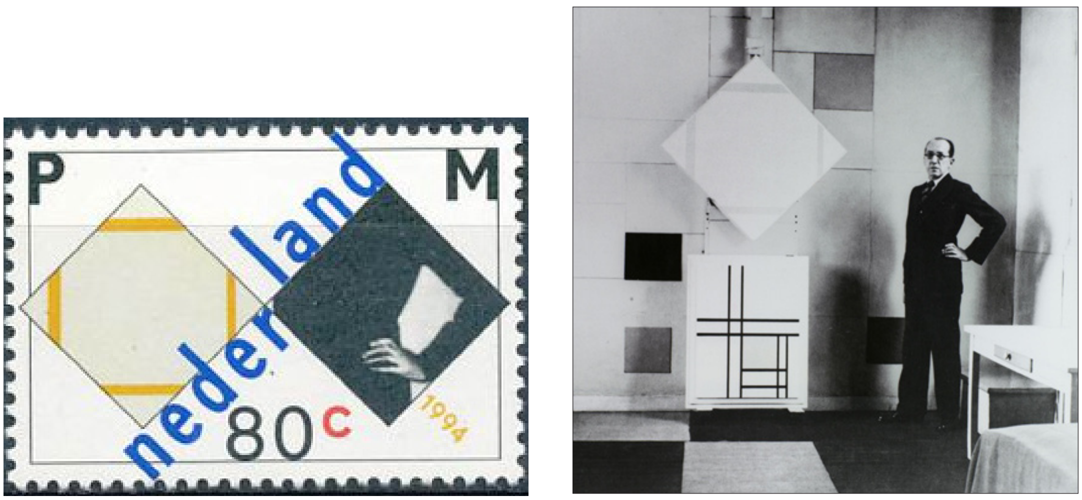

FIGURE 4.4.2: (left panel) "Rhomboid with yellow lines" [308]: Netherlands 1994, *Scott* 851; (right panel) Photograph of Piet Mondrian with two of his paintings in his Paris atelier, c. October 1933 [200, p. 47].

The stamp in Figure 4.4.2 (left panel), which the *Scott Catalogue* [308] calls "Rhomboid with yellow lines", features two "rhomboids" which are actually squares each tilted at 45°. The Dutch *Postzegelcatalogus* [227] identifies the stamp as "Compositie met gele lijnen" (Composition with yellow lines).

We believe that this stamp is based on the photograph[26] (Figure 4.4.2, right panel) by Charles Karsten of Piet Mondrian (1872–1944)[27] in his atelier in Paris at 26, rue du Départ (near the Gare Montparnasse), c. October 1933. Mondrian's 1933 painting "Lozenge Composition with Four Yellow Lines"[28] is shown on the left in the stamp and on the upper left in the photograph. Moreover, in the photograph is also shown (lower left) Mondrian's painting "Composition with Double Line and Yellow", unfinished. We believe that the original "Lozenge Composition with Four Yellow Lines" painting is in the Gemeentemuseum in The Hague which has a"marvellous series of works by Mondrian, ranging from moody Dutch landscapes to the sparkling Victory Boogie Woogie" [324].

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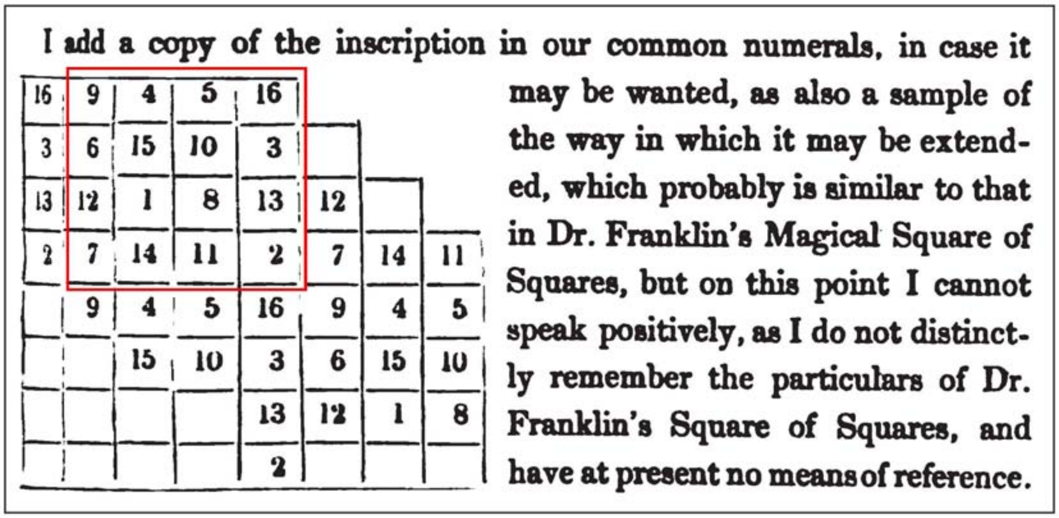
[26]Bax [200, p. 47] indicates that this photograph is "Courtesy of Karsten Archives, Nederlands Architectuurinstituut Rotterdam", while the image of this photograph online is accompanied by a description of the Mondrian/De Stijl exhibition at the Centre Pompidou in Paris (December 2010–March 2011). See also [287].

[27]The Dutch painter Pieter Cornelis Mondriaan (1872–1944), changed his name to Piet Mondrian in 1912.

[28]Mondrian's 1933 painting "Lozenge Composition with Four Yellow Lines", displayed in full colour by Bax [200, p. 241], is apparently the first by Mondrian using coloured lines [324].



I add a copy of the inscription in our common numerals, in case it may be wanted, as also a sample of the way in which it may be extended, which probably is similar to that in Dr. Franklin's Magical Square of Squares, but on this point I cannot speak positively, as I do not distinctly remember the particulars of Dr. Franklin's Square of Squares, and have at present no means of reference.

16	9	4	5	16			
3	6	15	10	3			
13	12	1	8	13	12		
2	7	14	11	2	7	14	11
	9	4	5	16	9	4	5
		15	10	3	6	15	10
				13	12	1	8
				2			

Figure 4.4.3: "Postscript" ending the article by Shortrede [90, p. 293 (1842)].

Shortrede ended his article [90, p. 293] with a "postscript" (Figure 4.4.3) presenting an arrangement of numbers which includes the magic square defined by $\mathbf{G}$ and 4 more magic squares defined by the magic matrices $\mathbf{G}_2$ (boxed in red), $\mathbf{G}_3, \mathbf{G}_4$, and $\mathbf{G}_5$, say:

$$\mathbf{G}_2 = \begin{pmatrix} 9 & 4 & 5 & 16 \\ 6 & 15 & 10 & 3 \\ 12 & 1 & 8 & 13 \\ 7 & 14 & 11 & 2 \end{pmatrix}; \quad \mathbf{G}_3 = \begin{pmatrix} 6 & 15 & 10 & 3 \\ 12 & 1 & 8 & 13 \\ 7 & 14 & 11 & 2 \\ 9 & 4 & 5 & 16 \end{pmatrix}; \tag{4.4.3}$$

$$\mathbf{G}_4 = \begin{pmatrix} 1 & 8 & 13 & 12 \\ 14 & 11 & 2 & 7 \\ 4 & 5 & 16 & 9 \\ 15 & 10 & 3 & 6 \end{pmatrix}; \quad \mathbf{G}_5 = \begin{pmatrix} 2 & 7 & 14 & 11 \\ 16 & 9 & 4 & 5 \\ 3 & 6 & 15 & 10 \\ 13 & 12 & 1 & 8 \end{pmatrix}. \tag{4.4.4}$$

We recall that a 4×4 $\mathbf{H}$-associated magic matrix is 4-pac (4-ply and with the alternate couplets property) and pandiagonal.

We find that

(1) $\mathbf{G}$ (4.3.1) has rank 3, index 1 (key $\kappa = 80$), is $\mathbf{H}$-associated, rhomboidal and EP,

(2) $\mathbf{G}_2$ has rank 3, index 3 (key $\kappa = 0$), is $\mathbf{H}$-associated, rhomboidal but not EP (index $\neq 1$),

(3) $\mathbf{G}_3$ has rank 3, index 1 (key $\kappa = 80$), is $\mathbf{H}$-associated, EP but not rhomboidal,

(4) $\mathbf{G}_4$ has rank 3, index 1 (key $\kappa = 80$), is $\mathbf{H}$-associated, rhomboidal but not EP,

(5) $\mathbf{G}_5$ has rank 3, index 3 (key $\kappa = 0$), is $\mathbf{H}$-associated, not rhomboidal and not EP (index $\neq 1$)

The magic matrix $\mathbf{M}_5$ (4.4.5) which defines the "Rouse Ball magic square" has rank 3, index 1 (key $\kappa = 48$), is $\mathbf{H}$-associated, rhomboidal but not EP ($\mathbf{M}_5^2$ not symmetric):

$$\mathbf{M}_5 = \begin{pmatrix} 15 & 10 & 3 & 6 \\ 4 & 5 & 16 & 9 \\ 14 & 11 & 2 & 7 \\ 1 & 8 & 13 & 12 \end{pmatrix}, \quad \mathbf{M}_5^2 = \begin{pmatrix} 363 & 251 & 287 & 255 \\ 311 & 295 & 323 & 227 \\ 227 & 323 & 295 & 311 \\ 255 & 287 & 251 & 363 \end{pmatrix} \tag{4.4.5}$$

The three magic matrices $\mathbf{G}_2, \mathbf{G}_4$ and $\mathbf{M}_5$ are rhomboidal in the same way as $\mathbf{G}$ in that each of these four magic matrices has the places of the numbers $1, 2, 3, 4$; $5, 6, 7, 8$; $9, 10, 11, 12$; and $13, 14, 15, 16$ forming a rhomboid with each having sides of lengths 2 and $\sqrt{5}$ units, and associated acute angle $\tan^{-1}(2) \simeq 63.435°$.

The 8×8 "Firth–Zukertort magic matrix", see (8.1.1) below,

$$\mathbf{Z} = \begin{pmatrix} 64 & 21 & 42 & \mathbf{3} & 37 & 16 & 51 & 26 \\ 38 & 15 & 52 & 25 & 63 & 22 & 41 & \mathbf{4} \\ 11 & 34 & 29 & 56 & 18 & 59 & \mathbf{8} & 45 \\ 17 & 60 & \mathbf{7} & 46 & 12 & 33 & 30 & 55 \\ 10 & 35 & 32 & 53 & 19 & 58 & \mathbf{5} & 48 \\ 20 & 57 & \mathbf{6} & 47 & 9 & 36 & 31 & 54 \\ 61 & 24 & 43 & \mathbf{2} & 40 & 13 & 50 & 27 \\ 39 & 14 & 49 & 28 & 62 & 23 & 44 & \mathbf{1} \end{pmatrix}, \tag{4.4.6}$$

is rhomboidal in a different (but similar) way. The places of the numbers $1, 2, 3, 4$; $5, 6, 7, 8$; $\ldots$, $61, 62, 63, 64$ each form a rhomboid of two different types, A and B, say, with 8 of each type. The 8 rhomboids of type A with sides of lengths 6 and $\sqrt{17}$ units, and associated acute angle $\tan^{-1}(4) \simeq 75.964°$, and the 8 rhomboids of type B with sides of lengths 2 and $\sqrt{17}$ units, all have the same associated acute angle $\tan^{-1}(4) \simeq 75.964°$. A rhomboid of type A (e.g., defined by **1,2,3,4** in red) has precisely 3 times the area of one of type B (e.g., defined by **5,6,7,8** in green).

As observed by Jelliss [279] the matrix

$$\mathbf{P} = \begin{pmatrix} 27 & 14 & 59 & 44 & 11 & 30 & 63 & 46 \\ 58 & 43 & 28 & 13 & 62 & 45 & 10 & 31 \\ 15 & 26 & 41 & 60 & 29 & 12 & 47 & 64 \\ 42 & 57 & 16 & 25 & 48 & 61 & 32 & 9 \\ \mathbf{1} & 24 & 53 & 40 & 17 & \mathbf{8} & 49 & 34 \\ 56 & 39 & \mathbf{4} & 21 & 52 & 33 & 18 & \mathbf{7} \\ 23 & \mathbf{2} & 37 & 54 & \mathbf{5} & 20 & 35 & 50 \\ 38 & 55 & 22 & \mathbf{3} & 36 & 51 & \mathbf{6} & 19 \end{pmatrix}, \tag{4.4.7}$$

which provides a solution to the knight's tour on an 8×8 chessboard, has the "squares and diamonds" property, so may be considered "rhomboidal" in that a square and a diamond are special cases of rhomboids with adjacent sides of equal length (in fact here equal to the length of a regular knight's move of type CSP2).

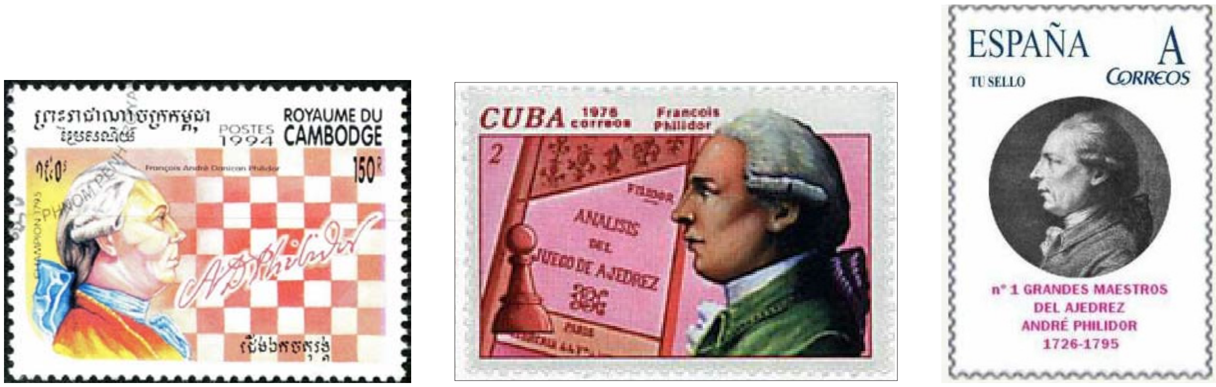


FIGURE 4.4.4: François-André Danican Philidor (1726–1795): Cambodia 1994, Cuba 1976, Spain 2009.

The solution defined by $\mathbf{P}$ is apparently the first published solution to the knight's tour with the "squares and diamonds" property and first appeared in an appendix by "F. P. H." (full name apparently unknown [279]) to the 6th English edition of *Studies of Chess* [87, p. 536 (1825)] by François-André Danican Philidor (1726–1795), a French composer who contributed to the early development of the opéra comique, and who was also regarded as the best chess player of his age; Philidor's book *Analyse du jeu des échecs* [82] was considered a standard chess manual for at least a century [324]; a Spanish version is illustrated in a stamp from Cuba 1976 (Figure 4.4.4).

As with the Firth–Zukertort magic matrix $\mathbf{Z}$ (4.4.6) there are two types of "rhomboids" in $\mathbf{P}$, with type A here being a square (e.g., defined by **5,6,7,8** in green) and type B being a diamond (e.g., defined by **1,2,3,4** in red). There are 8 squares and 8 diamonds each with sides of length $\sqrt{5}$ units. The diamonds have acute angle $90^\circ - 2\tan^{-1}(\frac{1}{2}) = \tan^{-1}(2) - \tan^{-1}(\frac{1}{2}) \simeq 37.870^\circ$.

The matrix $\mathbf{P}$ is not magic (the first semi-magic knight's tour was discovered in 1848 and there is no fully-magic knight's tour, see §8.3 below), though all the column totals are equal to 260, and so $\mathbf{P}$ has an eigenvalue equal to 260. As observed by "F. P. H." [87, p. 536] the matrix $\mathbf{P}$ has the interesting property that

$$\mathbf{P} - \mathbf{FPF} = 8\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix} = \mathbf{D}, \tag{4.4.8}$$

say. We note, however, that $\mathbf{P}$ is not $\mathbf{F}$-associated, in fact $\mathbf{P} + \mathbf{FPF}$ is nonsingular. "F. P. H." [87, p. 535] also gives another solution $\mathbf{P}_2$, say, to the knight's tour with the "squares and diamonds" property" and for which $\mathbf{P}_2 - \mathbf{FP}_2\mathbf{F} = 2(\mathbf{P} - \mathbf{FPF}) = 16\mathbf{D}$.

It seems that Karl Wenzelides (1770–1852)[29] was the first to publish [94, (1849)] a semi-magic knight's tour with the "squares and diamonds property". We define his tour by the matrix

$$\mathbf{P}_3 = \begin{pmatrix} 2 & 11 & 58 & 51 & 30 & 39 & 54 & 15 \\ 59 & 50 & 3 & 12 & 53 & 14 & 31 & 38 \\ 10 & 1 & 52 & 57 & 40 & 29 & 16 & 55 \\ 49 & 60 & 9 & 4 & 13 & 56 & 37 & 32 \\ 64 & 5 & 24 & 45 & 36 & 41 & 28 & 17 \\ 23 & 48 & 61 & 8 & 25 & 20 & 33 & 42 \\ 6 & 63 & 46 & 21 & 44 & 35 & 18 & 27 \\ 47 & 22 & 7 & 62 & 19 & 26 & 43 & 34 \end{pmatrix} \tag{4.4.9}$$

and note that

$$\mathbf{P}_3 - \mathbf{F}\mathbf{P}_3\mathbf{F} = 32 \begin{pmatrix} -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \end{pmatrix} = 32\mathbf{D}_3, \tag{4.4.10}$$

say. We note that the matrix $\mathbf{D}_3$ in (4.4.10) has a similar (but different) pattern to the matrix $\mathbf{D} = \mathbf{P} - \mathbf{FPF}$ in (4.4.8).

[29]Jelliss [280] notes that he learnt from Donald Knuth that Karl Wenzelides, 'polyhistorian', is listed in the *Biographisches Lexikon des Kaiserthums Oesterreich* [114] as born September 1770 in Troppau (now Opava in the Czech Republic), died 6 May 1852 in Nikolsburg (now Mikulov). He wrote poetry and music, besides works on the Bronze Age, etc; many of his books and letters were in the Troppauer Museum.

4.5. **The Euler algorithm.** Leonhard Euler (1707–1783) published two papers involving Graeco-Latin squares, a short paper [81, (1776)] and a long one [83, (1782)]; see also Klyve & Stemkoski [222]. In [81, (1776)], Euler showed that an $n \times n$ diagonal Graeco-Latin square with typical entry (a, b) can be turned into a classic fully-magic square by the following *Euler algorithm:*

$$\textit{replace the pair } (a, b) \textit{ with the number } n(a-1)+b, \tag{4.5.1}$$

or in matrix notation

$$\textit{replace the pair } (\mathbf{A}, \mathbf{B}) \textit{ with the matrix } \mathbf{M} = n(\mathbf{A}-\mathbf{E})+\mathbf{B}, \tag{4.5.2}$$

where $\mathbf{E}$ is the $n \times n$ matrix with each entry equal to 1. And then $\mathbf{M}$ defines a classic fully-magic square. We will call the matrices $\mathbf{A}$ and $\mathbf{B}$ *Euler basis matrices.*

The Ozanam–Grandin solution [250] to the Magic Card Puzzle is a *diagonal* Graeco-Latin square. With this notation, we have the Euler basis matrices:

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 2 & 3 \\ 3 & 2 & 4 & 1 \\ 4 & 1 & 3 & 2 \\ 2 & 3 & 1 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \end{pmatrix}, \tag{4.5.3}$$

and so

$$4(\mathbf{A}-\mathbf{E})+\mathbf{B} = \begin{pmatrix} 1 & 15 & 8 & 10 \\ 12 & 6 & 13 & 3 \\ 14 & 4 & 11 & 5 \\ 7 & 9 & 2 & 16 \end{pmatrix} = \mathbf{M}_1, \tag{4.5.4}$$

which defines a classic fully-magic square with magic sum 34. We will call $\mathbf{M}_1$ the *Ozanam–Grandin magic square*. It is interesting to note that here the Euler basis matrix $\mathbf{B} = \mathbf{A}'$, the transpose of $\mathbf{A}$.

We may compute the Euler basis matrices $\mathbf{A}$ and $\mathbf{B}$ from the $n \times n$ magic square matrix $\mathbf{M}$ by first forming $\mathbf{M} - \mathbf{E}$ and then expressing its elements to the base n. Specifically, we divide each element of $\mathbf{M} - \mathbf{E}$ by n and then $\mathbf{A} - \mathbf{E}$ contains the integer part and $\mathbf{B} - \mathbf{E}$ the remainder. For example, with the Ozanam–Grandin magic square, $n = 4$ and

$$\mathbf{M}_1 - \mathbf{E} = \begin{pmatrix} 0 & 14 & 7 & 9 \\ 11 & 5 & 12 & 2 \\ 13 & 3 & 10 & 4 \\ 6 & 8 & 1 & 15 \end{pmatrix}, \tag{4.5.5}$$

and so

$$\mathbf{A} - \mathbf{E} = \begin{pmatrix} 0 & 3 & 1 & 2 \\ 2 & 1 & 3 & 0 \\ 3 & 0 & 2 & 1 \\ 1 & 2 & 0 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{B} - \mathbf{E} = \begin{pmatrix} 0 & 2 & 3 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 1 & 3 \end{pmatrix}. \tag{4.5.6}$$

As pointed out by Chu [237] (see also Hodges [238, pp. 150–151]), it is not necessary that the orthogonal pair $(\mathbf{A}, \mathbf{B})$ define a diagonal Graeco-Latin square in order that $\mathbf{M} = 4(\mathbf{A} - \mathbf{E}) + \mathbf{B}$ be classic fully-magic. Neither of the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 4 & 4 \\ 4 & 4 & 1 & 1 \\ 3 & 2 & 3 & 2 \\ 2 & 3 & 2 & 3 \end{pmatrix} \quad \text{or} \quad \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 4 & 3 & 2 & 1 \end{pmatrix} \tag{4.5.7}$$

defines a Latin square, though $\mathbf{A}$ and $\mathbf{B}$ are orthogonal to each other. But here

$$4(\mathbf{A}-\mathbf{E})+\mathbf{B}=\begin{pmatrix}1 & 2 & 15 & 16\\ 13 & 14 & 3 & 4\\ 12 & 7 & 10 & 5\\ 8 & 11 & 6 & 9\end{pmatrix}=\mathbf{M}_2 \qquad (4.5.8)$$

is classic fully-magic! We will call $\mathbf{M}_2$ the *Chu magic square*.

Styan [240] (see also [250]), pointed out that of the 880 magic squares of order 4×4 precisely 144 yield Euler basis matrices $(\mathbf{A},\mathbf{B})$ which form a diagonal Graeco-Latin square. There are 144 solutions to the Magic Card Puzzle, excluding rotations and reflections, as observed by Dudeney [159, p. 216]. And every solution to the Magic Card Puzzle is a 4×4 diagonal Graeco-Latin square.

We will say that an $n\times n$ matrix is *row Latin* whenever the numbers $1,2,\ldots,n$ each occur precisely once in every one of the n rows, *column Latin* whenever the numbers $1,2,\ldots,n$ each occur precisely once in every one of the n columns, and *diagonal Latin* whenever the numbers $1,2,\ldots,n$ each occur precisely once in the two principal diagonals.

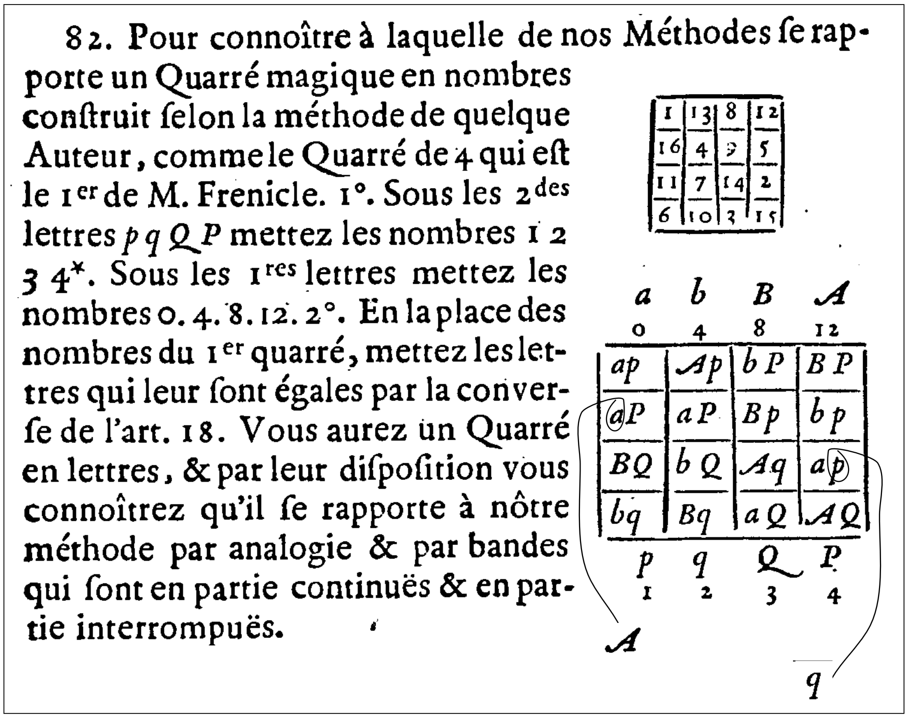
In the Chu magic square, the Euler basis matrices

$$\mathbf{A}=\begin{pmatrix}1 & 1 & 4 & 4\\ 4 & 4 & 1 & 1\\ 3 & 2 & 3 & 2\\ 2 & 3 & 2 & 3\end{pmatrix} \quad\text{and}\quad \mathbf{B}=\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\\ 4 & 3 & 2 & 1\end{pmatrix}. \qquad (4.5.9)$$

The matrix $\mathbf{A}$ here is column Latin but neither row nor diagonal Latin, while $\mathbf{B}$ is row Latin, but neither column nor diagonal Latin. Both $\mathbf{A}$ and $\mathbf{B}$ each contain the integers $1,2,3,4$ each four times.

We now present examples due to Joseph Sauveur (1653–1716), Murai Chūzen (1708–1797), and Walter William Rouse Ball (1850–1925).

4.5.1. *An example due to Joseph Sauveur (1653–1716).* The following example[30] given by Joseph Sauveur in 1709/1710 [78, p. 136, §82]:



82. Pour connoître à laquelle de nos Méthodes ſe rapporte un Quarré magique en nombres conſtruit ſelon la méthode de quelque Auteur, comme le Quarré de 4 qui eſt le 1er de M. Frenicle. 1°. Sous les 2des lettres *p q Q P* mettez les nombres 1 2 3 4*. Sous les 1res lettres mettez les nombres 0. 4. 8. 12. 2°. En la place des nombres du 1er quarré, mettez les lettres qui leur ſont égales par la converſe de l'art. 18. Vous aurez un Quarré en lettres, & par leur diſpoſition vous connoîtrez qu'il ſe rapporte à nôtre méthode par analogie & par bandes qui ſont en partie continuës & en partie interrompuës.

1	13	8	12
16	4	9	5
11	7	14	2
6	10	3	15

a	b	B	A
0	4	8	12
ap	Ap	bP	BP
aP	aP	Bp	bp
BQ	bQ	Aq	ap
bq	Bq	aQ	AQ
p	q	Q	P
1	2	3	4

FIGURE 4.5.2: Example from Sauveur [78, p. 136, §82] in 1709/1710.

With the coding as shown and with the two corrections, we find the Euler basis matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 2 & 3 \\ 4 & 1 & 3 & 2 \\ 3 & 2 & 4 & 1 \\ 2 & 3 & 1 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 1 & 4 & 4 \\ 4 & 4 & 1 & 1 \\ 3 & 3 & 2 & 2 \\ 2 & 2 & 3 & 3 \end{pmatrix}, \tag{4.5.10}$$

which are orthogonal to each other. The matrix **A** is row and column Latin but not diagonal Latin, and the matrix **B** is column and diagonal Latin but not row Latin. Moreover,

$$4(\mathbf{A} - \mathbf{E}) + \mathbf{B} = \begin{pmatrix} 1 & 13 & 8 & 12 \\ 16 & 4 & 9 & 5 \\ 11 & 7 & 14 & 2 \\ 6 & 10 & 3 & 15 \end{pmatrix} = \mathbf{J}_3 \tag{4.5.11}$$

defines the classic fully-magic square as given by Sauveur [78, p. 136, §82]. We will call $\mathbf{J}_3$ the *Sauveur magic matrix.*

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30 There seem to be 2 typos in the example as published, which we have corrected as shown in Fig. 4.5.2.

4.5.2. *An example due to Murai Chūzen (1708–1797).* A similar example is given by Mikami [130, p. 292], who refers to Murai Chūzen's *Sampō Dōshimon* [84] of 1781. This "was the first occasion [in Japan] of describing a general method for magic squares in a printed work, the writings of his predecessors, Takakazu Seki Kowa (1642–1708), Aoyama[31], Yoshisuke Matsunaga (1692–1744)[32], etc., being all recorded in manuscripts." [130, p. 291] "For even squares Murai gives only an example, where the arrangement as shown in the figure is indicated without any explanation." [130, p. 292] The figure given by Murai Chūzen as reported by Mikami [130, p. 292] corresponds to the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 4 & 1 & 1 & 4 \\ 3 & 2 & 2 & 3 \\ 2 & 3 & 3 & 2 \\ 1 & 4 & 4 & 1 \end{pmatrix}, \tag{4.5.12}$$

which are orthogonal to each other. The matrix **A** here is row and column Latin but not diagonal Latin, and the matrix **B** here is column and diagonal Latin but not row Latin, just like the example given by Sauveur [78, p. 136, §2] which we just considered. Moreover,

$$4(\mathbf{A} - \mathbf{E}) + \mathbf{B} = \begin{pmatrix} 4 & 5 & 9 & 16 \\ 7 & 14 & 2 & 11 \\ 10 & 3 & 15 & 6 \\ 13 & 12 & 8 & 1 \end{pmatrix} = \mathbf{M}_4 \tag{4.5.13}$$

defines a classic fully-magic square. We will call $\mathbf{M}_4$ the *Murai Chūzen magic matrix.*

4.5.3. *An example given by Walter William Rouse Ball (1850–1925).* Our last example is given by W. W. Rouse Ball (1911), starting (apparently) with the 5th edition of *Mathematical Recreations and Essays* [129, pp. 137, 161 (1911)], [145, pp. 137, 155 (1937)], [152, 163, **?**, pp. 193, 208]. In this example the Euler basis matrices

$$\mathbf{A} = \begin{pmatrix} 4 & 3 & 1 & 2 \\ 1 & 2 & 4 & 3 \\ 4 & 3 & 1 & 2 \\ 1 & 2 & 4 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 3 & 2 & 3 & 2 \\ 4 & 1 & 4 & 1 \\ 2 & 3 & 2 & 3 \\ 1 & 4 & 1 & 4 \end{pmatrix}. \tag{4.5.14}$$

The matrix **A** here is row and diagonal Latin but not column Latin, and **B** is column and diagonal Latin, but not row Latin. The matrices **A** and **B** are orthogonal to each other and the matrix

$$4(\mathbf{A} - \mathbf{E}) + \mathbf{B} = \begin{pmatrix} 15 & 10 & 3 & 6 \\ 4 & 5 & 16 & 9 \\ 14 & 11 & 2 & 7 \\ 1 & 8 & 13 & 12 \end{pmatrix} = \mathbf{M}_5 \tag{4.5.15}$$

defines a classic fully-magic square with magic sum 34. Moreover, $\mathbf{M}_5$ has the rhomboid property. We will call $\mathbf{M}_5$ the *Rouse Ball magic matrix.*

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[31] We have no information about this "Aoyama".

[32] Yoshisuke Matsunaga (1692–1744) was "a pupil of Seki's pupil" [193, p. 539].

4.5.4. *Bergholt's "semipandiagonal" magic matrices.* Bergholt [128] shows that a "semipandiagonal" 4×4 magic matrix can be expressed as the sum of two Latin squares, which Benson & Jacoby [165, p. 111] identify as

$$\mathbf{L}_1 = \begin{pmatrix} X & Y & Z & T \\ T & Z & Y & X \\ Y & X & T & Z \\ Z & T & X & Y \end{pmatrix}, \quad \mathbf{L}_2 = \begin{pmatrix} x & y & z & t \\ z & t & x & y \\ t & z & y & x \\ y & x & t & z \end{pmatrix}. \tag{4.5.16}$$

We find that these two Latin squares $\mathbf{L}_1$ and $\mathbf{L}_2$ are both magic matrices and that they are "orthogonal to each other" in that every (capital) letter in $\mathbf{L}_1$ is coupled uniquely with every (lower case) letter in $\mathbf{L}_2$, and so the pair $(\mathbf{L}_1, \mathbf{L}_2)$ form a Graeco–Latin square.

Benson & Jacoby [165, p. 111] note that the sum $\mathbf{B}_1 + \mathbf{B}_2$ generates all 432 semipandiagonal classic 4×4 magic squares including the 48 that are pandiagonal, and that these (Dudeney Type I) require

$$X + T = Y + Z \quad \text{and} \quad x + y = z + t. \tag{4.5.17}$$

It is easy to see that under the conditions (4.5.17) both $\mathbf{L}_1$ and $\mathbf{L}_2$ are also pandiagonal, $\mathbf{H}$-associated and 4-pac, and hence so is the sum $\mathbf{L}_1 + \mathbf{L}_2$.

4.5.5. *The Ollerenshaw magic matrix* **O**. We saw above using (4.1.2) that a 4×4 pandiagonal magic matrix is always 4-pac and **H**-associated. A 4×4 pandiagonal magic matrix is, however, not always EP. The "Ollerenshaw matrix" **O** (Figure TBC) is pandiagonal and 4-pac but not EP though both its Moore–Penrose inverse and group inverse are pandiagonal.

If in (4.5.16) we choose $X = 0,\ Y = 4,\ Z = 1,\ T = 5;\ \ x = 0,\ y = 10,\ z = 2,\ t = 8,$ then

$$\mathbf{L}_1 + \mathbf{L}_2 = \begin{pmatrix} 0 & 14 & 3 & 13 \\ 7 & 9 & 4 & 10 \\ 12 & 2 & 15 & 1 \\ 11 & 5 & 8 & 6 \end{pmatrix} = \mathbf{O}, \tag{4.5.18}$$

say. The magic matrix **O** defines the "Ollerenshaw magic square" given in Figure 4.5.1. The "most-perfect pandiagonal" magic matrix $\mathbf{P}_0$ given by Trenkler & Trenkler [202] is **O** with 1 added to each element. The Ollerenshaw magic matrix **O** is 4-pac, **H**-associated and pandiagonal; moreover it has index 1 but it is not EP and not rhomboidal.

FIGURE 4.5.1: Dame Kathleen Ollerenshaw (b. 1912) and the "Ollerenshaw magic square".

We may classify our examples as follows:

(1) $\mathbf{M}_1$ (Ozanam–Grandin) has rank 3, index 1 (key $\kappa = 80$), is **F**-associated, not rhomboidal and not EP,

(2) $\mathbf{M}_2$ (Chu) has rank 3, index 1 (key $\kappa = 40$), is not **V**-associated, not rhomboidal and not EP,

(3) $\mathbf{J}_3$ (Sauveur) has rank 3, index 1 (key $\kappa = 128$), is not **V**-associated, not rhomboidal and not EP,

(4) $\mathbf{M}_4$ (Murai Chūzen) has rank 3, index 1 (key $\kappa = 160$), is not **V**-associated, not rhomboidal and not EP,

(5) $\mathbf{M}_5$ (Rouse Ball) has rank 3, index 1 (key $\kappa = 48$), is **H**-associated, rhomboidal but not EP.

(6) **O** (Ollerenshaw) has rank 3, index 1 (key $\kappa = 64$), is **H**-associated, not rhomboidal and not EP.

We recall our classification above of the 5 magic matrices defined in Shortrede's "postscript":

(1) **G** (4.3.1) has rank 3, index 1 (key $\kappa = 80$), is **H**-associated, rhomboidal and EP,

(2) $\mathbf{G}_2$ has rank 3, index 3 (key $\kappa = 0$), is **H**-associated, rhomboidal but not EP (index $\neq 1$),

(3) $\mathbf{G}_3$ has rank 3, index 1 (key $\kappa = 80$), is **H**-associated, EP but not rhomboidal,

(4) $\mathbf{G}_4$ has rank 3, index 1 (key $\kappa = 80$), is **H**-associated, rhomboidal but not EP,

(5) $\mathbf{G}_5$ has rank 3, index 3 (key $\kappa = 0$), is **H**-associated, not rhomboidal and not EP (index $\neq 1$)

We recall that a 4×4 **H**-associated magic matrix is 4-pac and pandiagonal.

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5. Four 16×16 CSP2-magic matrices

Planck (1916, p. 469, Fig. 10) presents a 16×16 magic square and observes that it has 64 CSP2-magic paths (our Figure 5.1). We will denote this magic square by the matrix $\mathbf{X}_1$ and find that, in addition, to it being CSP2-magic it is also 4-pac and hence pandiagonal with rank 3, and is keyed with index 1. Moreover, $\mathbf{X}_1$ is $\mathbf{V}$-associated with $\mathbf{V} = \mathbf{I}_4 \otimes \mathbf{H}_4$, implying that the top left, top right, bottom left, and bottom right 4×4 submatrices are all $\mathbf{H}$-associated and hence pandiagonal.

In 1916, the Editor (Paul Carus) of *The Monist* reports in [134, vol. 26, pp. 315–316] that Frederic A. Woodruff[33] "has sent three original [classic] magic squares, one each of orders $8, 12$ and 16. Neither of the two smaller squares is CSP2-magic but the 16×16 (our Figure 5.2), which we denote by $\mathbf{X}_2$, is CSP2-magic. Moreover the matrix $\mathbf{X}_2$ is also 4-pac with rank 3, and is keyed with index 1, and is $\mathbf{F}$-associated. In addition, we find that $\mathbf{X}_1$ and $\mathbf{X}_2$ are EP.

OPEN QUESTION 5.1. Does there exist an $\mathbf{F}$-associated 8×8 magic matrix that is pandiagonal and CSP2-magic? We believe that all our 46080 Caïssan beauties are $\mathbf{H}$-associated: a classic $\mathbf{H}$-associated magic matrix cannot also be $\mathbf{F}$-associated. As we just noted above the 16×16 matrix $\mathbf{X}_2$ is pandiagonal and CSP2-magic, as is the 16×16 matrix $\mathbf{X}_3$ due to Woodruff (1917, p. 397, Fig. 725; our Figure 5.3 below).

In a "follow-up" paper, Woodruff [137, p. 397, (1917)] observes that the 16×16 magic square given in his Fig. 725 (our Figure 5.3), which we denote by $\mathbf{X}_3$ can readily be changed into a 4-pac, pandiagonal "balanced, quartered", and "Franklin" magic square by one transposition, as shown in his Fig. 730 (our Figure 5.4), our matrix $\mathbf{X}_4$. By this change it ceases to be $\mathbf{F}$-associated and CSP2-magic, but acquires other "ornate" features besides becoming a "Franklin" square. From these comments by Woodruff [137, p. 397, (1917)] we conclude, and indeed confirm, that $\mathbf{X}_3$ is CSP2-magic and that $\mathbf{X}_4$ is not.

We define a Franklin square as follows:

DEFINITION 5.1. We define an $n \times n$ magic matrix with magic sum m, usually classic and often 8×8, to be a "Franklin square" and to have the "Franklin property" whenever each of the $4n$ "Franklin-bent diagonals" (with wrap-around) are magic with sum m. The "Franklin-bent diagonals" are paths of n numbers with any of the following shapes and orientations:

```
# - - - - - - -    # - - - - - - #    - - - - - - - -    - - - - - - - #
- # - - - - - -    - # - - - - # -    - - - - - - - -    - - - - - - # -
- - # - - - - -    - - # - - # - -    - - - - - - - -    - - - - - # - -
- - - # - - - -    - - - # # - - -    - - - - - - - -    - - - - # - - -
- - - # - - - -    - - - - - - - -    - - - # # - - -    - - - - # - - -
- - # - - - - -    - - - - - - - -    - - # - - # - -    - - - - - # - -
- # - - - - - -    - - - - - - - -    - # - - - - # -    - - - - - - # -
# - - - - - - -    - - - - - - - -    # - - - - - - #    - - - - - - - #
```

[33] We know very little about Frederic A. Woodruff, but we believe he was born in 1855.

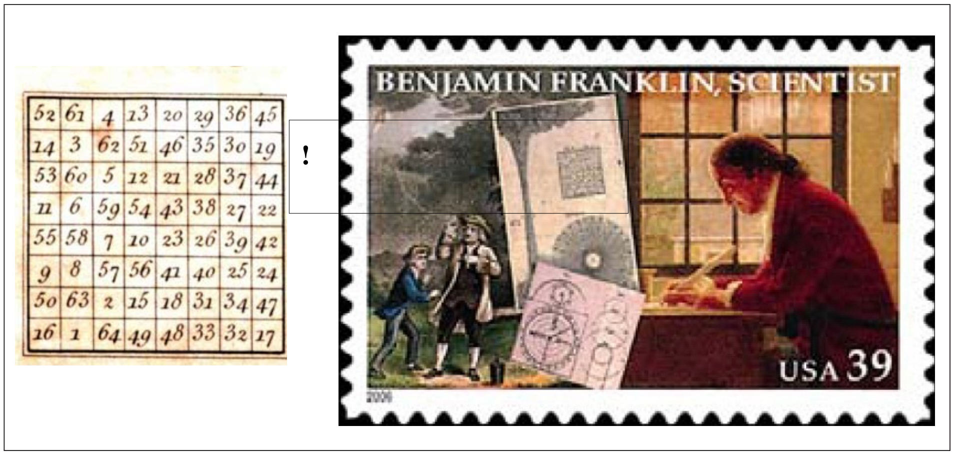


FIGURE 5.0: (top panel) Franklin bent dagonals;
(bottom panel) Franklin magic square and Benjamin Franklin: USA 2006, *Scott* TBC.

We assume that the meaning intended by Woodruff [137, p. 397, (1917)] for the term "quartered" means that (at least) all the four corner 8×8 submatrices of a 16×16 magic matrix are magic, in that the numbers in all the rows, columns and two main diagonals add up to the same magic sum, which is then necessarily half that of the magic sum for the parent 16×16 magic matrix. Andrews [135, p. 175, (1917)] uses the term "quartered" in connection with the semi-magic 8×8 Beverley matrix $\mathbf{B}$ (his Fig. 281, our Figure TBC), which we discuss below in Section TBC, and where the four corner 4×4 submatrices are all semi-magic.

We do not know the precise meaning intended by Woodruff [137] for the term "balanced" but suggest that in connection with "quartered" it means that any special ("ornate") properties that any one of the four corner 8×8 submatrices of a 16×16 magic matrix may have also hold for the other three.

Andrews [135, ch. XV, pp. 376–414, (1917)] presents 5 articles with the heading "Ornate magic squares" and from these we infer that "ornate magic squares" are "special' magic squares" like those that are

(1) pandiagonal,
(2) CSP2-magic,
(3) CSP3- magic,
(4) 4-pac,
(5) $\mathbf{V}$-associated for some involutory matrix $\mathbf{V}$, and/or have
(6) the "Franklin property", i.e., $4n$ magic "Franklin-bent diagonals".

Woodruff [137, p. 397] noted that $\mathbf{X}_4$ "contains 9 magic subsquares of order 8×8, each of which is pandiagonal". We observe, in addition, that each of these 9 magic submatrices is 4-pac and hence pandiagonal with rank 3. Moreover, each 8×8 submatrix is $\mathbf{V}$-associated with $\mathbf{V} = \mathbf{I}_4 \otimes \mathbf{F}_2$.

Following the properties given in *The Edinburgh Encyclopædia* [89, (1830)] of a 16×16 magic square (which is not explictly given there), we find that in $\mathbf{X}_4$ there are also 16 magic subsquares of order 4×4 and 4 magic subsquares of order 12×12, and that these 20 magic subsquares are also all 4-pac, and hence pandiagonal with rank 3, and are $\mathbf{V}$-associated with $\mathbf{V} = \mathbf{I}_{n/2} \otimes \mathbf{F}_2$ and, respectively, $n = 4, 12$.

Commenting on the article by Woodruff [137], Andrews [135, p. 404] says that

> Woodruff [137, p. 397] presents a remarkable magic[34] such a magic" [his Fig. 725 (our Figure 5.3, matrix $\mathbf{X}_3$)] of order 16 which is pandiagonal, CSP2-magic, 4-pac, and $\mathbf{F}$-associated, a combination of ornate properties which has probably never been accomplished before in this order of square, and it is constructed moreover by a unique method of his own devising.

We confirm that the magic matrix $\mathbf{X}_3$ [Woodruff's Fig. 725 (our Figure 5.3)] is

(1) pandiagonal and CSP2-magic,

(2) 4-pac,

(3) $\mathbf{F}$-associated, and has

(4) rank 3 and index 1, but not EP,

and that $\mathbf{X}_4$ [Woodruff's Fig. 730 (our Figure 5.4)] has 29 magic subsquares of orders $4, 8$ or 12 and that all these 30 magic matrices (including $\mathbf{X}_4$) are

(1) pandiagonal but *not* CSP2-magic,

(2) 4-pac,

(3) $\mathbf{V}$-associated with $\mathbf{V} = \mathbf{I}_q \otimes \mathbf{F}_2$, and so each 2×2 "block" is $\mathbf{F}$-associated, but the full 16×16 matrix $\mathbf{X}_4$ is not $\mathbf{F}$-associated; here $q = 2, 4, 6, 8$, half the order of the magic matrix involved: $4, 8, 12, 16$, and have

(4) rank 3 and index 1, but not EP, and have

(5) the "Franklin property", i.e., $4n$ magic "Franklin-bent diagonals".—TBC

[34] In the literature of the early 20th (or late 19th) century, we have often found "a magic square" to be referred to as just "a magic".

1	32	241	240	193	224	49	48	177	176	65	96	113	112	129	160
242	239	2	31	50	47	194	223	66	95	178	175	130	159	114	111
16	17	256	225	208	209	64	33	192	161	80	81	128	97	144	145
255	226	15	18	63	34	207	210	79	82	191	162	143	146	127	98
13	20	253	228	205	212	61	36	189	164	77	84	125	100	141	148
254	227	14	19	62	35	206	211	78	83	190	163	142	147	126	99
4	29	244	237	196	221	52	45	180	173	68	93	116	109	132	157
243	238	3	30	51	46	195	222	67	94	179	174	131	158	115	110
12	21	252	229	204	213	60	37	188	165	76	85	124	101	140	149
251	230	11	22	59	38	203	214	75	86	187	166	139	150	123	102
5	28	245	236	197	220	53	44	181	172	69	92	117	108	133	156
246	235	6	27	54	43	198	219	70	91	182	171	134	155	118	107
8	25	248	233	200	217	56	41	184	169	72	89	120	105	136	153
247	234	7	26	55	42	199	218	71	90	183	170	135	154	119	106
9	24	249	232	201	216	57	40	185	168	73	88	121	104	137	152
250	231	10	23	58	39	202	215	74	87	186	167	138	151	122	103

FIGURE 5.1: 16 × 16 magic square from Planck [44, p. 469, Fig. 10], Planck matrix $\mathbf{X}_1$.

1	224	49	240	145	80	161	128	113	176	65	160	225	64	209	16
254	35	206	19	110	179	94	131	142	83	190	99	30	195	46	243
4	221	52	237	148	77	164	125	116	173	68	157	228	61	212	13
255	34	207	18	111	178	95	130	143	82	191	98	31	194	47	242
10	215	58	231	154	71	170	119	122	167	74	151	234	55	218	7
245	44	197	28	101	188	85	140	133	92	181	108	21	204	37	252
11	214	59	230	155	70	171	118	123	166	75	150	235	54	219	6
248	41	200	25	104	185	88	137	136	89	184	105	24	201	40	249
8	217	56	233	152	73	168	121	120	169	72	153	232	57	216	9
251	38	203	22	107	182	91	134	139	86	187	102	27	198	43	246
5	220	53	236	149	76	165	124	117	172	69	156	229	60	213	12
250	39	202	23	106	183	90	135	138	87	186	103	26	199	42	247
15	210	63	226	159	66	175	114	127	162	79	146	239	50	223	2
244	45	196	29	100	189	84	141	132	93	180	109	20	205	36	253
14	211	62	227	158	67	174	115	126	163	78	147	238	51	222	3
241	48	193	32	97	192	81	144	129	96	177	112	17	208	33	256

FIGURE 5.2: 16 × 16 magic square from Woodruff [134, p. 316, Fig. 3], Woodruff matrix $\mathbf{X}_2$.

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
1	1	1	192	97	224	41	152	73	248	113	208	17	176	89	232	57	136	16
2	9	128	193	32	161	88	233	56	137	16	177	112	209	40	153	72	249	8
3	13	133	60	229	92	173	20	205	116	245	76	149	44	221	100	189	4	4
4	5	252	69	156	37	212	109	180	13	140	53	236	85	164	29	196	125	12
5	6	6	187	102	219	46	147	78	243	118	203	22	171	94	227	62	131	11
6	14	123	198	27	166	83	238	51	142	11	182	107	214	35	158	67	254	3
7	10	130	63	226	95	170	23	202	119	242	79	146	47	218	103	186	7	7
8	2	255	66	159	34	215	106	183	10	143	50	239	82	167	26	199	122	15
9	15	135	58	231	90	175	18	207	114	247	74	151	42	223	98	191	2	2
10	7	250	71	154	39	210	111	178	15	138	55	234	87	162	31	194	127	10
11	3	3	190	99	222	43	150	75	246	115	206	19	174	91	230	59	134	14
12	11	126	195	30	163	86	235	54	139	14	179	110	211	38	155	70	251	6
13	12	132	61	228	93	172	21	204	117	244	77	148	45	220	101	188	5	5
14	4	253	68	157	36	213	108	181	12	141	52	237	84	165	28	197	124	13
15	8	8	185	104	217	48	145	80	241	120	201	24	169	96	225	64	129	9
16	16	121	200	25	168	81	240	49	144	9	184	105	216	33	160	65	256	1

FIGURE 5.3: Fig. 725 from Woodruff [137, p. 397], Woodruff matrix $\mathbf{X}_3$.

	1	16	13	4	5	12	9	8	7	10	11	6	3	14	15	2
1	1	136	89	224	41	176	113	248	73	208	17	152	97	232	57	192
16	121	256	33	168	81	216	9	144	49	184	105	240	25	160	65	200
13	132	5	220	93	172	45	244	117	204	77	148	21	228	101	188	61
4	252	125	164	37	212	85	140	13	180	53	236	109	156	29	196	69
5	6	131	94	219	46	171	118	243	78	203	22	147	102	227	62	187
12	126	251	38	163	86	211	14	139	54	179	110	235	30	155	70	195
9	135	2	223	90	175	42	247	114	207	74	151	18	231	98	191	58
8	255	122	167	34	215	82	143	10	183	50	239	106	159	26	199	66
7	130	7	218	95	170	47	242	119	202	79	146	23	226	103	186	63
10	250	127	162	39	210	87	138	15	178	55	234	111	154	31	194	71
11	3	134	91	222	43	174	115	246	75	206	19	150	99	230	59	190
6	123	254	35	166	83	214	11	142	51	182	107	238	27	158	67	198
3	133	4	221	92	173	44	245	116	205	76	149	20	229	100	189	60
14	253	124	165	36	213	84	141	12	181	52	237	108	157	28	197	68
15	8	129	96	217	48	169	120	241	80	201	24	145	104	225	64	185
2	128	249	40	161	88	209	16	137	56	177	112	233	32	153	72	193

FIGURE 5.4: Fig. 730 from Woodruff [137, p. 397], Woodruff matrix $\mathbf{X}_4$.

6. The n-queens problem and 11×11 Caïssan magic squares

The n-queens problem concerns the placement of n queens on an $n \times n$ chessboard so that no two are *en prise,* i.e., no two attack one another, and the maximum number is 8 for the usual 8×8 chessboard [**?**]. This problem was apparently first considered for 8 queens on an 8×8 chessboard by [93] and answered by [95]. The numbers of different ways that n queens can be so arranged on an $n \times n$ chessboard are given in Sloane's sequence A000170 [262]:

$$1, 0, 0, 2, 10, 4, 40, 92, 352, 724, 2680, \ldots$$

and so there are 92 different ways for $n = 8$ queens to be so placed on an 8×8 chessboard and 2680 different ways for $n = 11$ queens on an 11×11 chessboard.

6.1. The 11×11 Planck matrix P.

An 11×11 magic square was presented by [37, p. 97, Fig. I] giving 11 completely disjoint solutions for the 11-queens problem on an 11×11 chessboard. One of these solutions is to place the 11 queens in the square shown in Fig. 6.1 in positions $1, 2, \ldots, 11$ (red boxes), a second solution places the 11 queens in positions $12, 13, \ldots, 22$, and so on, providing 11 completely disjoint solutions in all.

Moreover Planck [37] noted that 11×11 is the smallest such arrangement that is, in addition, a "Caïssan magic square", a pandiagonal magic square in which all knight's paths in the same direction (with wraparound) are magic (magic sum 671). We call such paths "magic knight's paths".

We have not found any other such 11×11 Caïssan magic square which has embedded therein 11 completely disjoint solutions to the 11-queens problem. And none at all for $n > 11$.

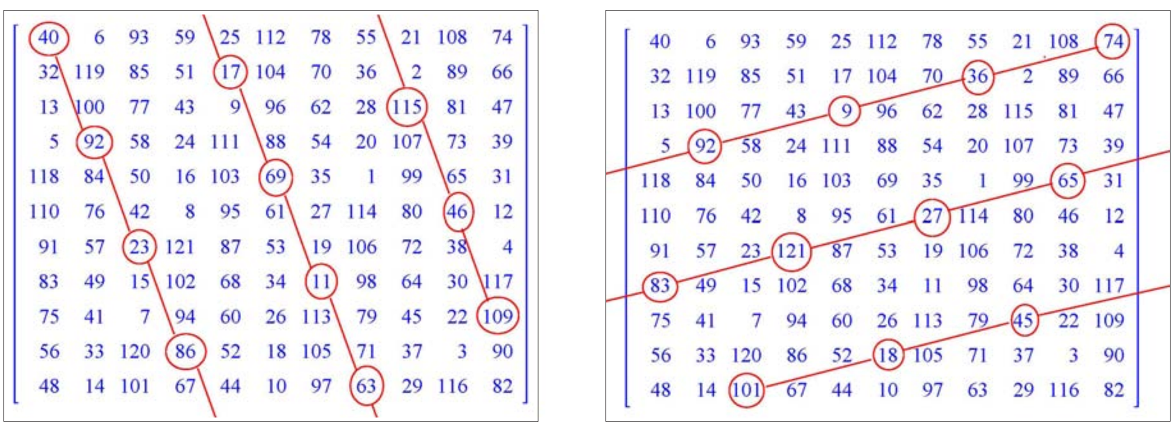


Figure 6.1: Special-Caïssan 11×11 magic square from [37, p. 97, Fig. I], with two special-knight's (CSP3) move magic paths indicated by red circles (left panel) and red boxes (right panel).

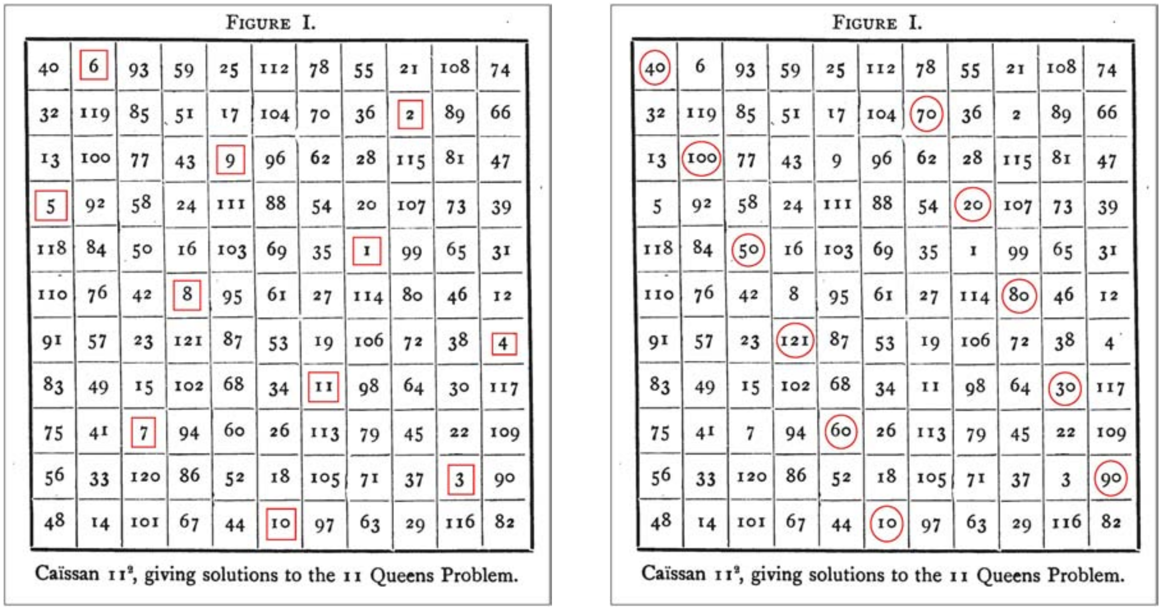


FIGURE 6.1.2: Special-Caïssan 11×11 magic square from [37, p. 97, Fig. I], with 11 solutions to the 11-queens problem, one solution indicated with red boxes (left panel); and a magic knight's path (CSP2) with red circles (right panel).

We denote the magic square in Fig. 6.1.2 by the Planck matrix

$$\mathbf{P} = \begin{pmatrix} 40 & 6 & 93 & 59 & 25 & 112 & 78 & 55 & 21 & 108 & 74 \\ 32 & 119 & 85 & 51 & 17 & 104 & 70 & 36 & 2 & 89 & 66 \\ 13 & 100 & 77 & 43 & 9 & 96 & 62 & 28 & 115 & 81 & 47 \\ 5 & 92 & 58 & 24 & 111 & 88 & 54 & 20 & 107 & 73 & 39 \\ 118 & 84 & 50 & 16 & 103 & 69 & 35 & 1 & 99 & 65 & 31 \\ 110 & 76 & 42 & 8 & 95 & 61 & 27 & 114 & 80 & 46 & 12 \\ 91 & 57 & 23 & 121 & 87 & 53 & 19 & 106 & 72 & 38 & 4 \\ 83 & 49 & 15 & 102 & 68 & 34 & 11 & 98 & 64 & 30 & 117 \\ 75 & 41 & 7 & 94 & 60 & 26 & 113 & 79 & 45 & 22 & 109 \\ 56 & 33 & 120 & 86 & 52 & 18 & 105 & 71 & 37 & 3 & 90 \\ 48 & 14 & 101 & 67 & 44 & 10 & 97 & 63 & 29 & 116 & 82 \end{pmatrix}, \qquad (6.1.1)$$

which we will call the 11×11 Planck-CMM, and which is nonsingular, and $\mathbf{F}$-associated in that all the elements of the matrix $\mathbf{P} + \mathbf{FPF}$ are equal, where $\mathbf{F}$ here is the 11×11 flip matrix. It follows that $\mathbf{P}^{-1}$ is also $\mathbf{F}$-associated and hence magic with magic sum $\mathrm{tr}\mathbf{P}^{-1} = \mathrm{tr}\mathbf{F}\mathbf{P}^{-1} = 1/\mathrm{tr}\mathbf{P} = 1/671$. As noted by Planck [37] the matrix $\mathbf{P}$ is pandiagonal but we find, however, that the inverse $\mathbf{P}^{-1}$ is not pandiagonal!

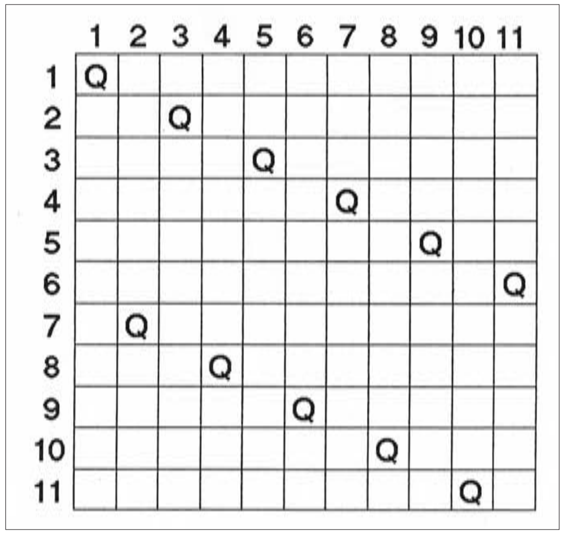


FIGURE 6.1.3: Knight's-path (CSP2) solution to the 11-queens problem on an 11×11 chessboard [183, Fig. 1].

A simple knight's-path (CSP2) solution to the 11-queens problem on an 11×11 chessboard was given [183, Fig. 1], see our Figure 6.3. We represent this solution by the simple knight's path (CSP2) selection matrix $\mathbf{K}$

$$\mathbf{K} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1\\ 0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0 \end{pmatrix} \tag{6.1.2}$$

We find that the 11×11 Planck magic matrix $\mathbf{P}$ is Caïssan since both $\mathbf{KP}$ and $\mathbf{PK}'$ are pandiagonal:

$$\mathbf{KP} = \begin{pmatrix} 40 & 6 & 93 & 59 & 25 & 112 & 78 & 55 & 21 & 108 & 74 \\ 13 & 100 & 77 & 43 & 9 & 96 & 62 & 28 & 115 & 81 & 47 \\ 118 & 84 & 50 & 16 & 103 & 69 & 35 & 1 & 99 & 65 & 31 \\ 91 & 57 & 23 & 121 & 87 & 53 & 19 & 106 & 72 & 38 & 4 \\ 75 & 41 & 7 & 94 & 60 & 26 & 113 & 79 & 45 & 22 & 109 \\ 48 & 14 & 101 & 67 & 44 & 10 & 97 & 63 & 29 & 116 & 82 \\ 32 & 119 & 85 & 51 & 17 & 104 & 70 & 36 & 2 & 89 & 66 \\ 5 & 92 & 58 & 24 & 111 & 88 & 54 & 20 & 107 & 73 & 39 \\ 110 & 76 & 42 & 8 & 95 & 61 & 27 & 114 & 80 & 46 & 12 \\ 83 & 49 & 15 & 102 & 68 & 34 & 11 & 98 & 64 & 30 & 117 \\ 56 & 33 & 120 & 86 & 52 & 18 & 105 & 71 & 37 & 3 & 90 \end{pmatrix}, \tag{6.1.3}$$

$$\mathbf{PK}' = \begin{pmatrix} 40 & 93 & 25 & 78 & 21 & 74 & 6 & 59 & 112 & 55 & 108 \\ 32 & 85 & 17 & 70 & 2 & 66 & 119 & 51 & 104 & 36 & 89 \\ 13 & 77 & 9 & 62 & 115 & 47 & 100 & 43 & 96 & 28 & 81 \\ 5 & 58 & 111 & 54 & 107 & 39 & 92 & 24 & 88 & 20 & 73 \\ 118 & 50 & 103 & 35 & 99 & 31 & 84 & 16 & 69 & 1 & 65 \\ 110 & 42 & 95 & 27 & 80 & 12 & 76 & 8 & 61 & 114 & 46 \\ 91 & 23 & 87 & 19 & 72 & 4 & 57 & 121 & 53 & 106 & 38 \\ 83 & 15 & 68 & 11 & 64 & 117 & 49 & 102 & 34 & 98 & 30 \\ 75 & 7 & 60 & 113 & 45 & 109 & 41 & 94 & 26 & 79 & 22 \\ 56 & 120 & 52 & 105 & 37 & 90 & 33 & 86 & 18 & 71 & 3 \\ 48 & 101 & 44 & 97 & 29 & 82 & 14 & 67 & 10 & 63 & 116 \end{pmatrix}. \tag{6.1.4}$$

We note that the magic path marked with red circles in Fig. 6.1.2 (right panel) is the main diagonal of the matrix $\mathbf{KP}$ in (6.1.3).

6.2. **The 11×11 La Loubère–Demirörs matrix L.** We now consider an 11×11 classic magic square constructed by the method of La Loubère[35] (Fig. 6.2.1, left panel) and (right panel) an 11×11 magic Latin square with solutions to the 11-queens problem constructed therefrom by [183, Fig. 2, 4], see our Figure 6.2.2.

68	81	94	107	120	1	14	27	40	53	66
80	93	106	119	11	13	26	39	52	65	67
92	105	118	10	12	25	38	51	64	77	79
104	117	9	22	24	37	50	63	76	78	91
116	8	21	23	36	49	62	75	88	90	103
7	20	33	35	48	61	74	87	89	102	115
19	32	34	47	60	73	86	99	101	114	6
31	44	46	59	72	85	98	100	113	5	18
43	45	58	71	84	97	110	112	4	17	30
55	57	70	83	96	109	111	3	16	29	42
56	69	72	95	108	121	2	15	28	41	54

2	4	6	8	10	1	3	5	7	9	11
3	5	7	9	11	2	4	6	8	10	1
4	6	8	10	1	3	5	7	9	11	2
5	7	9	11	2	4	6	8	10	1	3
6	8	10	1	3	5	7	9	11	2	4
7	9	11	2	4	6	8	10	1	3	5
8	10	1	3	5	7	9	11	2	4	6
9	11	2	4	6	8	10	1	3	5	7
10	1	3	5	7	9	11	2	4	6	8
11	2	4	6	8	10	1	3	5	7	9
1	3	5	7	9	11	2	4	6	8	10

FIGURE 6.2.1: (left panel) 11×11 classic magic square and (right panel) an 11×11 magic Latin square with solutions to the 11-queens problem [183, Fig. 2, 4].

We denote the 11×11 classic magic square (Fig. 6.2.1, left panel) by the matrix

$$\mathbf{L} = \begin{pmatrix} 68 & 81 & 94 & 107 & 120 & 1 & 14 & 27 & 40 & 53 & 66 \\ 80 & 93 & 106 & 119 & 11 & 13 & 26 & 39 & 52 & 65 & 67 \\ 92 & 105 & 118 & 10 & 12 & 25 & 38 & 51 & 64 & 77 & 79 \\ 104 & 117 & 9 & 22 & 24 & 37 & 50 & 63 & 76 & 78 & 91 \\ 116 & 8 & 21 & 23 & 36 & 49 & 62 & 75 & 88 & 90 & 103 \\ 7 & 20 & 33 & 35 & 48 & 61 & 74 & 87 & 89 & 102 & 115 \\ 19 & 32 & 34 & 47 & 60 & 73 & 86 & 99 & 101 & 114 & 6 \\ 31 & 44 & 46 & 59 & 72 & 85 & 98 & 100 & 113 & 5 & 18 \\ 43 & 45 & 58 & 71 & 84 & 97 & 110 & 112 & 4 & 17 & 30 \\ 55 & 57 & 70 & 83 & 96 & 109 & 111 & 3 & 16 & 29 & 42 \\ 56 & 69 & \mathbf{82} & 95 & 108 & 121 & 2 & 15 & 28 & 41 & 54 \end{pmatrix} \qquad (6.2.1)$$

with the element in the $(11, 3)$ position corrected to 82 from 72.

[35]Simon de La Loubère (1642–1729), a French diplomat, writer, mathematician and poet. De la Loubère brought to France from his Siamese travels a very simple method for creating n-odd magic squares, known as the "Siamese method" or the "de La Loubère method", which apparently was initially brought from Surat, India by a *médecin provençal* by the name of M. Vincent. [324]

We find that $\mathbf{L}$ is nonsingular, and $\mathbf{F}$-associated and so $\mathbf{L}^{-1}$ is also $\mathbf{F}$-associated and hence magic. Moreover, $\mathbf{L}$ is pandiagonal and in addition we find that the inverse $\mathbf{L}^{-1}$ is pandiagonal! Furthermore we find that $\mathbf{L}$ is Caïssan in that both $\mathbf{KL}$ and $\mathbf{LK}'$ are pandiagonal, but that $\mathbf{L}^{-1}$ is not Caïssan!

We denote the 11×11 magic Latin square in Fig. 6.2.1 (right panel) by the matrix

$$\mathbf{Q}_3 = \begin{pmatrix} 2 & 4 & 6 & 8 & 10 & 1 & 3 & 5 & 7 & 9 & 11 \\ 3 & 5 & 7 & 9 & 11 & 2 & 4 & 6 & 8 & 10 & 1 \\ 4 & 6 & 8 & 10 & 1 & 3 & 5 & 7 & 9 & 11 & 2 \\ 5 & 7 & 9 & 11 & 2 & 4 & 6 & 8 & 10 & 1 & 3 \\ 6 & 8 & 10 & 1 & 3 & 5 & 7 & 9 & 11 & 2 & 4 \\ 7 & 9 & 11 & 2 & 4 & 6 & 8 & 10 & 1 & 3 & 5 \\ 8 & 10 & 1 & 3 & 5 & 7 & 9 & 11 & 2 & 4 & 6 \\ 9 & 11 & 2 & 4 & 6 & 8 & 10 & 1 & 3 & 5 & 7 \\ 10 & 1 & 3 & 5 & 7 & 9 & 11 & 2 & 4 & 6 & 8 \\ 11 & 2 & 4 & 6 & 8 & 10 & 1 & 3 & 5 & 7 & 9 \\ 1 & 3 & 5 & 7 & 9 & 11 & 2 & 4 & 6 & 8 & 10 \end{pmatrix}, \tag{6.2.2}$$

which we find is also nonsingular, Caïssan and $\mathbf{F}$-associated. And so the inverse $\mathbf{Q}_3^{-1}$

$$\mathbf{Q}_3^{-1} = \frac{1}{726} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 67 & -65 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 67 & -65 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 67 & -65 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 67 & -65 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 67 & -65 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -65 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 67 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 67 & -65 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 67 & -65 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 67 & -65 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 67 & -65 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 67 & -65 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \tag{6.2.3}$$

is also $\mathbf{F}$-associated and hence magic. Moreover we find that $\mathbf{Q}_3^{-1}$ is Caïssan!

We note that

$$\mathbf{KQ}_3^{-1} = \frac{1}{726}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 67 & -65 \\ 1 & 1 & 1 & 1 & 1 & 67 & -65 & 1 & 1 & 1 & 1 \\ 1 & 67 & -65 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 67 & -65 & 1 \\ 1 & 1 & 1 & 1 & 67 & -65 & 1 & 1 & 1 & 1 & 1 \\ 67 & -65 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 67 & -65 & 1 & 1 \\ 1 & 1 & 1 & 67 & -65 & 1 & 1 & 1 & 1 & 1 & 1 \\ -65 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 67 \\ 1 & 1 & 1 & 1 & 1 & 1 & 67 & -65 & 1 & 1 & 1 \\ 1 & 1 & 67 & -65 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{6.2.4}$$

and

$$\mathbf{Q}_3^{-1}\mathbf{K}' = \frac{1}{726}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & -65 & 1 & 1 & 1 & 1 & 67 \\ 1 & 1 & 1 & 1 & -65 & 1 & 1 & 1 & 1 & 67 & 1 \\ 1 & 1 & 1 & -65 & 1 & 1 & 1 & 1 & 67 & 1 & 1 \\ 1 & 1 & -65 & 1 & 1 & 1 & 1 & 67 & 1 & 1 & 1 \\ 1 & -65 & 1 & 1 & 1 & 1 & 67 & 1 & 1 & 1 & 1 \\ -65 & 1 & 1 & 1 & 1 & 67 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 67 & 1 & 1 & 1 & 1 & 1 & -65 \\ 1 & 1 & 1 & 67 & 1 & 1 & 1 & 1 & 1 & -65 & 1 \\ 1 & 1 & 67 & 1 & 1 & 1 & 1 & 1 & -65 & 1 & 1 \\ 1 & 67 & 1 & 1 & 1 & 1 & 1 & -65 & 1 & 1 & 1 \\ 67 & 1 & 1 & 1 & 1 & 1 & -65 & 1 & 1 & 1 & 1 \end{pmatrix}. \tag{6.2.5}$$

are both pandiagonal.

7. The 15 × 15 Ursus Caïssan-magic square

Ursus [7, (1881)] gives a 15 × 15 special-Caïssan magic square constructed from the "seed matrix"

$$\mathbf{D} = \begin{pmatrix} 8 & 0 & 13 \\ 5 & 12 & 4 \\ 11 & 7 & 3 \\ 10 & 2 & 9 \\ 1 & 14 & 6 \end{pmatrix}, \tag{7.1}$$

which is "semi-magic" in that the 5 row-totals are all equal to 21 and the 3 column totals are all equal to 35, with $5 \times 21 = 3 \times 35 = 105$. Moreover, if we delete rows 2 and 4 then we obtain

$$\mathbf{D}^* = \begin{pmatrix} 8 & 0 & 13 \\ 11 & 7 & 3 \\ 1 & 14 & 6 \end{pmatrix}, \tag{7.2}$$

which is magic, with the numbers in all rows, columns and the two main diagonals adding to 21.

Let $\mathbf{E}_{p,q}$ denote the $p \times q$ matrix with every entry equal to 1. Then the 15 × 15 matrix

$$\mathbf{U}_{15} = 15\mathbf{E}_{3,5} \otimes \mathbf{X} + \mathbf{E}_{5,3} \otimes \mathbf{X}' + \mathbf{E}_{15,15}$$

$$= \begin{pmatrix}
129 & 6 & 207 & 131 & 2 & 204 & 126 & 12 & 206 & 122 & 9 & 201 & 132 & 11 & 197 \\
76 & 193 & 68 & 78 & 195 & 61 & 88 & 188 & 63 & 90 & 181 & 73 & 83 & 183 & 75 \\
179 & 110 & 49 & 175 & 112 & 59 & 170 & 109 & 55 & 172 & 119 & 50 & 169 & 115 & 52 \\
159 & 36 & 147 & 161 & 32 & 144 & 156 & 42 & 146 & 152 & 39 & 141 & 162 & 41 & 137 \\
16 & 223 & 98 & 18 & 225 & 91 & 28 & 218 & 93 & 30 & 211 & 103 & 23 & 213 & 105 \\
134 & 5 & 199 & 130 & 7 & 209 & 125 & 4 & 205 & 127 & 14 & 200 & 124 & 10 & 202 \\
84 & 186 & 72 & 86 & 182 & 69 & 81 & 192 & 71 & 77 & 189 & 66 & 87 & 191 & 62 \\
166 & 118 & 53 & 168 & 120 & 46 & 178 & 113 & 48 & 180 & 106 & 58 & 173 & 108 & 60 \\
164 & 35 & 139 & 160 & 37 & 149 & 155 & 34 & 145 & 157 & 44 & 140 & 154 & 40 & 142 \\
24 & 216 & 102 & 26 & 212 & 99 & 21 & 222 & 101 & 17 & 219 & 96 & 27 & 221 & 92 \\
121 & 13 & 203 & 123 & 15 & 196 & 133 & 8 & 198 & 135 & 1 & 208 & 128 & 3 & 210 \\
89 & 185 & 64 & 85 & 187 & 74 & 80 & 184 & 70 & 82 & 194 & 65 & 79 & 190 & 67 \\
174 & 111 & 57 & 176 & 107 & 54 & 171 & 117 & 56 & 167 & 114 & 51 & 177 & 116 & 47 \\
151 & 43 & 143 & 153 & 45 & 136 & 163 & 38 & 138 & 165 & 31 & 148 & 158 & 33 & 150 \\
29 & 215 & 94 & 25 & 217 & 104 & 20 & 214 & 100 & 22 & 224 & 95 & 19 & 220 & 97
\end{pmatrix}, \tag{7.3}$$

with characteristic polynomial

$$\det(\lambda\mathbf{I} - \mathbf{U}_{15}) = \lambda^{10}(\lambda - 1695)(\lambda^2 - 13050)(\lambda^2 - 49950), \tag{7.4}$$

and so we interpret $\mathbf{U}_{15}$ to be "double-keyed" [255] with magic keys 13050 and 49950. Moreover $\mathbf{U}_{15}$ is $\mathbf{F}$-associated and hence its Moore-Penrose inverse $\mathbf{U}_{15}^{+}$ is $\mathbf{F}$-associated and so is magic. Furthermore $\mathbf{U}_{15}$ has rank equal to 5 and index equal to 1 and is EP and so its group inverse $\mathbf{U}_{15}^{\#}$ and Moore-Penrose inverse $\mathbf{U}_{15}^{+}$ coincide.

To see that $\mathbf{U}_{15}$ is special-Caïssan of type CSP4, it suffices to show that both $\mathbf{UQ}$ and $\mathbf{QU}$ are pandiagonal, where $\mathbf{U}$ is the CSP4-selection matrix

$$\mathbf{S} = \begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0
\end{pmatrix}. \tag{7.5}$$

We find that $\mathbf{SU}_{15} =$

$$\begin{pmatrix}
129&6&207&131&2&204&126&12&206&122&9&201&132&11&197\\
16&223&98&18&225&91&28&218&93&30&211&103&23&213&105\\
164&35&139&160&37&149&155&34&145&157&44&140&154&40&142\\
174&111&57&176&107&54&171&117&56&167&114&51&177&116&47\\
76&193&68&78&195&61&88&188&63&90&181&73&83&183&75\\
134&5&199&130&7&209&125&4&205&127&14&200&124&10&202\\
24&216&102&26&212&99&21&222&101&17&219&96&27&221&92\\
151&43&143&153&45&136&163&38&138&165&31&148&158&33&150\\
179&110&49&175&112&59&170&109&55&172&119&50&169&115&52\\
84&186&72&86&182&69&81&192&71&77&189&66&87&191&62\\
121&13&203&123&15&196&133&8&198&135&1&208&128&3&210\\
29&215&94&25&217&104&20&214&100&22&224&95&19&220&97\\
159&36&147&161&32&144&156&42&146&152&39&141&162&41&137\\
166&118&53&168&120&46&178&113&48&180&106&58&173&108&60\\
89&185&64&85&187&74&80&184&70&82&194&65&79&190&67
\end{pmatrix} \tag{7.6}$$

and that $\mathbf{U}_{15}\mathbf{S} =$

$$\begin{pmatrix}
129&2&206&132&6&204&122&11&207&126&9&197&131&12&201\\
76&195&63&83&193&61&90&183&68&88&181&75&78&188&73\\
179&112&55&169&110&59&172&115&49&170&119&52&175&109&50\\
159&32&146&162&36&144&152&41&147&156&39&137&161&42&141\\
16&225&93&23&223&91&30&213&98&28&211&105&18&218&103\\
134&7&205&124&5&209&127&10&199&125&14&202&130&4&200\\
84&182&71&87&186&69&77&191&72&81&189&62&86&192&66\\
166&120&48&173&118&46&180&108&53&178&106&60&168&113&58\\
164&37&145&154&35&149&157&40&139&155&44&142&160&34&140\\
24&212&101&27&216&99&17&221&102&21&219&92&26&222&96\\
121&15&198&128&13&196&135&3&203&133&1&210&123&8&208\\
89&187&70&79&185&74&82&190&64&80&194&67&85&184&65\\
174&107&56&177&111&54&167&116&57&171&114&47&176&117&51\\
151&45&138&158&43&136&165&33&143&163&31&150&153&38&148\\
29&217&100&19&215&104&22&220&94&20&224&97&25&214&95
\end{pmatrix} \tag{7.7}$$

are both pandiagonal.

8. Firth's "Magic Chess Board" and Beverley's "Magic Knight's Tour"

Our motivation in this section is the one-page "article" (Figure 8.1.1) entitled "The Magic Chess Board, invented by W. Firth, dedicated to Dr. Zukertort" [*sic*], included at the end of the rare booklet [111] entitled *The Magic Square*[36] by "W. A. Firth", published in 1887. We believe that this "W. A. Firth" is William A. Firth (d. 1890) and that this "Dr. Zukertort" is Johannes Hermann Zukertort (1842–1888).

8.1. **The 8×8 Firth–Zukertort magic matrix.** Firth's one-page "article" (Figure 8.1.1) concerns an 8×8 magic square, which we define by the "Firth–Zukertort magic matrix"

$$\mathbf{Z} = \begin{pmatrix} 64 & 21 & 42 & 3 & 37 & 16 & 51 & 26 \\ 38 & 15 & 52 & 25 & 63 & 22 & 41 & 4 \\ 11 & 34 & 29 & 56 & 18 & 59 & 8 & 45 \\ 17 & 60 & 7 & 46 & 12 & 33 & 30 & 55 \\ 10 & 35 & 32 & 53 & 19 & 58 & 5 & 48 \\ 20 & 57 & 6 & 47 & 9 & 36 & 31 & 54 \\ 61 & 24 & 43 & 2 & 40 & 13 & 50 & 27 \\ 39 & 14 & 49 & 28 & 62 & 23 & 44 & 1 \end{pmatrix}. \tag{8.1.1}$$

Firth showed (Figure 8.1.1) that if the top left, top right, lower left and lower right 4×4 squares are stacked then they form a "magic cube" . An $n \times n \times n$ magic cube is the 3-dimensional equivalent of an $n \times n$ magic square, with the sum of the numbers in each row, each column, each "pillar" and the four main "space diagonals" equal to the same magic number. For more about magic cubes see, e.g., Benson & Jacoby [**?**] and Heinz & Hendricks [199].

We find that $\mathbf{Z}$ is $\mathbf{F}$-associated and so its Moore–Penrose inverse $\mathbf{Z}^+$ is $\mathbf{F}$-associated and hence magic. But $\mathbf{Z}$ is neither pandiagonal nor CSP2-magic. However, $\mathbf{Z}$ has rank 5, index 1 and is EP and hence $\mathbf{Z}$ has a group inverse $\mathbf{Z}^\#$ and this coincides with the Moore–Penrose inverse $\mathbf{Z}^+$. The characteristic polynomial is

$$\det(\lambda \mathbf{I} - \mathbf{Z}) = \lambda^3(\lambda - 260)(\lambda^4 - 2,129,920) \tag{8.1.2}$$

and so $\mathbf{Z}$ is keyed with the single "magic key of degree 2"

$$\kappa_2 = \tfrac{1}{4}(\mathrm{tr}\mathbf{Z}^4 - m^4) = 2,129,920 \tag{8.1.3}$$

and hence powers $\mathbf{Z}^{4p+1}$ are linear in the parent $\mathbf{Z}$ and so are all $\mathbf{F}$-associated and EP. We find that

$$\mathbf{Z}^{4p+1} = \kappa_2^p \mathbf{Z} + m(m^{4p} - \kappa_2^p)\bar{\mathbf{E}}, \quad p = 1, 2, \ldots \tag{8.1.4}$$

with magic sum $m = 260$ and where $\bar{\mathbf{E}}$ has every element equal to 1/8. When $p = 1$ (8.1.4) becomes

$$\mathbf{Z}^5 = \kappa_2 \mathbf{Z} + m(m^4 - \kappa_2)\bar{\mathbf{E}}; \quad m = 260,\ \kappa_2 = 2,129,920. \tag{8.1.5}$$

[36]Many thanks to Christian Boyer for alerting us to this booklet and to Jani A. Virtanen for finding a copy for us.

For the Ursus matrix $\mathbf{U}$, we have in parallel

$$\mathbf{U}^5 = \kappa^2\mathbf{U} + m(m^4 - \kappa^2)\bar{\mathbf{E}}; \quad m = 260,\ \kappa_1 = 2688, \tag{8.1.6}$$

where the single "magic key of degree 1"

$$\kappa_1 = \kappa = \tfrac{1}{2}(\mathrm{tr}\mathbf{U}^2 - m^2) = 2688. \tag{8.1.7}$$

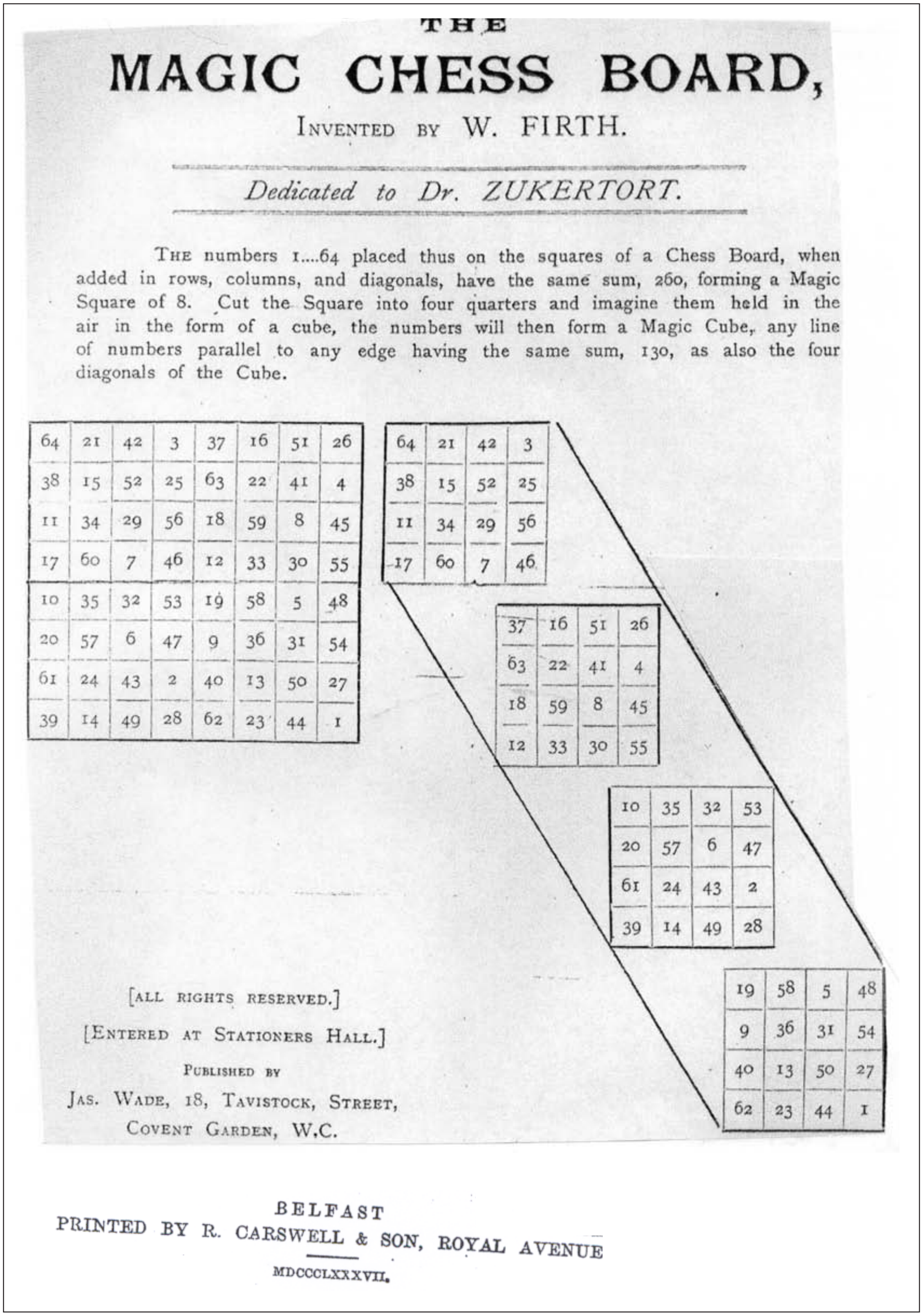
Since $\mathbf{Z}$ is EP we find that the Moore–Penrose and group inverses coincide and that (8.1.4) holds with $p = -1, -2, \ldots$, and so, e.g., with $p = -1$

$$(\mathbf{Z}^{+})^3 = (\mathbf{Z}^{\#})^3 = \frac{1}{\kappa_2}\mathbf{Z} + m\left(\frac{1}{m^4} - \frac{1}{\kappa_2}\right)\bar{\mathbf{E}}. \tag{8.1.8}$$

Firth dedicated his "Magic Chess Board" to "Dr. Zukertort", almost certainly the chess master Johannes Hermann Zukertort (1842–1888), who was one of the leading world chess players for most of the 1870s and 1880s[37], and who lost $7\frac{1}{2}$–$12\frac{1}{2}$ to Wilhelm Steinitz (1836–1900) in the 1886 inaugural World Chess Championship (played in New York, St. Louis and New Orleans). We know of no other distinguished chess player or chess Grandmaster[38] associated with a magic square (or magic cube or magic chessboard). The unnamed "most distinguished chess-player in England" who made the "greatest magic square extant" reported by William Beverly [*sic*] in 1889, see Figure 8.3.2 below, might well have been Zukertort, who died in London in 1888: Steinitz left England in 1883 and moved to New York, where he lived for the rest of his life.

[37]Many thanks to Tõnu Kollo for alerting us to this fact.

[38]The title Grandmaster is awarded to strong chess players by the Fédération Internationale des Échecs (FIDE) or World Chess Federation. Apart from "World Champion", Grandmaster is the highest title a chess player can attain. The earliest use of the term "Grandmaster" may have been in 1907 in the Ostend tournament when the term "Großmeister" was used, though informally, it was apparently used already in 1838 by the newspaper *Bell's Life in London, and Sporting Chronicle* in its chess column; *Bell's Life* was a British weekly sporting paper published as a pink broadsheet between 1822 and 1886. [324]



THE

MAGIC CHESS BOARD,

INVENTED BY W. FIRTH.

Dedicated to Dr. ZUKERTORT.

THE numbers 1....64 placed thus on the squares of a Chess Board, when added in rows, columns, and diagonals, have the same sum, 260, forming a Magic Square of 8. Cut the Square into four quarters and imagine them held in the air in the form of a cube, the numbers will then form a Magic Cube, any line of numbers parallel to any edge having the same sum, 130, as also the four diagonals of the Cube.

64	21	42	3	37	16	51	26
38	15	52	25	63	22	41	4
11	34	29	56	18	59	8	45
17	60	7	46	12	33	30	55
10	35	32	53	19	58	5	48
20	57	6	47	9	36	31	54
61	24	43	2	40	13	50	27
39	14	49	28	62	23	44	1

64	21	42	3
38	15	52	25
11	34	29	56
17	60	7	46

37	16	51	26
63	22	41	4
18	59	8	45
12	33	30	55

10	35	32	53
20	57	6	47
61	24	43	2
39	14	49	28

19	58	5	48
9	36	31	54
40	13	50	27
62	23	44	1

[ENTERED AT STATIONERS HALL.]

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FIGURE 8.1.1: "The Magic Chess Board, invented by W. Firth, dedicated to Dr. Zukertort", published at the end of the book entitled *The Magic Square* by W. A. Firth (1887).

8.2. **William A. Firth (c. 1815–1890) and Henry Perigal, Junior (1801–1898).** We know very little about "W. A. Firth" but note that the title page of *The Magic Square* [111] says "W. A. Firth, B. A., Cantab., late scholar of Emmanuel College, Cambridge, and mathematical master of St. Malachy's College, Belfast." The preface is signed with the address "31 Hamilton Street, Belfast" and it seems to us, therefore, almost certain that our "W. A. Firth" is the "William A. Firth" who died on 12 September 1890, since from the Public Record Office of Northern Ireland [113], we find that "The Will of William A. Firth, late of Belfast, Mathematical Professor, who died on 12 September 1890 at same place was proved at Belfast by Margaret Firth of Hamilton-street Belfast, Widow, one of the Executors." We do not know, however, when he was born or even when he got his B. A. degree from Emmanuel College, Cambridge. We believe this "W. A. Firth" is the "William Firth", whose book [109] entitled *Solutions to the mathematical questions in the examination for admission to the Royal Military Academy*[39], was published in London, July 1879.

In addition, we believe that our "W. A. Firth" is the "Mr. William A. Firth (Whiterock, Belfast)" who with "Mr. Henry Perigal were balloted for and duly elected" members of the Quekett Microscopical Club[40] on 22 July 1881. Henry Perigal, Junior (1801–1898), was an English amateur mathematician best known for his elegant dissection proof of Pythagoras's theorem, a diagram of which is carved on his gravestone[41]. Perigal also discovered a number of other interesting geometrical dissections and, though employed modestly for much of his life as a stockbroker's clerk, was well known in British scientific society.

   

FIGURE 8.2.1: (from left to right) St Malachy's College, Belfast; statue of St Malachy; Henry Perigal, Junior; John Thomas Quekett.

[39] The Royal Military Academy Sandhurst (RMAS), commonly known simply as Sandhurst, is the British Army officer initial training centre. The Academy's stated aim is to be: "the national centre of excellence for leadership" [324].

[40] The Quekett Microscopical Club, named after the English microscopist and histologist John Thomas Quekett (1815–1861), was founded in 1865 and is dedicated to optical microscopy, amateur and professional.

[41] "Henry Perigal's body was cremated and the ashes buried in the churchyard of the Church of St. Mary and St. Peter in Wennington, Essex (about 15 miles east of central London)." [**?**] For an obituary of Henry Perigal, see [**?**].

8.3. **Henry Perigal, Junior (1801–1898) and the first magic knight's tour.** Henry Perigal, Junior, introduced (on 29 March 1848) the article entitled "On the magic square of the knight's march", by William Beverley [*sic*], and published in *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science,* vol. 33, no. 220, pp. 101–105 (August 1848) [92]; this article included what is apparently the first "magic knight's tour", see Figure 8.4.1.

XVIII. *On the Magic Square of the Knight's March.* *By* WILLIAM BEVERLEY.

To the Editors of the Philosophical Magazine and Journal.

GENTLEMEN,

I INCLOSE for insertion in the Philosophical Magazine a very interesting MAGIC SQUARE, formed by numbering consecutively the *moves of the* KNIGHT *in the grand tour* of the chess-board. The *knight's march* has engaged the ingenuity of many eminent philosophers and mathematicians; but I believe that Mr. W. Beverley is the first who has solved the difficult problem of converting it into a *magic square.* The principle upon which he has effected it, seems to be somewhat akin to that invented by Dr. Roget, S.R.S., as explained in his paper on the *Knight's Move* in vol. xvi. of the Philosophical Magazine.

Yours very faithfully,
H. PERIGAL, Jun.

5 Smith Street, Chelsea,
March 29, 1848.

Fig. 1.

Columns.	9	10	11	12	13	14	15	16	
1	1	30	47	52	5	28	43	54	=260
2	48	51	2	29	44	53	6	27	=260
3	31	46	49	4	25	8	55	42	=260
4	50	3	32	45	56	41	26	7	=260
5	33	62	15	20	9	24	39	58	=260
6	16	19	34	61	40	57	10	23	=260
7	63	14	17	36	21	12	59	38	=260
8	18	35	64	13	60	37	22	11	=260
	260	260	260	260	260	260	260	260	

It will be easily observed that the march can be commenced from any of the corner or rooks' squares of the board; from any of the bishops' squares, or bishops' eighths; or from either of the rooks' thirds, or rooks' sixths.

WILLIAM BEVERLEY.

9 Upper Terrace, Islington,
June 5, 1847.

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SIR ROBERT KANE, M.D. M.R.I.A.

"Nec aranearum sane textus ideo melior quia ex se fila gignunt, nec noster vilior quia ex alienis libamus ut apes." JUST. LIPS. *Polit.* lib. i. cap. 1. Not.

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JULY—DECEMBER, 1848.

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RICHARD AND JOHN E. TAYLOR, RED LION COURT, FLEET STREET,
Printers and Publishers to the University of London;
SOLD BY LONGMAN, BROWN, GREEN, AND LONGMANS; SIMPKIN, MARSHALL AND CO.; S. HIGHLEY; WHITTAKER AND CO.; AND SHERWOOD, GILBERT, AND PIPER, LONDON: — BY ADAM AND CHARLES BLACK, AND THOMAS CLARK, EDINBURGH; SMITH AND SON, GLASGOW; HODGES AND SMITH, DUBLIN; AND WILEY AND PUTNAM, NEW YORK.

FIGURE 8.3.1: Excerpts from "On the magic square of the knight's march", by William Beverley [*sic*], introduced by H. Perigal, Jun., and published in *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science,* vol. 33, no. 220, pp. 101–105 (August 1848) [92].

The Greatest Magic Square Extant.

BY WILLIAM BEVERLY.

In veiw of the great interest that is manifested in that mysterious subject, the magic square, we present what is probably the finest and the most perfect example extant. It was made by the most distinguished chess-player of England.

260 ‖	260 ‖	260 ‖	260 ‖	260 ‖	260 ‖	260 ‖	260 ‖	= 2080 ‖
1	30	47	52	5	28	43	54	= 260
48	51	2	29	44	53	6	27	= 260
31	46	49	4	25	8	55	42	= 260
50	3	32	45	56	41	26	7	= 260
33	62	15	20	9	24	39	58	= 260
16	19	34	61	40	57	10	23	= 260
63	14	17	36	21	12	59	38	= 260
18	35	64	13	60	37	22	11	= 260

This square illustrates the Knight's Tour over the chess-board, in the game of chess, in which the knight plays to every square on the board, and touches it but once.

Every line of figures running up and down sums up 260.

Every line of figures running right and lelt sums up 260.

Divide the board into four quarters; then the rows and files of each quarter will sum up 130.

Divide the board into sixteen equal parts; the numbers that compose each sixteenth part will sum up 130.

It also follows that any sixteenth portion of the board, added to any other sixteenth portion, will sum up 260.

It also follows that any half row or file, added to any other half row or file, in the entire square, will sum up 260.

Take the files of numbers running up and down; the four central numbers of the file will sum up 130; and so of course the four remaining or outer numbers will sum up 130.

These are only some of the wonderful properties of this mysterious square. This is really a *magic square*; and incomparison, the ordinary square by this name sinks into insignificance.

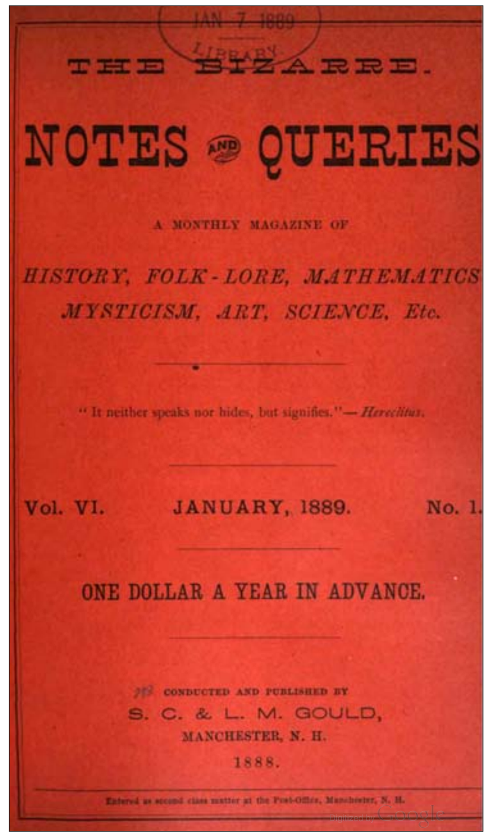

FIGURE 8.3.2: "The greatest magic square extant" by William Beverly [*sic*], published in *The Bizarre: Notes and Queries*, vol. 6, no. 1, pp. 224–225 (January 1889) [112].

This "semi-magic" square (neither of the two main diagonals add to the magic sum) was republished in 1889 as being "made by the most distinguished chess player of England" in the short article entitled "The greatest magic square extant" by William Beverly [*sic*], published in *The Bizarre: Notes and Queries*, vol. 6, no. 1, pp. 224–225 (January 1889) [92], see Figure 8.3.2. According to Jelliss [280]:

> The first magic knight's tour was composed in 1847 by a certain William Beverley, whose address was given as 9 Upper Terrace, Islington (London). *The Dictionary of National Biography* (Supplement 1901) has an extensive entry for William Roxby Beverley (born at Richmond, Surrey 1814?, died at Hampstead, London, 17 May 1889) who is probably the knight's tour Beverley, though our evidence for this is purely circumstantial (i.e., he was in the right area of London at the right time). He was a scene painter and designer of theatrical effects, and travelled round the country quite a lot in the course of this work. He is recorded as being in London from 1846 onwards, working at the Princess's Theatre, the Lyceum, Covent Garden and Drury Lane, and exhibited water colours at the Royal Academy. William Roxby Beverley had three older brothers, Samuel, Henry and Robert. Henry Roxby Beverley (1796–1863) controlled the Victoria Theatre, London for a short time, and died at 26 Russell Square, the house of his brother Mr William Beverley the eminent scene painter. (This address is currently an Annex of Birkbeck College, University of London.) Their father William Roxby (1765–1842) was an actor-manager and adopted Beverley as a stage name, after his home town, the old capital of the East Riding of Yorkshire. Upper Terrace no longer exists; it was that part of Upper Street where Islington Town Hall now stands. I could not trace Beverley's name there in the census records for 1851.

We note that William Roxby Beverley died in 1889 the year in which the magic knight's tour was republished (by William Beverly [*sic*]).

We will denote "magic square of the knight's march" = "greatest magic square extant" by the semi-magic matrix $\mathbf{B}$, which we find has rank 5 and index 1 but is not double-keyed like the Firth–Zukerort matrix $\mathbf{Z}$. Moreover, $\mathbf{B}$ is not pandiagonal, not CSP2-magic, not $\mathbf{V}$-associated (though $\mathbf{B}^+$ is semi-magic), and not 4-pac. The Beverley-matrix $\mathbf{B}$ does, however, have a "semi-alternate-couplets property" with the "Beverley-couplets matrix" $\mathbf{B}_\mathrm{c} = \mathbf{RB}$

$$\mathbf{B} = \begin{pmatrix} 1 & 30 & 47 & 52 & 5 & 28 & 43 & 54 \\ 48 & 51 & 2 & 29 & 44 & 53 & 6 & 27 \\ 31 & 46 & 49 & 4 & 25 & 8 & 55 & 42 \\ 50 & 3 & 32 & 45 & 56 & 41 & 26 & 7 \\ 33 & 62 & 15 & 20 & 9 & 24 & 39 & 58 \\ 16 & 19 & 34 & 61 & 40 & 57 & 10 & 23 \\ 63 & 14 & 17 & 36 & 21 & 12 & 59 & 38 \\ 18 & 35 & 64 & 13 & 60 & 37 & 22 & 11 \end{pmatrix}, \quad \mathbf{B}_\mathrm{c} = \mathbf{RB} = \begin{pmatrix} 49 & 81 & 49 & 81 & 49 & 81 & 49 & 81 \\ 79 & 97 & 51 & 33 & 69 & 61 & 61 & 69 \\ 81 & 49 & 81 & 49 & 81 & 49 & 81 & 49 \\ 83 & 65 & 47 & 65 & 65 & 65 & 65 & 65 \\ 49 & 81 & 49 & 81 & 49 & 81 & 49 & 81 \\ 79 & 33 & 51 & 97 & 61 & 69 & 69 & 61 \\ 81 & 49 & 81 & 49 & 81 & 49 & 81 & 49 \\ 19 & 65 & 111 & 65 & 65 & 65 & 65 & 65 \end{pmatrix}, \tag{8.3.1}$$

where $\mathbf{R}$ is the "couplet-summing" matrix (2.2.3). It is interesting to observe that rows 1 and 5 of $\mathbf{B}_\mathrm{c}$ are the same and with "alternate couplets" as are rows 3 and 7, which are just rows 1 and 5 shifted one to the right (wth wrap-around). We find that $\mathbf{B}_\mathrm{c}$ has rank 4 and index 3 with $\mathrm{rank}(\mathbf{B}_\mathrm{c}^3) = \mathrm{rank}(\mathbf{B}_\mathrm{c}^4) = 2$. Moreover,

$$\mathbf{B}_\mathrm{c}^3 = 260^3\mathbf{E} + 2^6 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -95 & 536 & 319 & -760 & 193 & -193 & 31 & -31 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 95 & -536 & -319 & 760 & -193 & 193 & -31 & 31 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (8.3.2)$$

where $\mathbf{E}$ is the 8×8 matrix with every entry equal to 1. The "Beverley-couplets matrix" $\mathbf{B}_\mathrm{c} = \mathbf{RB}$ is semi-magic with two nonzero eigenvalues, the magic sum 520 (= 2×260) and 36.

Beverley [92, (1848)] observed, see also Andrews [135, p. 175 (1917)] and Falkener [8, p. 325 (1892)], that the Beverley-matrix $\mathbf{B}$ is "quartered" in the sense that the four corner 4×4 submatrices are all semi-magic. Furthermore, Beverley [92] noted that the numbers in sixteen 2×2 submatrices—the four 2×2 submatrices of each of these four corner 4×4 submatrices all add to 130, half the magic sum 260 of the full matrix $\mathbf{B}$. In fact the numbers in thirty-five 2×2 submatrices of $\mathbf{B}$ add to 130, as indicated by the thirty-five elements equal to 130 in the "Beverley-double-couplets matrix" $\mathbf{B}_\mathrm{dc} = \mathbf{RBR}'$

$$\mathbf{B}_\mathrm{dc} = \mathbf{RBR}' = \begin{pmatrix} 130 & 130 & 130 & 130 & 130 & 130 & 130 & 130 \\ 176 & 148 & 84 & 102 & 130 & 122 & 130 & 148 \\ 130 & 130 & 130 & 130 & 130 & 130 & 130 & 130 \\ 148 & 112 & 112 & 130 & 130 & 130 & 130 & 148 \\ 130 & 130 & 130 & 130 & 130 & 130 & 130 & 130 \\ 112 & 84 & 148 & 158 & 130 & 138 & 130 & 140 \\ 130 & 130 & 130 & 130 & 130 & 130 & 130 & 130 \\ 84 & 176 & 176 & 130 & 130 & 130 & 130 & 84 \end{pmatrix}. \qquad (8.3.3)$$

For example, the 130 in row 4, column 5 of $\mathbf{B}_\mathrm{dc}$ means that the numbers $(56, 41; 9, 24)$ in the 2×2 submatrix of $\mathbf{B}$ defined by rows 4 and 5 and columns 5 and 6 add to 130.

Let a chess knight make a tour on an $n \times n$ chessboard whose squares are numbered from 1 to n^2 along the path of the knight (Figure 8.3.3). Then the tour is called a "magic knight's tour" if the resulting arrangement of numbers is a magic square. If the resulting matrix is "semi-magic", i.e., just the rows and columns sum to the magic sum, then we have a "semi-magic knight's tour". When the two main diagonals also sum to the magic number, then we have a "fully-magic knight's tour". Beverley's magic knight's tour on an 8×8 chessboard is semi-magic. In fact there is no fully-magic knight's tour on an 8×8 chessboard, as shown by an exhaustive computer enumeration of all possibilities by the international team of Günter Stertenbrink, Jean-Charles Meyrignac, and Hugues Mackay in 2003 (after 61.40 days of extensive computation). For more about the magic knight's tour see, e.g., Jelliss [280].

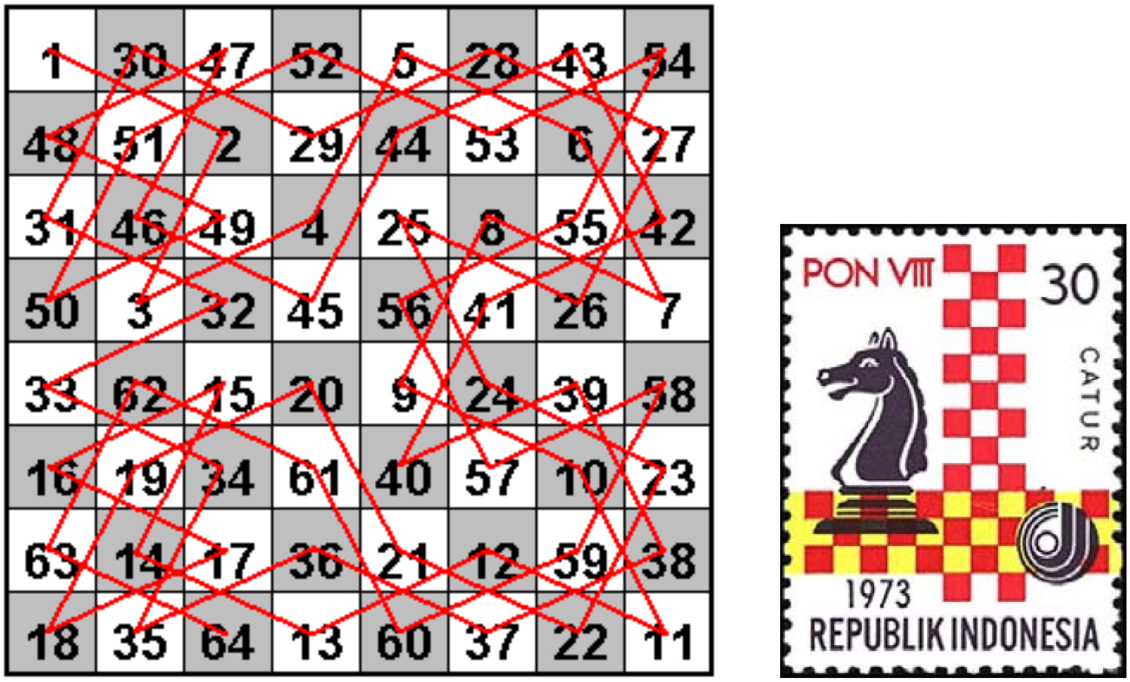

FIGURE 8.3.3: (left panel) first semi-magic knight's tour (Dan Thomasson 2001); (right panel) stamp for "8th National Sports Week": Indonesia 1973, *Scott* 847.

8.4. **Johannes Hermann Zukertort (1842–1888).** Johannes Hermann Zukertort was born on 7 September 1842 in Lublin, Congress Poland, the Polish state created on 3 May 1815 by the Congress of Vienna as part of the political settlement at the end of the Napoleonic Wars. Zukertort said that his mother was the Baroness Krzyanowska (Krzyżanovska)[42] According to the biography of Zukertort by Domański & Lissowski [219], he enrolled on 29 April 1861 at the University of Breslau (then in Prussia, now Wrocław in Poland) to study medicine; he later claimed that he had completed his degree, but this has been disputed—apparently his name was removed from the list of students on 9 February 1867 since he had not fulfilled all the requirements for the degree; nevertheless, he was later addressed with the title "Dr." In Breslau he met the German chess master Adolf Anderssen (1818–1879) and fell in love with chess. This new passion with chess did not prevent Zukertort from distinguishing himself in other ways. He became fluent in a wide range of languages (perhaps as many as 14). He fought for Prussia against Austria, Denmark, and France; was once left for dead on the battlefield; and was decorated for gallantry 9 times; and he was noted as a swordsman and marksman. He was an accomplished pianist and, for a while, a music critic. [324]

According to [219], on 17 April 1797 the Prussian Kaiser Francis II (Erwählter Römischer Kaiser Franz II) (1768–1835), who was the last Holy Roman Emperor (ruling from 1792–1806) decided that all Jews, in addition to the surname each of them already had if any (Jews living in the border regions of Poland at that time did not have surnames) should choose themselves an additional family name. However, the German-speaking officials often influenced the choices. Around 1804 some Austrian official in Lublin gave Moszko Lejbow, who was the grand father of our Johannes Hermann, Zukertort as the last name. What the reason was for this choice is not known, i.e., there is no information as to whether Moszko Lejbow was selling or baking sweet things, etc.

Johannes Hermann Zukertort died on 20 June 1888 from a cerebral haemorrhage. after playing a chess game in Simpson's Divan (in London on The Strand near the Savoy Hotel, now the restaurant Simpson's in the Strand). "Samuel Reiss's Grand Cigar Divan, which opened in 1828, soon became a thriving coffee house, almost a club, among London gentlemen with members paying one guinea a year for use of the facilities. Patrons smoked, read their newspapers at leisure, and played chess while reclining on divans. Right from its early years the house was a popular recreational chess venue, and games of chess were even frequently played against other local coffee houses, with runners hired to deliver each move as it was made.

In 1848, Reiss joined forces with the caterer John Simpson to expand the premises, renaming it 'Simpson's Grand Divan Tavern'. It was soon established as one of the top London restaurants noted for using solely British produce: sirloins of beef and saddles of mutton, served from silver-topped trolleys (some of which are still in use today), and carved at the table for each individual guest. It became an established attraction and patrons included Charles Dickens, William Gladstone, and Benjamin Disraeli. Just as Wimbledon is considered the home of tennis and Lord's the home of cricket, Simpson's can justifiably claim the equivalent title for chess." [324]

[42] Presumably this lady was *not* Tekla Justyna Krzyanowska, the mother of the famous Polish composer and virtuoso pianist Frédéric François Chopin (1810–1849), who had 3 sisters but no brothers.

FIGURE 8.4.1: (left panel) Simpson's Divan in "The Good Old Days" [?, p. 3];
(right panel) portrait of Zukertort in the early 1880s [324].

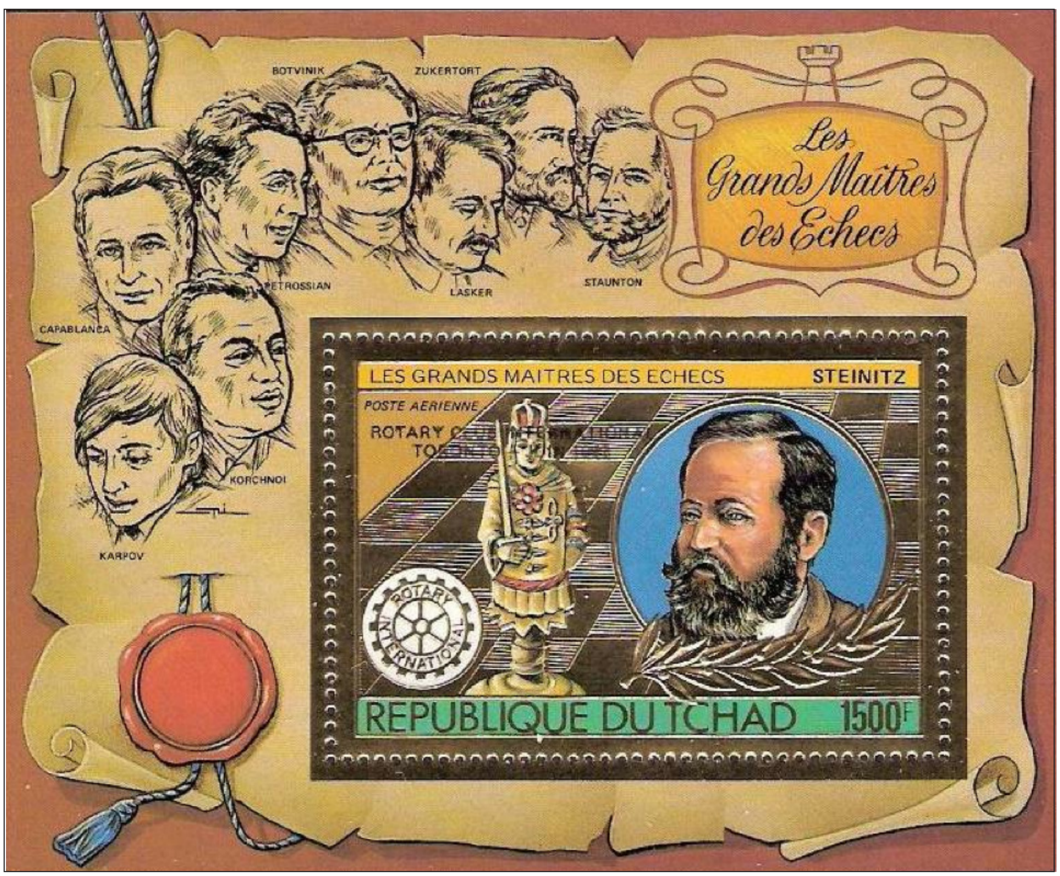

FIGURE 8.4.2: Souvenir sheet for "Les grands maîtres des échecs": Chad[43] 1982, *Scott* 433B.

[43]Chad (French: Tchad), officially known as the Republic of Chad, is a landlocked country in central Africa. It is bordered by Libya to the north, Sudan to the east, the Central African Republic to the south, Cameroon and Nigeria to the southwest, and Niger to the west.

The only stamps that we have found that are associated with Zukertort are a souvenir sheet for “Les grands maîtres des échecs” from Chad 1982 (Figure 4.3) and a pair of stamps from the Central African Republic 1983 (Figure 4.4)[44].

There is a portrait of Zukertort in the selvage of the souvenir sheet in Figure 8.4.2, which shows a single stamp with a portrait of Steinitz. Zukertort is shown in the selvage between Emanuel Lasker (1868–1941) and Howard Staunton (1810–1874) at the top centre, and proceeding anti-clockwise there are also portraits of Mikhail Moiseyevich Botvinnik (1911–1995), Tigran Vartanovich Petrosian[45] (1929–1984), José Raúl Capablanca y Graupera (1888–1942), Viktor Lvovich Korchnoi (b. 1931), and Anatoly Yevgenyevich Karpov (b. 1951). This souvenir sheet was part of a set of nine stamps (Chad 1982, *Scott* 427–433, 433A, 433B) which featured Capablanca, Karpov, Korchnoi, and Staunton as well as François-André Danican Philidor (1726–1795), Paul Charles Morphy (1837–1884), Boris Vasilievich Spassky (b. 1937), and Robert James “Bobby” Fischer (1943–2008), who are not shown in the selvage of the souvenir sheet (Figure 8.4.3).

FIGURE 8.4.3: “Les grands maîtres des échecs”: Central African Republic 1983, (left panel) Steinitz with chess position from a Spassky game, *Scott* 576, and (right panel) Spassky with a position from the 1886 Steinitz–Zukertort championship match, *Scott* 580.

There are many stamps which honour Steinitz, including the one shown in the souvenir sheet from Chad displayed in Figure 8.4.2. The only stamps *per se* that we have found associated with Zukertort are two (Figure 4.4) from a set issued by the Central African Republic in 1983. According to Edwards & Lubianiker [252] the chess position shown on the 300F stamp (Figure 8.4.3, right panel) is from the ninth game of the 1886 Steinitz–Zukertort championship match though the portrait there is of Boris Spassky; this game (with Zukertort white) was played on 10 February 1886; the complete game may now be replayed online. The 5F stamp (Figure 4.4, left panel) shows a portrait of Steinitz together with a position which derives from a game by Spassky played in 1953, nearly 50 years after Steinitz died. Presumably the intent was to put the Steinitz–Zukertort chess position on the 5F stamp and the Spassky chess position on the 300F stamp!

[44] The only stamp associated with Zukertort that is listed in the 1999 Second Edition of *Collect Chess on Stamps* [?] is the 300F stamp from the Central African Republic shown in Figure 8.4.2 (right panel).

[45] Spelled “Petrossian” on the souvenir sheet.

8.5. Kasparov–Karpov World Chess Championship Matches: 1984–1990.

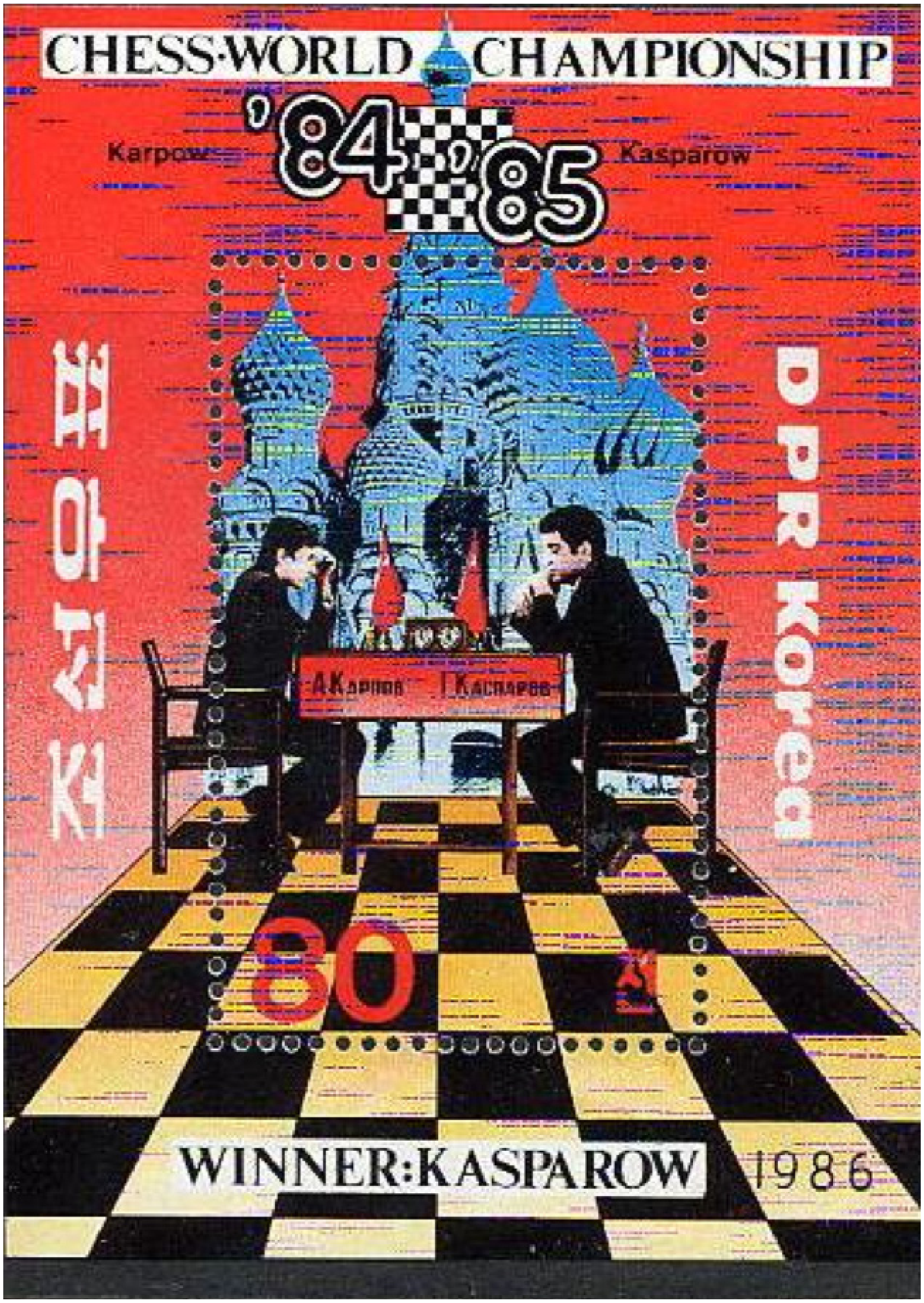

FIGURE 8.5.1: Souvenir sheet for the 1st Kasparov–Karpov World Chess Championship Match, Moscow 1984–1985: Democratic People's Republic of Korea: 1986, *Scott* 2549.

We find it very interesting that on 21–24 September 2009 just a few days before these words were first written [254] there took place in Valencia a rematch between Karpov, whose portrait is shown in the selvage in Figure 8.4.2 (the last portrait proceeding anti-clockwise from Zukertort) and Garry Kasparov (b. 1963: Garry Kimovich Weinstein), who won 9–3. This event took place exactly 25 years after the two players' legendary encounter at the World Chess Championship Match in Moscow 1984, which ended without result. After a few months break, the match continued in 1985 (in Moscow) with Kasparov winning with a score of 13–11, see Figure 8.5.1.

FIGURE 8.5.2: (left panel) 2nd Kasparov–Karpov World Championship Match, London & Leningrad 1986: Armenia 1996, *Scott* 537 (from a booklet pane of four stamps), and (right panel) souvenir sheet for the 4th Kasparov–Karpov World Championship Match, New York & Lyon 1990: Guinea 1992, *Scott* 1195.

FIGURE 8.5.3: Souvenir sheet with three stamps for the
3rd Kasparov–Karpov World Championship Match, Sevilla 1987: Surinam 1987, *Scott* 796.

The Second World Chess Championship Match between Kasparov and Karpov took place in 1986, hosted jointly in London and Leningrad; Kasparov won $12\frac{1}{2}$–$11\frac{1}{2}$. The position shown in the stamp (Figure 8.5.2, left panel) is from the 22nd game (Kasparov white), which was played on 3 October 1986; the complete game [252] may now be replayed online. There were just two more World Chess Championship Matches between Kasparov and Karpov: the Third in Sevilla 1987 (Figure 8.5.2), which ended in a draw with a score of 12–12, but Kasparov kept the title. The Fourth (and last) World Chess Championship Match between Kasparov and Karpov (Figure 8.5.2, right panel) was held in New York and Lyon in 1990, and again Kasparov won $12\frac{1}{2}$–$11\frac{1}{2}$. Shown in the selvage of the souvenir sheet in Figure 8.5.2 (right panel) are portraits of chess masters Johann Jacob (János Jakab) Löwenthal (1810–1876) and Pierre Charles Fournier de Saint-Amant (1800–1872), as well as of Lasker and Staunton, whose portraits are also in the selvage of the Chad souvenir sheet featured in Figure 8.4.2.

9. An illustrated bibliography on Caïssan magic squares and some related topics

9.1. Some publications by or connected with "Ursus": Henry James Kesson (b. c. 1844).

PUBLISHED EVERY WEEK, PRICE 6D., BY POST 6½D.

"THE QUEEN," The Lady's Newspaper, is the oldest and by far the best Lady's Newspaper in existence.

The Queen, The Lady's Newspaper & Court Chronicle, vol. 70, p. 142 (August 6, 1881).

CAISSAN MAGIC SQUARES.

By "URSUS."

ONCE UPON A TIME, when Orpheus was a little boy, long before the world was blessed with the "Eastern Question," there dwelt in the Balkan forests a charming nymph by name Caïssa. The sweet Caïssa roamed from tree to tree in Dryad meditation fancy free. As for trees, she was, no doubt, most partial

References

[1] [1844-Scotland] *Travels in Scotland,* by J. G. Kohl [Johann Georg Kohl (1808–1878)], translated from the German *Reisen in Schottland* into English by John Kesson (d. 1876), With notes by the translator, in correction or elucidation of Mr. Kohl's observations, pub. Bruce and Wyld, London, 1844[46]: online at Google Books. 12

[2] [1846-*Childhood*] *The Childhood of King Erik Menved : An historical romance,* by B. S. Ingemann [Bernhard Severin Ingemann (1789–1862)], translated from the Danish *Erik Menveds Bamdom* into English by John Kesson (d. 1876), Library of Foreign Romance and Nobel Newspaper, vol. VII, pub. Bruce and Wyld, London, 1846: online at Google Books. 12

[3] [1854-Kesson/*Cross*] *The Cross and the Dragon; or, The Fortunes of Christianity in China: with notices of the Christian missions and missionaries, and some account of the Chinese Secret Societies,* by John Kesson (of the British Museum), pub. Smith, Elder, and Co., London, 1854: online at Google Books. 12

[4] [1878-*Pocket*] *Pocket Guide to Chess,* by "Cavendish", Thos. De La Rue, London, 21 pp., 1878. [2nd ed. 1883, 5th ed. 1897–1922. Koninklijke Bibliotheek, 's-Gravenhage [Royal Library, The Hague] Request number 363.B.95.] 105

[5] [1879-Ursus] "Magic squares and circles", by "Ursus", *The Queen, The Lady's Newspaper & Court Chronicle,* vol. 65, p. 577 (June 21, 1879), vol. 66, pp. 4–5 (July 5, 1879), vol. 66, p. 66 [sic] (July 19, 1879), vol. 66, [within pp. 165–186/TBC, missing in McGill microfilm] (August 23, 1879) & vol. 66, p. 215 (6 September 6, 1879). 12

[6] [1880-Ursus] "Trees in rows", by "Ursus", *The Queen, The Lady's Newspaper & Court Chronicle,* vol. 67, p. 298 (April 3, 1880), vol. 67, pp. 350–351 (April 17, 1880), vol. 67, p. 474 (May 22, 1880), vol. 67, pp. 500–501 (May 29, 1880), vol. 68, p. 54 (July 17, 1880), vol. 68, p. 127 (July 24, 1880). 12

[7] [1881-Ursus] "Caïssan magic squares", by "Ursus", *The Queen: The Lady's Newspaper & Court Chronicle,* vol. 70, p. 142 (August 6, 1881), vol. 70, pp. 276–277 (September 10, 1881), vol. 70, p. 391 (October 15, 1881). GHC QCU-15jan11-opt online (7.5 mb, 39 pp.) at McGill. [Apparently the first article on "Caïssan magic squares" and discusses the subject in considerable detail: seminal!] 4, 6, 9, 12, 13, 15, 16, 17, 86, 105, 140

[8] [1892-Falkener] *Games Ancient and Oriental and How to Play Them: Being the Games of the Ancient Egyptians, the Hiera Gramme of the Greeks, the Ludus Latrunclulorum of the Romans and the Oriental Games of Chess, Draughts, Backgammon and Magic Squares,* by Edward Falkener [Edward Lyon Falkener (1814–1896], pub. Longmans, Green and Company, London, [5] + 366 pp., 1892: online at Google Books. " Unabridged and corrected republication" by Dover, New York, 1961 GHC. 95, 104, 105, 114

> Chapters XXX–XLII (30–42) on magic squares, pp. 269–356; Chapter XLI (41): "Indian magic squares", pp. 337–344. GHC IndianMagic2. Apparently reprinted by Bidev, Skopje [59].

[9] [1892-Falkener/*Field*-review] Unsigned review of *Games Ancient and Oriental and How to Play Them* by Edward Falkener [8], *The Field, The Farm, the Garden: The Country Gentleman's Newspaper,* no. 2055, p. 737 (May 14, 1892). [*Gentleman's* not *Gentlemen's*.] 105

> The chapters on .. Magic Squares," which finish the volume, though of interest in some respects, are disfigured by inaccuracies which impair their value. The definition of a magic square is wrong. It is defined to be "a square, the cells of which add up to the same amount, whichever way they are taken" (p. 269). The perpendicular and horizontal bands, and the diagonal of a magic square, afford a constant summation, and in more perfect squares, some other paths also. But this is not in "whichever way they are taken". It is impossible to construct a square which is magic in all given paths. Again, to form odd squares, "place the first number immediately below the centre, then place the others, one by one, in a diagonal line, inclining to the right" (p. 271). It is immaterial whether the inclination is right or left. The only effect is to reverse the position of the terms of the series. Then,

[46] We conjecture that Henry James Kesson was born c. 1844 and that the translator John Kesson (d. 1876) was his father.

when discussing Indian squares, Mr. Falkener remarks (p. 388) that Mr Kesson, who has treated of these squares in *The Queen,* says the name Caïssan squares was given to them by Sir William Jones . . . Mr Kesson says nothing of the kind: that gentleman knows better than anyone else that the adjective "Caïssan" was suggested to him by "Cavendish", who originated it[47]. Caïssa is Sir William Jones's fanciful goddess of chess, and as Indian squares, when perfect, include all the rooks', knights' and bishops' paths, that is the paths of all the chess pieces, —the name Caïssan is very appropriate (p. 338). And (pp. 338, 339), Mr Falkener observes that a Caïssan square of eight (sixty-four cells), is not so perfect in all its paths as one of ten (one hundred cells). A Caïssan square of a hundred cells is impossible of construction.

[10] [1892-Falkener/Finkel-review] "Edward Falkener: old board games for new", by Irving L. Finkel, review of [8], *Board Games Studies,* vol. 1, pp. 104–108 (1998): online GHC Falk-review. 8, 142

The review in *The Field* [reviewer not identified] is lengthy and hostile, and greatly upset the author [Falkener], indeed it formed the subject of correspondence between Falkener and Iltyd Nicholl, the editor[48] of *The Field*[49][9]. From this letter it is clear that the reviewer, who attacked Falkener very vigorously, was the "Mr Kesson" who had written articles on Magic Squares in 1879–1881 referred to by Falkener in his book, pp. 337–338. Nicholl, evidently an experienced editor, wrote consolingly (probably not for the first time in his career): "... you must take comfort in the thought that to be found fault with at such length is in itself a compliment. Probably your work has forestalled something of a similar nature which Mr Kesson himself contemplated, and that is an offence which some people can never forgive."

According to Iltyd Nicholl, "Kesson" was a *nom de plume,* deriving from the site called "Nassek," where a contention-producing magic-square had been earlier discovered over a gateway[50].

[11] [1894-Cavendish] *Recreations with Magic Squares: the eight queens' problem solved by magic squares and domino squares,* by "Cavendish" [Henry Jones (1831–1899)] pub. Thomas de la Rue, London, xii + 84 pp., 1894: online at Google Books, GHC Cavendish. 38, 42, 105, 141

"To Henry James Kesson [b. c. 1844], this book is cordially dedicated by his sincere friend, the Author." "The solution of the Eight queens' Problem by Magic Squares is believed to be new. That such a solution is possible was simultaneously suspected by Mr. Kesson and the Author. As regards the solution given, the Author desires to state that it is due to an analysis made by Mr. Kesson; all the Author claims respecting it is the mode of arrangement." [pp. xi–xii]

[12] [1894-Peasants] *The Peasants,* action song, in Staff and Tonic sol-fa Notations, music by Benjamin John W. Hancock, words by H. J. Kesson, pub. Beal & Co., London, 1894.

[13] [1895-Shoe] *The Magic Shoe,* an operetta-cantata for young people (Tonic sol-fa), music by Benjamin John W. Hancock, words by H. J. Kesson, pub. J. Curwen & Sons, London, 35 pp., 1895?

[14] [1901-Boating] *A Boating Song,* music by Benjamin John W. Hancock, words by H. J. Kesson, Choruses for equal Voices no. 389, pub. J. Curwen & Sons, London (printed in Leipzig), 4 pp., 1901.

[15] [1901-Railway] *The Railway Journey,* a school song, music by Benjamin John W. Hancock, words by H. J. Kesson, Choruses for equal Voices, no. 390, pub. J. Curwen & Sons, London (printed in Leipzig), 4 pp., 1901.

[47]The first use of Caïssan squares by "Cavendish" that we have found is in the 1894 book [11], published 13 years after the seminal article by "Ursus" [7]. Cavendish's *The Pocket Guide to Chess* [4] was first published in 1878–TBC.

[48]According to *Wikipedia* [324] Frederick Toms was editor of *The Field* from 1888–1899.

[49]*The Field, The Country Gentleman's Newspaper* is the world's oldest country and field sports magazine, published continuously since 1853. [324]

[50]Presumably this Nassek = Nasik? We do not know of any magic square been discovered over a gateway in Nasik. We find it interesting that "Kesson" is "Nassek" (almost) backwards but we doubt if "Kesson" is a *nom de plume.*

(left panel) The Lincoln Imp [16]; (right panel) Lincoln Cathedral: Montserrat 1978, *Scott* 387.

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[16] [1904-LincolnImp] *The Legend of the Lincoln Imp,* by H. J. Kesson (Ursus), pub. J. W. Ruddock & Sons, Lincoln, [16] p. 15 cm., 1904. [Reprinted 1907, 1911, 1919, 1922, 1923, 1925, 1927, 1930. Apparently also reprinted 1965. Book is listed in "Curwen's 1895 Publications" as by "Ursus", price 4d., postage 2d. Dedicated "To my friend E. B. K. D." Booklet signed by "H. J. Kesson (Ursus)" on title page. My file=ImpUrsus.] 106, 147, 148

[17] [1974-Horn] *The Victorian Country Child,* by Pamela Horn, pub. The Roundwood Press, Kineton, 1974. [Reprinted: Illustrated History Paperback Series) Alan Sutton Publishing, Ltd. (September 1997), ISBN-13: 978-0750914994.] 12

> Report for Austrey[51] School in Warwickshire for 1890: "Discipline is very well maintained. The children read unusually well, in a narural voice and with good intonation. Handwriting also deserves praise; Spelling very fair, as is the mechanical Arithmetic. The first and second standards know their tables well, but the third and upwards should solve easy problems and do better in mental arithmetic. English fails, parsing being poor in the fourth standard. The school has been much improved by the addition of a class room, improvement of play-ground, etc. More pictures are needed. Considering that the children live near the school, the attendance should be much more regular than it is."
>
> *Master:* Henry James Kesson, Trained Two Years; Certificated First Class. Emily Kesson, Sewing Mistress and General Assistant. [From p. 213 (1985), p. 250 (1997 printing)].

[18] [2000-Jessie] *Jessie Kesson: writing her life, a biography,* by Isobel Murray, pub. Canongate Books Ltd., Edinburgh, 2000. TBC-ILL

> Surviving a deprived, institutionalised childhood in the workhouse of Inverness, Jessie Kesson took on a huge variety of jobs in London and became a skilled social worker as well as a respected and accomplished writer.

[51] Austrey is a village at the northern extremity of the county of Warwickshire, near Newton Regis and No Man's Heath, and close to the Leicestershire villages of Appleby Magna, Norton-juxta-Twycross and Orton on the Hill.

9.2. **Some publications by or about Andrew Hollingworth Frost (1819–1907).**

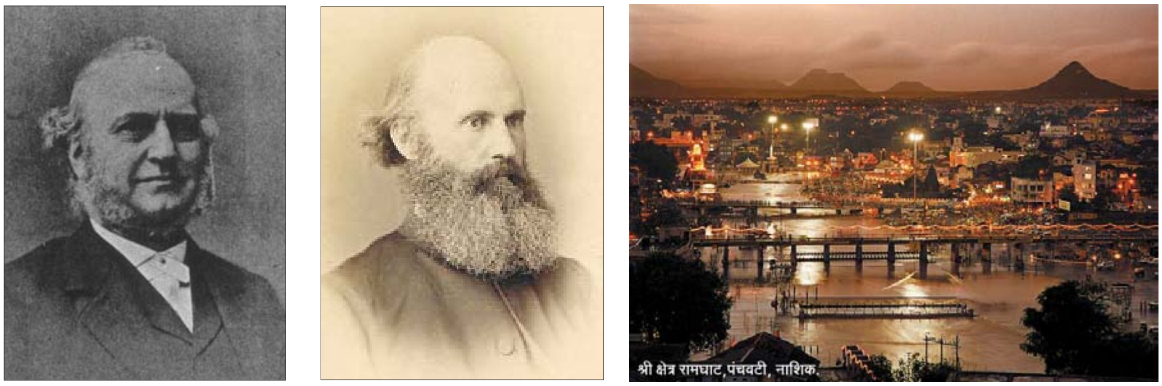

FIGURE 9.2.1: (left panel) Percival Frost (1817–1898), (centre) Andrew Hollingworth Frost (1819–1907)[52], (right) Nasik (Nashik)[53] and the Godavari River.

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[19] [1857-Frost] Article on Nasik squares, title TBC.] by A. .H. F., *Cambridge Mathematical Journal,* 1857 = vol. 6? TBC: cited by [28, p. 215], [31, p. 311].

> *The Cambridge Mathematical Journal* and its successors, *The Cambridge and Dublin Mathematical Journal,* and *The Quarterly Journal of Pure and Applied Mathematics,* were a vital link in the establishment of a research ethos in British mathematics in the period 1837–1870. From the beginning, the tension between academic objectives and economic viability shaped the often precarious existence of this line of communication between practitioners. Utilizing archival material, this paper presents episodes in the setting up and maintenance of these journals during their formative years. "*The Cambridge Mathematical Journal* and its descendants: the linchpin of a research community in the early and mid-Victorian age" [210].
>
> *Cambridge and Dublin Mathematical Journal,* Rosenthall has: v.1–9 (1846–1854).
>
> *The Quarterly Journal of Pure and Applied Mathematics,* Rosenthall has: v.1–50 (1857–1927). Apparently no articles by Frost in vol. 1 (1857).

[20] [1865-Frost] "Invention of magic cubes, and construction of magic squares possessing additional properties", by Rev. A. H. Frost (communicated by Rev. Percival Frost), *The Quarterly Journal of Pure and Applied Mathematics,* vol. 7, no. 25 (February 1865), pp. 92–96 & vol. 7, no. 26 (June 1865), pp. 97–102: GHC Frost-1865/F1. Falk-F1. VBB-87. ["I shall call them Nasik squares." (p. 94).] Complete vol. 7 (1866): online at Google Books, GHC QJM-v7/1866; no. 26 = June 1865; no. 27 = October 1865; no. 28 = February 1866. 6, 7, 17

> Mr. A. Frost, however, has investigated a very elegant method of constructing squares, in which not only do the rows and columns form a constant sum, but also the same constant sum is obtained by *the same number of summations in the directions of the diagonals* —the square being moved in the direction of the sides till the opposite sides are in contact, in order to supply a number of figures in the direction parallel to the diagonal equal to the number of figures in the aide. In this kind of square (an example of which is here given (2),) the whole number of summations is four times the number of figures in the side. I will now briefly mention the principal steps by which Mr. A. Frost obtained his general method of construction; and, in order to distinguish the squares which he has investigated from the ordinary magic squares, I shall call them Nasik Squares, and his cubes Nasik Cubes.

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[52] Portrait in 1870, from the Church Mission Society archives; via Christian Boyer [32].

[53] Nasik is in the northwest of Maharashtra state (180 km from Mumbai and 220 km from Pune), India, and India Security Press in Nasik is where a wide variety of items like postage stamps, passports, visas, and non-postal adhesives are printed: photograph of Nasik online at nashikchandi.com

[21] [1866-Frost] "Supplementary note on Nasik cubes", by Andrew Frost, *The Quarterly Journal of Pure and Applied Mathematics,* vol. 8, no. 29 (month-TBC 1866), p. 74: GHC Frost-1866/F2 Falk-F2. VBG-88. Complete vol. 8 (1867): online at Google Books, GHC QJM-v8/1867. 17

[22] [1876a-Frost] "On the knight's path", by A. H. Frost, *The Quarterly Journal of Pure and Applied Mathematics,* vol. 14, no. 54 (month 1876), pp. 123–125: GHC Frost-1876a. Complete vol. 14 (1877): online at Google Books, GHC QJM-v14/1877.

[23] [1876b-Frost] "A simple method of tracing paths of a knight over the squares of 5, 6, 7, 8, and their extension to higher squares", by A. H. Frost, *The Quarterly Journal of Pure and Applied Mathematics,* vol. 14, no. TBC (month-TBC 1876), pp. 354–359: Frost-1876b. ["simple method" not "simple way".] Complete vol. 14 (1877): online at Google Books, GHC QJM-v14/1877.

[24] [1877a-Frost] "On the general properties of Nasik squares", by A. H. Frost, *The Quarterly Journal of Pure and Applied Mathematics,* vol. 15, no. 57 (June 1877), pp. 34–49: GHC Frost-1877.pdf, VBB-91. [8×8 "Nasik square": p. 48.] Complete vol. 15 (1878): online at Google Books, GHC QJM-v15/1878 6, 9, 17, 108

[25] [1877b-Frost] "On the general properties of Nasik cubes", by A. H. Frost, *The Quarterly Journal of Pure and Applied Mathematics,* vol. 15, no. 57 (June 1877), pp. 93–96 & vol. 15, no. 58 (November 1877), pp. 97–123: GHC Frost-1877b.pdf, VBB-92. [no. 59 = April 1878.] Complete vol. 15 (1878): online at Google Books, GHC QJM-v15/1878. 6, 108

[26] [1877c-Frost] "Description of Plates 3 to 9", by A. H. Frost, *The Quarterly Journal of Pure and Applied Mathematics,* vol. 15, no. 60 (June 1878), pp. 366–368: GHC PlatesGarni.pdf, VBB-93. [Plates 2–5 at end of no. 58 (November 1877); Plates 6–9 at end no. 59 (April 1878): Google-scanned copy incomplete. Paper copy in Rosenthall TBC, plates 1 & 2 also interesting?] 6, 108

[27] [1877d-Frost] "On the construction and properties of Nasik squares and cubes", by A. H. Frost, Cambridge, 1877. TBC: Falk-F3. Preprint of [24, 25, 26]—TBC?

[28] [1882-Frost/EB9] "Magic square", by A. H. F., In *The Encyclopædia Britannica: A Dictionary of Arts, Sciences, and General Literature,* Ninth Edition, vol. 15, pp. 213–216, Popular Reprint, pub. Henry G. Allen, New York, 1882 & Charles Scribner's Sons, New York, 1883. 17, 107

[29] [1896-Frost] "The construction of Nasik squares of any order", by Rev. A. H. Frost, *Proceedings of the London Mathematical Society,* vol. 27, no. 1, pp. 487–518: online at Oxford University Press Journals Digital Archive & Highwire Press Oxford University Press Archive. 6, 7

> The object of this paper is to give a method by which Nasik squares of the nth order can be formed for all values of n; a Nasik square being defined to be "A square containing n cells in each side, in which are placed the natural numbers from 1 to n^* in such an order that a constant sum $\frac{1}{2}n(n^2+1)$ (here called W) is obtained by *adding the numbers on n of the cells, these cells lying in a variety of directions defined by certain laws."*

[30] [1899-Percival/obit] "Dr. Percival Frost", by Dr. George Bruce Halsted, *The American Mathematical Monthly,* vol. 6, nos. 8–9, pp. 189–191 (August–September 1899): online at JSTOR. [Includes a portrait (p. 190).] 7, 142

[31] [1911-Frost/EB11] "Magic square", by A. H. F. (with "Fennell's Magic Ring" by C. A. M. F.), In *The Encyclopædia Britannica: A Dictionary of Arts, Sciences, Literature and General Information,* 11th edition, vol. 17: Lord Chamberlain to Mecklenburg, pp. 310–313, pub. Cambridge University Press, 1911: GHC EB11v17-C1a. online & reset at wikisource GHC Frost-EB11-1911-Wikisource. 107

[32] [2009-Frost/bio-Boyer] "Andrew Hollingworth Frost (1819–1907), Biographical sketch" by Christian Boyer: online at the Multimagic squares website by Christian Boyer, GHC CBR-Frost, 3 pp. Undated but c. 2009. 107, 140

9.3. **Some publications by or about Charles Planck (1856–1935).**

FIGURE 9.3.1: Charles Planck, c. 1903 [39] (left panel), c. 1897 [35] (right panel).

FIGURE 9.3.2: (left panel) Edward Nathan Frankenstein (1840–1913) [36, p. 233]; (centre & right panels) Benjamin Glover Laws (1861–1931) [36, p. 129], [324]; William Symes Andrews (1847–1929), "Pioneer electrical engineer and former associate" of Thomas Alva Edison (1847–1931) [143]: Mali 2009 [308].

[33] [1886-Planck/ChessProblem] *The Chess Problem; text-book with illustrations containing four hundred positions selected from the works of H. J. C. Andrews, E. N. Frankenstein, B. G. Laws and C. Planck,* pub. Cassell, London, vi, [2], + 332 pp. (1886); Revised edition, pub. Tynron Press, Scotland (1989); reprinted by Graham Brash (Pte) Ltd., Singapore (2001) ISBN 981-218-051-6. [Missing signature in copy borrowed by ILL from Vancouver Public Library TBC-ILL.]

'The Chess Problem" discussion by C. Planck, pp. 1–80, followed with Problems by H. J. C. Andrews (pp. 83–116), by E. N. Frankenstein (pp. 119–150), by B. G. Laws (pp. 153–206), and by C. Planck (pp. 209–261). "Coauthors" of Charles Planck are Henry John Clinton Andrews (1828–1887), and

[34] [1888-Planck/EMC] "Magic squares, cubes, and quadrics—magic square for 1888—oddly even roots—answer to "W.T.P." (Query 64688)—Nasik and ply-squares" by Charles Planck (article signed "C. Planck" [*sic*]), *The English Mechanic and World of Science*[54]*,* vol. 47, no. 1199, p. 60 (March 16, 1888): GHC CPK-17. Whole volume 47 online (601 pp., 140.9 mb) at Google Books, GHC EMC47.pdf.

Article # [28511], signed C.P. (identified as C. Planck by Andrews [135, p. 363]).

[35] [1897-Planck/bio-Bouquet] "Charles Planck, M.A., Cantab., L.R.C.P., M.R.C.S." In *The Chess Bouquet* (compiled by Frederick Richard Gittins), pub. Feilden, McAllan & Co., London, 1897) [36, pp. 86–87 (1897)], GHC CPK-03f. 109

Unsigned biography of Charles Planck with a portrait.

[36] [1897/ChessBouquet] *The Chess Bouquet: or, The book of the British composers of chess problems, with portraits, biographical sketches, essays on composing and solving, and over six hundred problems, being chiefly selected masterpieces, to which is added portraits and sketches of the chief chess editors of the United Kingdom* (compiled by Frederick Richard Gittins), pub. Feilden, McAllan & Co., London, 1897: online (265 pp., 27.2 mb.) at Google Books GHC ChessBouquet. Reprinted by Nabu Press (2010, 284 pp., ISBN-13: 978-1146241571), for sale at amazon.com US$21.85. GOC 109, 110, 142, 147, 148

[37] [1900-Planck/n-queens] "The n queens problem," by C. Planck, *The British Chess Magazine,* vol. 20, no. 3, pp. 94–97 (March 1900): GHC Planck-queens2; complete vol. 20 online at Google Books. [No hyphen in title.] 23, 79, 80

[38] [1902-Planck/Nature] "Magic squares of the fifth order," by C. Planck, *Nature,* vol. 65, no. 1692, p. 509 (April 3, 1902): online at the Nature Publishing Group, GHC CPK-Nature.pdf.

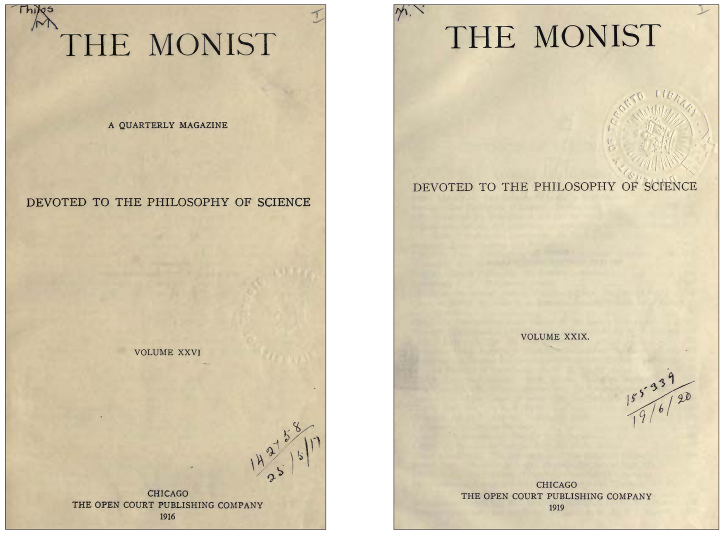
Letter to the Editor concerning the number of 5×5 magic squares estimated by [123] published on March 13, 1902; Planck's letter is dated March 15, 1902.

[39] [1903-Planck-bio/BCM] *The British Chess Magazine,* vol. 23, p. 194 (May 1903): GHC CPK-04. The whole volume 23 online at Google Books (557 pp., 33 mb., GHC BCM23).] 109

Untitled, unsigned biography of Charles Planck with a portrait.

[40] [1905-PathNasiks] *The Theory of Path Nasiks,* by C. Planck, printed [apparently for private circulation] by A. J. Lawrence, Printer, Rugby [England], [1] + 18 pp. (1905). Copy from the British Library, GHC CPK-18. 13

[54]Full journal title: *The English Mechanic and World of Science, with which are incorporated "The Mechanic", "Scientific Opinion", and "The British & Foreign Mechanic", illustrated with numerous practical engravings.*


THE MONIST

A QUARTERLY MAGAZINE

DEVOTED TO THE PHILOSOPHY OF SCIENCE

VOLUME XXVI

CHICAGO
THE OPEN COURT PUBLISHING COMPANY
1916

THE MONIST

DEVOTED TO THE PHILOSOPHY OF SCIENCE

VOLUME XXIX.

CHICAGO
THE OPEN COURT PUBLISHING COMPANY
1919

FIGURE 9.3.3: (left panel) Journal title page from *The Monist,* vol. 26 (1916) and (right panel) vol. 29 (1919): The words *A Quarterly Magazine Devoted* in journal subtitle changed to just *Devoted*.

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[41] [1910-Planck/Monist-v20a] "The construction of magic squares and rectangles by the method of "complementary differences" by W. S. Andrews [& C. Planck]," *The Monist: A Quarterly Magazine Devoted to the Philosophy of Science,* vol. 20, pp. 434–444 (1910): GHC CPK-10a. Whole vol. 20 (662 pp., 61.2 mb), online at the Internet Archive: GHC Monist-v20a. Article reset and republished (signed W. S. A.) in Andrews [135, in ch. XI: Sundry constructive methods, pp. 257–266]: GHC CPK-10b.

Article signed "W. S. Andrews, Schenectady, N. Y." but footnote at end of title: "This article has been compiled almost entirely from correspondence received by the writer from Dr. Planck, and in a large part of it the text of his letters has been copied almost verbatim. Its publication in present form has naturally received his sanction and endorsement. W. S. A." Article starts with "We are indebted to Dr. C. Planck for a new and powerful method for producing magic squares ...".

[42] [1910-Planck/Monist-v20] "Four-fold magics" by C. Planck, *The Monist: A Quarterly Magazine Devoted to the Philosophy of Science,* vol. 20, pp. 617–630 (1910): GHC CPK-11a. Article reset and republished (signed C. P.) in Andrews [135, in ch. XIV: Magic octahedroids, pp. 363–375]: GHC CPK-11b. Whole vol. 20 (662 pp., 61.2 mb), online at the Internet Archive: GHC Monist-v20a.

[43] [1912-Planck/Monist-v22] "The theory of reversions" by C. Planck, *The Monist: A Quarterly Magazine Devoted to the Philosophy of Science,* vol. 22, pp. 53–81 (1912): GHC CPK-12a. Whole vol. 22 (654 pp., 88.8 mb), online at the Internet Archive, GHC Monist-v22. Article reset and republished (signed C. P.) in Andrews [135, as ch. XII: The theory of reversions, pp. 295–320]: GHC CPK-12b.

[44] [1916a-Planck/Monist] "General rule for constructing ornate magic squares of orders $\equiv 0 \pmod 4$" by C. Planck, *The Monist: A Quarterly Magazine Devoted to the Philosophy of Science,* vol. 26, pp. 463–470 (1916): GHC CPK-13a. Whole vol. 26 (652 pp., 62.3 mb) online at the Internet Archive, GHC Monist-v26a. Article reset and republished (signed C. P.) in Andrews [135, in ch. XV: Ornate magic squares, pp. 376–383]: GHC CPK-13b. 13, 77

[45] [1916b-Planck/Monist] "Ornate magic squares of composite odd orders", by C. Planck (article signed "C. Planck" [*sic*]), *The Monist: A Quarterly Magazine Devoted to the Philosophy of Science,* vol. 26, pp. 470–476 (1916): GHC CPK-14a. Whole vol. 26 (652 pp., 62.3 mb) online at the Internet Archive, GHC Monist-v26a. Article reset and republished in Andrews [135, in ch. XV: Ornate magic squares, pp. 383–389]: GHC CPK-14b. 13

[46] [1919-Planck/Monist] "Pandiagonal magics of orders 6 and 10 with minimal numbers", by C. Planck (article signed C. P.), *The Monist: Devoted to the Philosophy of Science,* vol. 29, pp. 307–316 (1919): GHC CPK-15a. Whole vol. 29 (654 pp., 90.0 mb) online at the Internet Archive, GHC Monist-v29a. Article published in 1919 and not reprinted in Andrews [135, 2nd ed., 1917]. 21

[47] [1935a-Planck/obit-*The Times*] "Charles Planck, M.A., M.R.C.S., L.R.C.P.", *The Times,* June 21, 1935, issue 47096, p. 1, col. A "Deaths", article CS17247445: online at *The Times* Digital Archive 1785–1985 at Cengage Learning, GHC CPK-5a. 112

[48] [1935b-Planck/obit-BMJ] "Charles Planck, M.A., M.R.C.S., L.R.C.P.", *The British Medical Journal,* vol. 1, no. 3886, p. 1344 (June 29, 1935): online at JSTOR, GHC CPK-05b; also online at Highwire Press and online at PubMed Central.

Unsigned obituary of Charles Planck indicating that he was born in "Dhiapore" [*sic*], India "while his father was serving in the Indian Medical Service". For more about "Dhiapore" see [273].

[49] [1935c-Planck/obit-BCM] "Charles Planck, M.A., M.R.C.S., L.R.C.P.", *The British Chess Magazine,* vol. 55, pp. 386–388 (1935): snippets online at Google Books, GHC (of snippets) CPK-obit-BCM-best.pdf. [Obituary of Charles Planck, apparently unsigned, building upon the obituary [47].]

[50] [1935d-Planck/obit-BJP] "Charles Planck," [obituary] by G. H. H.-S. [George Hastie Harper-Smith (1877–1954)], *The Journal of Mental Science,* vol. 81, no. 334, p. 748 (July 1935): online at BJPsych: *The British Journal of Psychiatry*[55], GHC CPK-06. [Paper copy of whole vol. 81 in McGill Redpath Storage bin 5844.]

[51] [2011-Planck/bio-Boyer] "Dr. Charles Planck (1856–1935): biographical sketch" by Christian Boyer. online at the Multimagic squares website by Christian Boyer, updated 2011, GHC CPK-09; marked-up CPK-09x. 140

[55]"*The British Journal of Psychiatry* was originally founded in 1853 as *The Asylum Journal* and was known as the *The Journal of Mental Science* from 1858 to 1963. The complete archive of contents between 1855 and 2000 is now available online."

9.4. **Some publications by or connected with Pavle Bidev (1912–1988).**

FIGURE 9.4.1: (left panel) Igalo, Croatia (where apparently Bidev lived); (right panel) Royal Library, The Hague (where many of Bidev's publications are available).

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[52] [1907-G12/Cashmore] "Chess magic squares: on magic squares constructed using chess moves", by M. Cashmore, *Report of the South African Association for the Advancement of Science, Cape Town,* vol. 3, pp. 83–90 (1907). [Reprint (apparently by Pavle Bidev, c. 1975) at Koninklijke Bibliotheek, 's-Gravenhage [Royal Library, The Hague] Request number XSR 564. GHC G12/Cashmore. KB1-G10. [May have originally been published in 1905.] 45, 49, 54, 55, 114, 115, 154

[53] [1965-Bidev/Nâsik] *Die indischen magische Quadrat des Typus Nâsik als mathematische Grundlage des Schachspiels, etc.,* by Pavle Bidev, Writer on Chess, Skopje, ff. 46, 6: plates. 30 cm. (typewritten), c. 1965. [OCLC 558002563, copy in British Library.]

[54] [1969-Bidev/Živa] "Divinatorische Herkunft und kosmische Symbolik des Schachs und damit verwandter Spiele", by P. Bidev, *Živa Antika/Antiquité Vivante,* vol. 19, no. 1 (month-TBC, 1969), pp. 71–111. [Snippets online at Google Books: GHC Ziva-best. KB1-G11 from reprint at KBH.]

[55] [1970-Bidev/Dürer] *Albrecht Dürers magische Quadrat und die darin vorhanden Pfade von Schachfiguren,* by Pavle Bidev, Veit & Co., Leipzig, pp. 307–311 (1970). [OCLC 672029542, copy in Univ. Calgary Library. WorldCat entry says "book" but gives pp. 307–311.] TBC-ILL

[56] [1972-Bidev/Simbol-Croatian] *Šah simbol kosmosa : geneza šaha od kineske astrologije do indijske mistike : sa mnogim slikama i dijagrmima u tekstu,* in Croatian, by Pavle Bidev; preface by Max Euwe [Machgielis Euwe (1901–1981)], pub. Organizacioni Komitet Šahovkih Olimpijada, 229 pp., Skopje, 1972, OCLC 68728809. Croatian version of [**?**]. Copy at the Koninklijke Bibliotheek, 's-Gravenhage [Royal Library, The Hague] Request number XSM 440. KB2.

[57] [1972-Bidev/Simbol-Russian] *Šahot simbol na kosmosot : geneza na šahot od kineskata astrologija do indiskata mistika,* by Pavle Bidev, preface by Max Euwe, pub. Makedonska Kniga, Skopje, 254 pp., 1972, OCLC 71511361. Russian version of [**?**]. Copy at the Koninklijke Bibliotheek, 's-Gravenhage [Royal Library, The Hague] Request number XSP 601. KB2. English section = KB1-G01

[58] [1975-Bidev/Ferz] "Die 132 magischen 5×5 Quadrate durch den Schritt des Ferz", by Pavle Bidev pub. Bidev, Skopje, 1975. KB1-G14.

[59] [1975-Bidev/Falkener] "Indian magic squares", by Edward Falkener, pub. Bidev, Skopje, 6 pp., 1975. [Apparently a reprint of Chapter 14 from Falkener [8], copy at Koninklijke Bibliotheek, 's-Gravenhage [Royal Library, The Hague] Request number XSR 565. TBC-KBH.] 104

[60] [1975-Bidev/Cashmore-reprint] "Chess magic squares by M. Cashmore", pub. Bidev, Skopje, 1975. [A "reprint" of [52].] KB1-G12.

[61] [1975-Bidev/magischen] "Geschichte der Entdeckung des Schachs im magischen Quadrats und des magischen Quadrat im Schach," by Pavle Bidev, *Schachwissenschaftliche Forschungen: Beiträge zur Kultur- und Geistesgeschichte des Schachspiels,* pp. 120–131, January 5, 1975. [Pub. 1972–1975. OCLC 72972646 Hrsg.: Egbert Meissenburg, Winsen/Luhe.]

[62] [1977-Bidev/Bonus] "Vom genetischen Code-Algorithmus des indischen Schach im magischen Quadrat der Acht – Neueste Erkenntnisse" [in German: "On genetic code-algorithms of Indian chess in 8×8 magic squares – Newest findings"] by Pavle Bidev, In *Bonus Socius: Bijdragen tot de cultuurgeschiedenis van het schaakspel en andere bordspelen, jubileumuitgave voor Meindert Niemeijer ter gelegenheid van zijn 75ste verjaardag* [in Dutch: *Bonus Socius*[56] *: Contributions to the Cultural History of Chess and Other Board Games, Festschrift for Meindert Niemeijer*[57]] *on the occasion of his 75th birthday*], samengesteld door [compiled by] C. Reedijk [Cornelis Reedijk (1921–2000)] & K. W. Kruijswijk [Karel Wendel Kruijswijk (b. 1921)] pub. Onder auspiciën van de Koninklijke Bibliotheek, 's-Gravenhage, 296 pp., 1977. Four copies at Koninklijke Bibliotheek, 's-Gravenhage [Royal Library, The Hague] Request number NL 91 D 1010, Sch D 4138, ND 1982/60007, 14002198. GHC Bidev-Bonus/7may11b.pdf (with partial translation into English). KB2. 114, 149

[63] [1978-Bidev/BCM] "Did chess originate in China: my answer to Mr. A. S. M. Dickins [162]", by Pavle Bidev, *The British Chess Magazine,* vol. 98, no. 7, pp. 295–298 (July 1978), reprinted: Meissenburg, Winsen/Luhe. GHC BidevPlus-BCM-opt.pdf (right-hand edge of p. 298 cut off). KB2. 114

[64] [1978-Whyld] "Quotes & Queries, no. 3942: Arising from Professor Bidev's article [63] ...", by K. Whyld [Kenneth Whyld (1926–2003)], *The British Chess Magazine,* vol. 98, no. TBC, pp. 521–522 (month-TBC, 1978) GHC BidevPlus-BCM-opt.pdf [Includes "the" 8×8 Caïssan magic square. ... Bidev [62] says that the pioneering work was done ... by a London mathematician named Kesson who, under the pen-name Ursus, wrote a series of articles, "Magic squares, and Caïssan magic squares" in *The Queen,* ... Copy via PDL from BLL. 13, 14, 124, 126, 153

[65] [1979-Bidev/Wohin] "Wohin führt die zylindrische Verkreisung/Verkugelung des panmagischen Quadrates? - reductio ad absurdum der grossen panmagischen Idee!", by Pavle Bidev, pub. Bidev, Igalo[58], 1979. KB1-G06.

[56] *Bonus Socius* is the earliest known collection of chess problems, written in the 13th century by Nicolas de Nicolaï of Lombardy." Includes chess problem, c. 1266, from *Bonus Socius:* online at Puzzles: a chess lesson written by Joe Hurd. "The first composed chess problem was by the caliph Mutasim Billah of Baghdad around 840 A.D. The earliest known European collections of chess problems were copied at the English monasteries of Abbotsbury and Cerne Abbey in Dorset around 1250. In 1295 Nicholas de St. Nicholai wrote the *Bonus Socius,* the first great compilation of chess problems": online at Chess.com. Good companion (Bonus socius); XIIIth century manuscript collection of chess problems; Author: James F Magee; R. Biblioteca nazionale centrale (Florence, Italy). Publisher: Florence, Printed by the Tipografia Giuntina, 1910. OCLC 225173 TBC-ILL. Notes: "The manuscript has been wrongly attributed to Nicholaus St. Nicholai. cf. p. 6."

[57] Meindert Niemeijer (1902-1987) was een Nederlandse schaakcomponist, mecenas en verzamelaar van schaakliteratuur. Zijn bibliotheek vormde de basis van een van de grootste collecties van schaakboeken ter wereld. "[Dutch chess composer, generous patron and collector of chess literature. His library formed the basis of one of the greatest collections of chess books in the world.]" [324]

[58] Igalo is in Croatia, near Dubrovnik, and is where former Yugoslav leader Josip Broz Tito (1892–1980) had his summer villa. [324]

[66] [1980-Bidev/Schach-Nasiks] "Schach-Nasiks der Ordnungen vier, fünf, sieben, acht und neun : ihre elementare Schacheigenschaften, ohne eingehende An[a]lysen", by Pavle Bidev pub. Bidev, Skopje, 1980.

[67] [1981-G10/BidevCashmoreKritik] "Bedenken gegen Anzahlbestimmung von Schach-Nasiks 8×8: Kritik der Arbeit von M. Cashmore, Capetown 1907: Chess magic squares" [52] (in German: "Thiughts aaist the total counts of $8times8$ Nasik chess magic squares: critque of the work by M. Cashmore, Capetown 1907: Chess magic squares" [52]), by Pavle Bidev, typewritten manuscript (in German), 16 unnumbered pages, issued "Igalo YU', c. 1981. [Koninklijke Bibliotheek, 's-Gravenhage [Royal Library, The Hague] Request numberXSN 931. GHC G10/gegenCashmore.] KB1-G10. 46, 47, 54

[68] [1981-Bidev/panmagische]"Der panmagische Torus 8×8 und die panmagische Ebene 8×8: wer verdient den Vorzug für schach-magischquadratische Analysen? : eine theoretische Auseinandersetzung mit Prof. E. M. Bruins, Amsterdam," by Pavle Bidev, pub. Igalo, 1981.

[69] [1982-Bidev/hypermagics] "Magic and panmagic squares 8×8 explain the origin of chess : chess magics an unexplored field of hypermagics," by Pavle Bidev, pub. Bidev, Igalo, 1982. KB1-G09. In English!!

[70] [1986-Bidev/*Stammt*] *Stammt Schach aus Alt Indien oder China? Teil I, Deutsch, Teil II, Englisch = Did chess originate in China or India: chess and magic squares* by Pavle Bidev, Selbstverlag, Igalo 1986, iii, [i], 304, 88 pp. 302 pages in German and 88 pages in English. ["Expounds his theories about origin of chess related to 8×8 magic squares." [**?**]. OCLC 22604668, copy at the Cleveland Public Library and at the Koninklijke Bibliotheek, 's-Gravenhage [Royal Library, The Hague] Request number XSB 742. NA-ILL. KB2. Apparently includes (pp. 24, 46–48, 82–85]) reprint of Cashmore (1907) [52]. 55

[71] [1987-BCM/Bidev] "How old is chess?", by Pavle Bidev, *The British Chess Magazine,* vol. 107, no. 5, pp. 214–217 (May 1987). GHC BidevPlus-BCM-opt.pdf (right-hand edge of p. 216 cut off).

[72] [1996-Bidev/Pennell] "How magic squares explain the chess games of India and China", by Pavle Bidev, In *The Origin of Chess,* edited by Mike Pennell, Supplement to *The Chess Collector,* July 1996, pp. TBC. [Bidev died in 1988.]

9.5 Some other publications about Caïssan magic squares and related topics

[73] [1533-Agrippa] *De occulta philosophia libri tres,* by Henrici Cornelii Agrippae ab Nettesheym, pub. Soter, Köln, July 1533. TBC 139

[74] [1539-Cardano] *Practica arithmetice et mensurandi singularis : in qua que preter alias coninentur, versa pagina demonstrabit* by Hieronimi C. Cardani medico mediolanonsis. pub. Mediolani : Io. Antonins Castellioneus medidani imprimebat, impensis Bernardini Calusci, 312 pp., 1539. [McGill C266p 1539 (By Consultation) Osler.] TBC 32, 140

[75] [1567-Paracelsus] *Archidoxa Magica,* attributed to Paracelsus [Theophrastus Bombastus von Hohenhiem (1493–1541)]. TBC 32, 162

[76] [1691-La Loubère/*royaume*] *Du royaume de Siam, Tome Second: Contenant plusieurs Piéces détachées,* by [Simon] de La Loubère, Envoyé extraordinaire du ROY auprés du Roy de Siam en 1687 & 1688, pub. Chez La Veuve de Jean Baptiste Coignard, Imprimeur & Libraire ordinaire du Roy, et Jean Baptiste Coignard, Imprimeur & Libraire ordinaire du Roy, Paris, 1691. 116

[77] [1693-La Loubère/*Kingdom*] *The Kingdom of Siam,* by Simon de La Loubère, in English translated from French [76, (1691)], reprint edition, pub. Oxford University Press, 1986. [See vol. 2, pp. 227–247.] MMHC 15

[78] [1710-Sauveur] "Construction générale des quarrés magiques", by Joseph Sauveur, *Mémoires de Mathématique & de Physique de l'Académie Royale des Sciences de Paris,* Année MDCCX [1710], pp. 92–138. 69, 70

[79] [1763-Jones/Caïssa] "Caissa, or The game at chess, a poem" by Sir William Jones, written in the year 1763, In Sir William Jones's *Poems* [80, pp. 149–170]: GHC Jones-1772-Caissa. See also [257]. 7, 116, 129, 134, 140, 146

[80] [1772-Jones/*Poems*] *Poems, consisting chiefly of translations from the Asiatick languages, to which are added two essays: I. On the poetry of the Eastern nations, II. On the arts, commonly called imitative,* by Sir William Jones, pub. Oxford University Press, 1772: online (249 pp., 6.2 mb) at Google Books: GHC Jones-1772.pdf. See also [79, 257]. 116, 129, 134, 140

[81] [1776-Euler/quadratis magicis]"De quadratis magicis" (in Latin), by Leonhard Euler, Talk delivered to the St. Petersburg Academy of Sciences on October 17, 1776, and first published in *Leonhardi Euleri Commentationes arithmeticae collectae, auspiciis Academiae imperialis scientiarum petropolitanae*, vol. 2 (1849), pp. 593–602. [Eneström index no. E795. Reprinted in *Leonhardi Euleri Opera Omnia, Series I: Opera Mathematica,* vol. I.7, pp. 441–457. Online (with commentary and references) in *The Euler Archive*. Translated from Latin into English as "On magic squares" by Jordan Bell, 4 December 2004.] 67, 155

[82] [1777-Philidor] *Analyse du jeu des échecs,* by A. D. Philidor [François-André Danican Philidor (1726–1795)], nouvelle édition considérablement augmentée, London, 1777: online at Google Books, GHC PhilidorFrench. [First published in French in 1749 and in English in 1750: 6th English edition 1825 [87].] 65, 117

[83] [1779-Euler/quarrés magiques]"Recherches sur une nouvelle espèce de quarrés magiques" (in French), by Leonhard Euler, Read to the St. Petersburg Academy of Sciences on March 8, 1779, *Verhandelingen uitgegeven door het zeeuwsch Genootschap der Wetenschappen te Vlissingen*, vol. 9 (Middleburg 1782), pp. 85–239. [Eneström index no. E530. Reprinted in *Commentationes Arithmeticae*, 2 (1849), 302–361,

and in *Euleri Opera Omnia, Series I: Opera Mathematica,* vol. I.7, pp. 291–392. Translated from French into English as "Investigations on a new type of magic square" by Andie Ho & Dominic Klyve (working draft, 70 pp., ©2007), online open-access (with commentary and references) in *The Euler Archive.*] 67

[84] [1781-Murai] *Sampō Dōshimon* = *Sanpo Dojimon* (in Japanese), by Murai Chūzen (1781). [Title in English: *Mathematical Methods of Pupils' Questions.* Author is Murai Chūzen (1708–1797). Book first published 1781; reprinted by Kotensugakushoin, Tokyo, 1936.] 70

[85] [1803-Philidor/Jones] *Studies of Chess: containing Caïssa, a poem by Sir William Jones; a systematic introduction to the game; and the whole analysis of chess,* composed by Mr. A. D. Philidor, with critical remarks in two volumes, printed for Samuel Bagster, London, 1803: online at Google Books. GHC PhilidorPrattJones.pdf. [English translation, attributed to Peter Pratt (fl. 1799–1822), of *Analysis du jeu des échecs* [82]; 6th edition [87].]

[86] [1810-Jones/*Works*] *The Poetical Works of Sir William Jones : with the life of the author,* by Richard Dagley, pub. J. Nichols & Son, London, 1810. 1

[87] [1825-Philidor/Pratt] *Studies of Chess, containing a Systematic introduction to the game, and The analysis of chess,* by A. D. Philidor [François-André Danican Philidor (1726–1795)], 6th [English] edition, with very considerable additions, printed for Samuel Bagster, London, 1825: online at Google Books, GHC Philidor.pdf. [English edition (attributed in McGill catalogue MUSE to Peter Pratt, fl. 1799–1822) of *Analyse du jeu des échecs* [82] which first appeared in French in 1749 and in English in 1750. Appendix entitled "Two original problems and solutions", by F.P.H., pp. 535–536.] 65, 116, 117, 150

[88] [1826-*Dutzend*] *Ein Dutzend mathematischer Betrachtungen,* by Clemens Rudolph Ritter von Schinnern, pub. Geistinger, Vienna, 1826.

[89] [1830-Brewster] "Arithmetic: 1. Magic Squares," unsigned but by 'an able correspondent', in the Science section of *The Edinburgh Encyclopædia,* vol. 17, pp. 564–573 & Plate CCCCLXXXIV, edited by David Brewster, pub. William Blackwood, Edinburgh, 1830. 75, 140

[90] [1842-Shortreede/Shortrede] "On an ancient magic square, cut in a temple at Gwalior", by "Captain Shortreede" [*sic*], *Journal of the Asiatic Society of Bengal, New Series,* vol. 11, no. 124, pp. 292–293 (1842). [Author = Major-General Robert Shortrede (1800–1868) [100].] 58, 61, 63, 154

[91] [1844/1845-Newton] *A New Method of Ascertaining Interest and Discount, at various rates, both simple and compound, and of interest on indorsements: also a few magic squares of a singular quality,* by Israel Newton, of Norwich, Vt., pub. E. P. Walton and Sons, printers, Montpelier VT, 16 pp., 1845. [Author = Deacon Israel Newton (1763–1856) was [124, p. 228] "the inventor of the well-known medical preparations widely known as 'Newton's Bitters', 'Newton's Pills', etc, and sold extensively for many years throughout New England and New York". GHC from American Antiquarian Society Library, Worcester MA (via Haston Library, Franklin VT). Includes 6 magic squares (pp. 15–16: NewtonMagic-opt): "By I. Newton, Sept. 28, 1844, in the 82d year of his age." One magic square is 16×16 (2.4.15).] 32, 33, 121

[92] [1848-Beverley] "On the magic square of the knight's march", by William Beverley [*sic*], with an introduction by H. Perigal, Jun. [*sic*], *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, Third Series,* vol. 33, no. 220, pp. 101–105 (August 1848): GHC Beverley-complete. [Author may be William Roxby Beverley (c. 1814–1889). Reprinted in *The Chess Player's Chronicle,* vol. 9, pp. 344–347 (1848/1849) and in *Contributions to Kinematics,* edited by Henry Perigal, Junior, London, pp. 1–5 (1854). See also Beverly [112].] 92, 94, 95, 118, 120, 154

[93] [1848-Bezzel] "Vor einige Zeit wurden uns von einem Schachfreunde [Max Friedrich Wilhelm Bezzel (1824–1871)] zwei Fragen vorgelegt ...", *Schachzeitung der Berliner Schachgesellschaft,* vol. 3, p. 363 (1848). GHC Schach.pdf 79, 134, 135, 139

> "I. Wie viele Steine mit der Wirksamkeit der Dame können auf das im Uebrigen leere Brett in der Art aufgestellt werden, dass keiner den andern angreift und deckt,, und wie müssen sie aufgestellt werden?
>
> "SCHACHZEITUNG, 1848, 3rd year, Berlin, Veit & Comp. Max Lange's copy with the title page copied in his hand and boldly signed by him on endpaper before the title page. paper excellent, but pp 401 to 433 lacking. Content fine except for a few underlinings in colored pencil. US$495": online at Caissa Editions Bookstore Dale A. Brandreth, Box 151, Yorklyn, DE 19736 USA dbrandreth3 (at) comcast (dot) net.]

[94] [1849-Wenzelides] "Bemerkungen über den Rösselsprung", by "C. W." [Karl Wenzelides (1770–1852)], *Schachzeitung der Berliner Schachgesellschaft,* vol. 4, pp. 41–93 (1849). [Article is dated 1 October 1848.] TBC 66

[95] [1850-Nauck] "Briefwechseln mit allen für alle", by Franz Nauck, *Illustrirte Zeitung (Leipzig),* vol. 15, no. 377, p. 182 (September 21, 1850). GHC-TBC 79, 154

> "The eight-queens problem was posed again by Franz Nauck in the more widely read, *Illustrirte Zeitung (Leipzig),* in its issue of June 1, 1850. Four weeks later Nauck presented 60 different solutions. In the September issue he corrected himself and gave 92 solutions but he did not offer a proof that there are not more." [251, p. 269]

[96] [1850-Gauss/Schumacher] Correspondence between Gauß [Johann Carl Friedrich Gauß (1777–1855)], and Schumacher [Heinrich Christian Schumacher (1780–1850)], September 1–27, 1850: Gauß an Schumacher, Laufde. No. 1307 (No. d. Briefs 550), 1310 (551), 1312 (552); Schumacher an Gauß, 1308 (758), 1309 (759), 1311 (760). Reprinted in *Briefwechsel zwischen C. F. Gauss und H. C. Schumacher*, herausgegeben von C. A. F. Peters [Christian August Friedrich Peters (1806–1880)], Sechster Band [vol. 6], Druck von Gustav Esch, Altona, pp. 105–122 (1865). GHC Brief-best2. 143, 150, 152

> *Briefwechsel zwischen C. F. Gauss und H. C. Schumacher*, vol. 5 & 6 online at Hathi Trust; Vol. 1 online at Google Books: GHC Briefwechsel-v1 (19.9mb, 462 pp); Vol. 1 & 2 online at Google Books: GHC vol. 1 & 2: Brief12 (40.6 mb, 953 pp).

[97] [1854-Perigal/*Kinematics*] "On the magic square of the knight's march," by William Beverley [92, (1848)], In *Contributions to Kinematics,* edited by Henry Perigal, Junior, pub. London, pp. 1–5, (1854).

[98] [1860/1861-Bellavitis] "Disposizione sullo scacchiere di otto regine", by G. Bellavitis [Giusto Conte Bellavitis (1803–1880), "qui a donné aussi les 92 solutions", Lucas [110, p. 60], *Atti del Reale Istituto Veneto di Scienze, Lettere ed Arti, Serie 3,* vol. 6, p. 134 (March 1861); pp. 434–435 & vol. 15 (1869–1870), pp. 844–845 [Ahrens [121, v1/#244] . TBC-ILL/TBC-NAN/copy in Leiden! 139

[99] [1862-Jaenisch/v1] *Traité des applications de l'analyse mathématique au jeu des échecs, précédé d'une introduction à l'usage des lecteurs soit étrangers aux échecs, soit peu versés dans l'analyse,* vol. 1, by C. F. de Jaenisch [Carl Friedrich Andreyevich von Jaenisch (1813–1872)], Saint-Pétersbourg, 1862: online at Google Books, GHC Jaenisch-v1. 146

> "On trouve encore la solution complète du problème des huit reines dans le premier volume" [116, p. 214], see pp. 118–135 GHC Jaenisch-v1/118-135.

[100] [1868-Shortrede/obit] Unsigned untitled obituary of Major-General Robert Shortrede (1800–1866), *Monthly Notices of the Royal Astronomical Society,* vol. 29, pp. 120–121 (1868). 58, 117, 154

[101] [1869-Lionnet] "Question 963", by Lionnet, *Nouvelles annales de mathématiques, série 2,* vol. 8, p. 560 (1869): in McGill/RoseLib, GHC Lionnet-best. 148

> 963. 1° Écrire les n premiers nombres entiers 1, 2, 3,..., n, sur une même ligne, de telle sorte que la différence entre deux quelconques de ces nombres ne soit pas égale à celle de leurs rangs sur cette ligne;
>
> 2° Combien le problème admet-il de solutions?
>
> Sur un échiquier composé de n^2 cases, placer n reines de manière qu'aucune d'elles ne soit en prise par l'une des (n — 1) autres est la même question posée en d'autres termes. (LIONNET.)

Short statement of the n-queens problem in two parts: (1) Give a solution, and (2) How many solutions are there? Possibly the first such statement published. Author surely is François-Joseph Lionnet (1805–1884) [102], but maybe F. J. "E." Lionnet (E = Eustache ?) [261].

[102] [1869/1885-Lionnet-bio] "Notice sur la vie et les travaux de François-Joseph Lionnet", by A. Marre [Eugène Aristide Marre (1823–1918)], *Bullettino di Bibliografía e di Storia delle Scienze matematiche e fisiche, pubblicato da B. Boncompagni,* vol. 18, pp. 424–440 (1885). ISSN 1123-5209, JFM 18.0020.02. 119

[103] [1873/1874-Carpenter] "The eight queens problem, or how to place eight queens upon the board without being en prise", by Geo. E. Carpenter, *Brownson's Chess Journal, Dubuque (Iowa),* vol. 5, no. 35 ff. (1873 & 1874): see also Carpenter [120, (1900)]. [#305 in Ahrens [139, vol. 2].] TBC 121, 154

[104] [1874-Glaisher] "On the problem of the eight queens", by J. W. L. Glaisher [James Whitbread Lee Glaisher (1848–1928)], *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, Fourth Series,* vol. 48, no. 120, pp. 457–467 (December 1874), GHC Glaisher2. [Whole vol. 48 online at Google Books.] 121, 143

[105] [1874-Günther] "Zur Mathematisches Theorie des Schachbretts", by Siegmund Günther [Adam Wilhelm Siegmund Günther (1848–1923)], *Archiv der Mathematik und Physik*, vol. 56, pp. 281–292 (1874). online at Google Books: GHC Gun74a. 146

[106] [1874-Linde] *Geschichte und Litteratur des Schachspiels,* by Antonius van der Linde [Antonius van der Linde (1833–1897)], pub. Julius Springer, Berlin 1874, 2 vols. Reprinted by Olms, Zürich, 1981, 524 pp. ["Das Problem des Rösselsprungs", vol. 2, pp. 101–111 & "Caïssa" by Hermann Lehner (1873), vol. 2, p. 292. From Jelliss [280]: vol. 1, p. 245 (diagrams), pp. 292–295 (history), vol. 2, pp. 101–111 (bibliography), also pp. 337–338. : GHC Linde-scans3. 147, 148

[107] [1874-Pauls] "Das Maximalproblem der Damen auf dem Schachbrete", by E. Pauls, *Deutsche Schachzeitung, Organ für das Gesammte Schachleben,* vol. 29, pp. 129–134, 257–267 (1874). 134, 154

[108] [1876-Günther] *Vermischte Untersuchungen zur Geschichte der mathematischen Wissenschaften,* by Siegmund Günther, pub. B. G. Teubner, Leipzig, 1876; reprinted: Sändig, Wiesbaden, 1968: "Historische Studien über die magischen Quadrate, ch. 4, pp. 188-270: GHC Guenther-BTS.

[109] [1879-Firth/*Solutions*] *Solutions to the mathematical questions in the examination for admission to the Royal Military Academy,* by William Firth, BA (Cambridge) [*sic*], pub. Harington, London, 38 pp., 1879. 91

[110] [1882-Lucas/*Recréations*] *Recréations mathématiques,* by Édouard Lucas [François Édouard Anatole Lucas (1842–1891)], pub. Gauthier-Villars, Paris, xxv + 258 pp., 1882: online at Google Books, GHC Lucas-best2 (8mb, 304 pp). "Quatrième recréation: Le problème des huit reines au jeu des échecs, pp. 58–86. GHC Lucas-best. 118, 148

[111] [1887-Firth/*Magic Square*] *The Magic Square,* by W. A. [*sic*] Firth, printed by R. Carswell & Son, Belfast, 19 pp. & 3 folded sheets in pocket inside back cover (including "The Magic Chess Board, invented by W. Firth, dedicated to Dr. Zukertort, pub. Jas. Wade, Covent Garden, 1887. (See Figure TBC.) [Title page says "W. A. Firth, B. A., Cantab., late scholar of Emmanuel College, Cambridge, and mathematical master of St. Malachy's College, Belfast."] 88, 91, 153, 154

[112] [1889-Beverly/Beverley] "The greatest magic square extant" by William Beverly [*sic*], *The Bizarre: Notes and Queries*, vol. 6, no. 1, pp. 224–225 (January 1889): GHC Beverley-complete. [Same 8×8 semi-magic square as given by Beverley [92, (1848)]. Author may be William Roxby Beverley (c. 1814–1889).] 93, 117, 154

[113] [1890-Firth/will]: "The will of William A. Firth, late of Belfast, Mathematical Professor, who died on 12 September 1890 at same place was proved at Belfast by Margaret Firth of Hamilton-street Belfast, Widow, one of the Executors." Retrieved online from the Department of Culture, Arts and Leisure, Public Record Office of Northern Ireland, 66 Balmoral Avenue, Belfast BT9 6NY, Northern Ireland: online access not available on 6 May 2011. 91

[114] [1891-*Lexikon*] *Biographisches Lexikon des Kaiserthums Oesterreich,* edited by by Constantin von Wurzbach, Der große österreichische Hausschatz 2, pub. Zamarski, Vienna, 1856–1891. 66

[115] [1895-Cavendish/*Whist Table*] *The Whist Table: A Treasury of Notes on the Royal Game,* by "Cavendish," C. Mossop, A. C. Ewald, Charles Hervey, and other distinguished players, to which is added "Solo Whist and tsp rules", by Abraham S. Wilks, the whole edited by "Portland" (J. Hogg), pub. Charles Scribner's Sons, New York, 1895: online Google eBook. GHC WhistTable.pdf. [Frontispiece: Portrait of "Cavendish".] 141

[116] [1895-Lucas/*amusante*] *L'arithmétique amusante,* by Édouard Lucas, pub. Gauthier-Villars, Paris, viii + 266 pp., 1895. [MMHC QA95 L93. Reprinted in Elibron Classics series, ©Adamant Media Corporation, 2006, ISBN-13: 978-0543941589 pbk: Note IV §I Les huit dames *eq seq.*, pp. 210–260. Also reprinted by Kessinger Publishing, LLC (June 2, 2008) ISBN-13: 978-1436639804 amazon.com US$32.64, and by Nabu Press (May 14, 2010) ISBN-13: 978-1149426692 amazon.com US$21.85. Also online at Google Books (6.4mb, 284 pp.) GHC Lucas-1895/amusante.] 118, 148

[117] [1897-McClintock] "On the most perfect forms of magic squares, with methods for their production", by Emory McClintock [John Emory McClintock (1840–1916)], *American Journal of Mathematics,* vol. 19, no. 2, pp. 99–120 (1897): online at JSTOR, GHC McClintock-1897. 17, 22, 23, 25

[118] [1899-Perigal/obit] "Obituary notice: Henry Perigal", *Monthly Notices of the Royal Astronomical Society,* vol. 59, pp. 226–228 (1899): online at the Royal Astronomical Society. 150

[119] [1899-Sprague] "On the eight queens problem", by T. B. Sprague [Thomas Bond Sprague (1830–1920)], *Proceedings of the Edinburgh Mathematical Society,* vol. 17 (Seventeenth Session, Fourth Meeting, February 1899), pp. 43–68: online at Cambridge University Press: GHC Sprague2. 121, 152

[120] [1900-Carpenter] “On the N queens problem, Or how to place N queens on a board of N squares on a side so that no queen shall interfere with the action of any other” (published in 7 parts: February–September 1900, 50 pp.), by Geo. E. Carpenter (signed: Tarrytown, New York) [George Edward Carpenter (b. 1844)], *The British Chess Magazine,* vol. 20, no. 2, pp. 42–48 (February 1900); no. 4, pp. 133–137 (April 1900); no. 5, pp. 181–183 (May 1900); no. 6, pp. 223–225 (June 1900); no. 7, pp. 264–267 (July 1900); no. 8, pp. 300–304 (August 1900); no. 9, pp. 344–364 (September 1900): GHC Carpenter-best, online at Google Books: GHC BCM-v20. [Builds on [104, 119]. see also Carpenter [103, (1873/1874)]. Portrait in *Brentano's Chess Monthly,* TBC.] 119, 154

> **1643 George Edward Carpenter,** son of Ward,[1096] was born at Tarrytown, N. Y., March 25, 1844, and grew up in his father's office, where he fitted himself for the profession in which he is now engaged. For several years he taught school in the village, and then became partner with his father as civil engineer and surveyor under the firm name of Ward Carpenter & Son. He is familiar with every acre of the old town, having surveyed and laid out the greater part of it. The maps accompanying this work are from his hands, and to him we are indebted for the genealogy of his particular family. Mr. Carpenter is a *bachelor*, *wedded* to his hobby of chess, in which he has attained the position of “champion.” See Bretano's Chess Monthly, which also has his portrait.

[121] [1901-Ahrens/v1] *Mathematische Unterhaltungen und Spiele* [Band I], by W. Ahrens [Wilhelm Ernst Martin Georg Ahrens (1872–1927)], pub. B. G. Teubner, Leipzig, 1901: MMC QA95 A28 (GBC). Reprinted by Nabu Press, 412 pp., March 15, 2010, ISBN-13 978-1147308495. GHC Ahrens/v1-1901-BTS. 118, 154

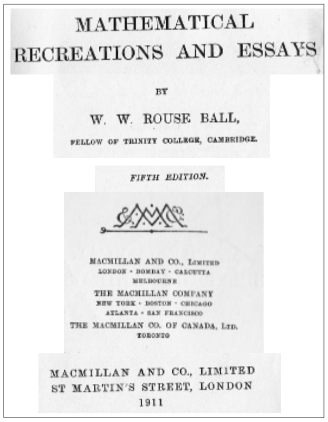
[122] [1901-IHWYST] “Sacred magic square of seven, or The seal of Jesu-Maria, the XT. Sigillium XTI”, by “I. H. W. Y. S. T.”, from the notes on “The Gospel of the Twelve”, *Altruism and Idealism, Notes and Queries: A Monthly of History, Folk-Lore, Mathematics, Literature, Art, Arcane Societies, Etc.,* vol. 19, p. 257 (1901).

[123] [1902-MacMahon] “Magic squares and other problems upon a chess-board,” by P. A. MacMahon [Percy Alexander MacMahon (1854–1929)], *Nature,* vol. 65, no. 1689, pp. 447–452 (March 13, 1902): online at the Nature Publishing Group, GHC MacMahon-Nature.pdf. Reprinted in [ch. 19, paper [60], pp. TBC][178] TBC-ILL (v2). See also [126] 110, 122, 149

Original article unsigned but a footnote on p. 447 says “A discourse delivered at the Royal Institution on Friday evening, February 14, by Major P. A. MacMahon, F.R.S.”

[124] [1905-*Norwich*] *A History of Norwich, Vermont* (Published by Authority of the Town) with portraits and illustrations, by M. E. Goddard [Merritt Elton Goddard] & Henry V. Partridge [Henry Villiers Partridge], pub. The Dartmouth Press, Hanover NH., 276 pp., 1905: online at Google Books, GHC (335 pp., 7.2mb) A_history_of_Norwich_Vermont.pdf. [Norwich University is a private university now located in Northfield VT. The university was founded in 1819 at Norwich VT as the American Literary, Scientific and Military Academy by military educator and former superintendent of West Point, Captain Alden B. Partridge (1785–1854), the father of co-author Henry Villiers Partridge (1839–1920) and a contemporary of Israel Newton (1763–1856) [91].] 33, 117, 150, 154

[125] [1905-Ball] *Mathematical Recreations and Essays*, 4th edition, by W. W. Rouse Ball, pub. Macmillan, London, xvi+ 388 pp., 1905. [Translated into French [?]. 1st & 2nd editions: *Mathematical Recreations and Problems of Past and Present Times*, February & May 1892 [?]; 3rd–10th editions: *Mathematical Recreations and Essays*, 1896–1937 [129, 145]; 11th–13th editions (with H. S. M. Coxeter), 1939–1987 [152, 163, ?]; see also Singmaster [221].] 125, 126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FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

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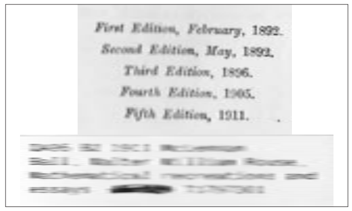
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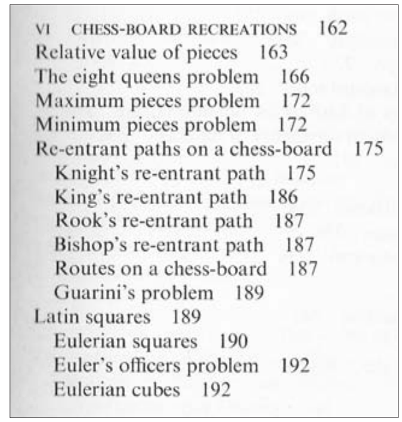
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[126] [1906-MacMahon] "Magic squares and other problems on a chessboard," by P. A. MacMahon, *Proceedings of the Royal Institution of Great Britain,* vol. 17, pp. 60–63 (1906). JFM 37.0994.08. Reference also given as vol. 17, no. 96, pp. 50–61 (4 February 1892) by Marder [147, p. 3] & Swetz [242, p. 200] and as vol. 17, pp. 50–63 (1902), Koninklijke Bibliotheek, 's-Gravenhage [Royal Library, The Hague]-Request number360 D 179. less likely since Vols. 13-24, 1890/92-1925; see also [123]. Not included in the "The Bibliography of the Mathematical Papers of Percy Alexander MacMahon" [169, pp. xix–xxix].] 121, 149

[127] [1910-Bennett] "The eight queens problem", by G. T. Bennett [Geoffrey Thomas Bennett (1868–1943)], *The Messenger of Mathematics,* vol. 39, p. 19–21 (1910): MRC, GHC Bennett-best. 123, 139

> "It is proved that it is impossible to combine on the same board more than six sets of queens satisfying the conditions of the problem." See also [132].

[128] [1910-Bergholt] "The magic square of sixteen cells: a new and completely general formula", by Ernest Bergholt [Ernest Bergholt (1856–1925)], *Nature: A Weekly Illustrated Journal of Science*, vol. 83, no. 2117, pp. 368–369 (May 26, 1910). 71, 154

[129] [1911-Ball] *Mathematical Recreations and Essays,* by W. W. Rouse Ball, 5h edition, pub. Macmillan, London, xvi + 492 pp., & "Errata et Addenda", 2 pp. [First edition, 1892; Second 1892; Third 1896; Fourth 1905. Ch. VI Chess-board recreations, pp. 109–136 (The eight queens problem, pp. 113–119; this chapter added since the Fourth edition. MMHC QA95 B2. See also 13th edition [164].] 70, 122, 125, 126, 139

[130] [1913-Mikami] *The Development of Mathematics in China and Japan,* by Yoshio Mikami, Abhandlungen zur Geschichte der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen begruendent von Moritz Cantor, XXX Heft. B. G. Teubner, Leipzig; G. E. Stechert & Co., New York; William & Norgate, London. [Author = Yoshio Mikami (1875–1950). Reprinted by Chelsea, New York, 1961 & Martino Publishing, Mansfield Centre, Connecticut, 2004. "The 2nd (reprint) edition (Chelsea 1974) consists of the complete text of the first edition, 1913, with an appendix on Soroban Calculation, by Rikitaro Fujisawa, first published in 1912."] 70, 149

[131] [1913-Murray] *A History of Chess,* by H. J. R. Murray [Harold James Ruthven Murray (1868–1955)], pub. Oxford University Press, 1913 (ISBN-13: 978-0198274032: used copy: amazon.com US$44.95);

reprinted 1962; for Oxbow Books, Oxford [2002?], ISBN 0-19-827403-3) Concordia University Library GV1317 M8 2002, GHC) & by Benjamin Press (1985, 900 pp., ISBN-13: 978-0198274032 ISBN-13: 978-0936317014). [For Caïssa, see pp. 793, 874.] 10, 149

[132] [1914-Gosset] "The eight queens problem", by Thorold Gosset, *The Messenger of Mathematics,* vol. 44, p. 48 (1914): MRC, GHC Gosset-best. 122, 154

> "Four years ago [127] proved that it is impossible to combine on the same board more than six sets of queens satisfying the conditions of the problem. A much simpler proof of this may be of interest."

[133] [1916-Kohtz] "Von der Erfindung des Schachspiels," by Johannes Kohtz (1843–1918), In *Handbuch des Schachspiels,* by Paul Rudolf von Bilguer (1815–1840) & Tassilo von Heydebrand und der Lasa, edited by Carl Schlechter, 8th edition, Olms, Leipzig, pp. 17–61. 1916. [Reprinted 2008 by Ishi Press ISBN-13 978-0923891404] ILL # 590823, pdg. 139, 146

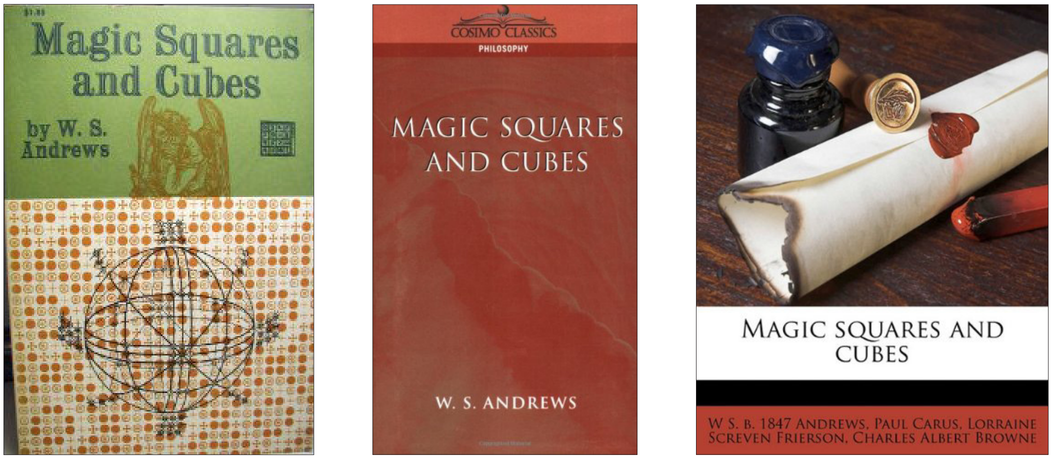
> This is probably the most famous chess book ever written and is certainly the most influential chess book ever written. Paul Rudolf von Bilguer (1815–1840) was a German chess master and chess theoretician from Ludwigslust, Mecklenburg-Schwerin. He was regarded as one of the strongest chess players in the world when he lived. Tragically, Bilguer died on 16 September 1840 just before reaching his 25th birthday, so his great promise was never realized except though this book. Von Der Lasa continued his work and it was published in 1843. Bilguer is listed as the sole author and Von Der Lasa appears only as the author of the Introduction, but it is widely believed that much of the work was by Von Der Lasa, especially because of other contributions by Von Der Lasa to chess literature later on.

[134] [1916-Woodruff/Editor] "Four-ply pandiagonal associated magic squares", by "Editor" [Paul Carus], *The Monist: A Quarterly Magazine Devoted to the Philosophy of Science,* vol. 26, pp. 315–316. ["Frederic A. Woodruff has sent us three original magic squares, ...; apparently not reprinted in *Magic Squares and Cubes* WSA-1917.] 13, 74, 77

[135] [1917-Andrews] *Magic Squares and Cubes,* by W. S. Andrews, with chapters by other writers. Second edition, revised and enlarged, pub. Open Court, Chicago (1917): online (428 pp., 24.7 mb, GHC WSA2-1917-Moews) at David Moews's home page (scan courtesy of Paul C. Moews). Reprinted by Dover (viii + 419 pp., 1960, GHC), Cosimo Classics (viii + 419 pp., 2004, GHC). Revised version of [**?**, (1908)]. 13, 15, 75, 76, 95, 110, 111, 112, 124, 139, 154

> "Author" is William Symes Andrews (1847–1929); "other writers", in addition to Charles Planck, are Charles Albert Browne, Jr., (1870–1947), Paul Carus (1852–1919), Lorraine Screven Frierson (1861–1936)[59], H. M. Kingery, D. F. Savage, Harry A. Sayles, Frederic A. Woodruff (b. 1855), and John Worthington.

[59] Lorraine Screven Frierson (1861–1936) was a noted conchologist but we know very little about the other "other writers".



Magic Squares and Cubes, outside front cover from reprints by
(left to right) [135]: Dover (1960), Cosimo Classics (2004). [**?**]: Nabu (2010).

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[136] [1917-Dudeney] *Amusements in Mathematics*, by H. E. Dudeney [Henry Ernest Dudeney (1857–1930)], Thomas Nelson, London, 1917. [Reprinted by Dover, New York, 1958, 1970 GHC. "Magic Square Problems", pp. 119–127. Problem 300 "The eight queens", pp. 89, 215; "Magic square probems", pp. 119–121 (includes 12 Dudeney types of 4×4 magic squares (p. 120) from *The Queen.*

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[137] [1917-Woodruff/Andrews] "The construction of ornate magic squares of orders $8, 12$ and 16 by tables", by "F. A. W." [Frederic A. Woodruff], In *Magic Squares and Cubes* [135, pp. 390–404]. [Apparently not published in *The Monist.*] 13, 74, 75, 76, 78

[138] [1918-Foster] "Puzzles of arrangement: mental calistenics for the T.B.M. and the W.R.W. (War-Racked-Woman)", by R. F. Foster, *Vanity Fair,* vol. 9, no. 5, pp. 65 & 90 (March 1918). 13

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[143] [1929-Andrews/obit] "W. S. Andrews dies; an Edison pioneer", *The New York Times,* July 2, 1929, p. 22: online at ProQuest Historical Newspapers *The New York Times* (1851–2007), GHC CPK-WSA-obit. 109, 141

[144] [1930-Kraitchik] *La mathématique des jeux ou Récréations mathématiques,* by M. Kraitchik [Maurice Kraitchik (1882–1957)], pub. Imprimerie Stevens Frères, Brussels, 1930. [Translated into English and revised as Kraitchik [173]. Borrowed via ILL from Queen's University, Kingston, Ontario.] 127

[145] [1937-Ball] *Mathematical Recreations and Essays,* 4th reprint of the 10th edition, pub. Macmillan, New York, 1937 & Kessinger Publishing, Whitefish MT, 2004, xiii + 380 pp. [10th edition first published: 1937. Chapter VII (pp. 137–161) "Magic Squares" reprinted as [209]. 1st & 2nd editions: *Mathematical Recreations and Problems of Past and Present Times*, February & May 1892 [**?**]; 3rd–10th editions: *Mathematical Recreations and Essays*, 1896–1937 [125, 129]; 11th–13th editions (with H. S. M. Coxeter), 1939–1987 [152, 163, **?**]; see also Singmaster [221].] 70, 122, 130, 139

[146] [1938-*Scripta*] "Gauss's arithmetization of the problem of a Queen", by Jekuthiel Ginsburg, *Scripta Mathematica: A Quarterly Journal Devoted to the Philosophy, History, and Expository Treatment of Mathematics,* vol. 5, pp. 63–66 (January 1938). ["The poor man got the benefit of a crumb inadvertently dropped from the table of the Prince."]

[147] [1940-Marder1] *The Magic Squares of Benjamin Franklin: the first of a series of four papers describing the technique of Leonhard Euler applied to the Lahireian method of forming magic squares of all sizes under the general title "The Intrinsic Harmony of Number", Three entirely new methods of producing "The Bent Diagonals of Benjamin Franklin",* by Clarence C. Marder, pub. Edmond Byrne Hackett, The Brick Row Book Shop, New York, 1940. GHC: MarderKA1-opt. 13, 15, 122, 142

[148] [1941-Marder2] *Magic squares of the Fifth and Seventh Orders, a new application of the method used by Claude Gaspar Bachet de Mezeriac : the second of a series of four papers describing the technique of Leonhard Euler applied to the Lahireian method of forming magic squares of all sizes under the general title "The Intrinsic Harmony of Number",* by Clarence C. Marder, pub. Edmond Byrne Hackett, The Brick Row Book Shop, New York. 1941. GHC: MarderKA2-opt.

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[151] [1950-Schwerdtfeger] *Introduction to Linear Algebra and the Theory of Matrices* by Hans Schwerdtfeger, pub. P. Noordhoff N. V., Groningen, 280 pp., 1950; 2nd edition: 289 pp., 1961 (see pp. 130–131). 29, 152

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[153] [1956-Heath/*Euclid*] *The Thirteen Books of Euclid's Elements,* translated from the text of Heiberg, with introduction and commentary by Sir Thomas L. Heath, Second Edition [unabridged], revised with additions, vol. I, Introduction and Books I, II, pub. Dover, New York, 1956. GHC 62, 141

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[159] [1970-Dudeney] *Amusements in Mathematics,* Henry Ernest Dudeney, [Second] reprint edition. Dover, New York, 1970. 68, 141

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[172] [1981-GarnerHerzberg] "On McCarty's queen squares", by Cyril W. L. Garner & Agnes M. Herzberg, *The American Mathematical Monthly*, vol. 88, no. 8, pp. 612–613 (1981). online at JSTOR: GHC GarnerAgnes.

[173] [1981-Kraitchik] *Mathematical Recreations*, 2nd revised edition, by Maurice Kraitchik (1882-1957), pub. Dover, New York, 328 pp., 1981 GHC. [Original version in French: [144]. Original English version: W. W. Norton, New York, 1942; reprinted by Dover: 1953, 1981.] 125

> "Ranging from ancient Greek and Roman problems to modern applications and techniques, this book features 250 lively puzzles and problems, with solutions. Both beginners and advanced mathematicians will appreciate its variety of numerical pastimes, which include unusual historic problems from medieval European, Arabic, and Hindu sources."

[174] [1982-Oshaw/order4] "Magic squares of order four[60]", by [Dame] Kathleen Ollerenshaw & [Sir] Hermann Bondi, *Proceedings of the Royal Society, London, Series A: Mathematical, Physical and Engineering Sciences*, 306 (1495), 443–532 (1982). [GHSC] 139, 147, 148, 149

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[176] [1984-Pólya/Collected-v4] *George Pólya: Collected Papers, Volume IV: Probability; Combinatorics; Teaching and Learning in Mathematics,* edited by Gian-Carlo Rota, Mathematicians of Our Time, vol. 22, pub. MIT Press, 1984. MSHC QA3 P73 v.4 124, 127

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[60]This remarkable paper presents a history of the work on 4×4 magic squares. Over 300 years ago Frénicle listed all 880 of the 4×4 magic squares, which he found by exhaustive search. (One supposes that besides exhausting the possibilities, Frénicle was a bit exhausted, too.) Besides presenting a history, this paper also presents a much more analytical construction of the squares. In fact, two different methods are used. The paper concludes with a complete list of the 880 magic squares set down in an order that takes structure into account. [Review by D. A. Klarner, MR703622 (84i:05031)].

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> "This paper currently (April 16, 2010) contains 324 references (originally in BibTeX format) to articles dealing with or at least touching upon the well-known n-queens problem.' The literature is not totally clear about the exact article in which the n-queens problem is first stated, but the majority of the votes seems to go to [93]."

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(31 May 2010), and in press for publication in its *Proceedings* (Antonella Cupillari, ed.), preprint (25 pp., with images in colour) online at McGill, file last edited on 31 October 2010. [ISSN 0825-5924. Talk and article builds upon the two articles by Loly & Styan in *Chance* [258, 259].] 4, 136, 148

[270] [2010/2011-Split] "Magic Moore–Penrose inverses and philatelic magic squares with special emphasis on the Daniels–Zlobec magic square", by Ka Lok Chu, S. W. Drury, George P. H. Styan & Götz Trenkler, *Croatian Operational Research Review,* vol. 2, pp. 4–13 (2011): online at McGill (with images in colour; complete vol. 2: online in Zagreb. [Paper based on the invited talk presented (by George P. H. Styan) in the Session devoted to the 70th birthday of Professor Sanjo Zlobec at the 13th International Conference on Operational Research (KOI-2010) Split, Croatia, 30 September 2010. Extended beamer file (including 3×3 Tesla PMS) modified on 28 October 2010, 27 pp., with images in colour): online at McGill. Talk and article build upon the three articles by Loly & Styan [258, 259, 269].] 4, 153

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[280] [2011-Jelliss/History] "History of Magic Knight's Tours," by George Peter Jelliss: online at Mayhematics, 12 pp., GHC Transferred to this 'Mayhematics' site February–March 2011. 66, 94, 95, 119, 146

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Caïssan squares: the magic of chess

George P. H. Styan[1]

[1] *McGill University, Canada, email:styan@math.mcgill.ca*

Keywords: alternate couplets property, bibliography, Caïssa, EP, 4-ply, involution-associated magic matrices, involutory matrix, knight-Nasik, magic key, most-perfect, pandiagonal, philatelic items, postage stamps, rhomboid, “Ursus”.

We study various properties of $n \times n$ Caïssan magic squares. A magic square is Caïssan whenever it is pandiagonal and knight-Nasik, so that all paths of length n by a chess bishop are magic (pandiagonal) and by a (regular) chess knight are magic (CSP2-magic).

Following the seminal 1881 article [4] by “Ursus” in *The Queen,* we show that 4-ply magic matrices, or equivalently magic matrices with the “alternate-couplets” property, have rank at most equal to 3. We also show that an $n \times n$ magic matrix $\mathbf{M}$ with rank 3 and index 1 is EP if and only if $\mathbf{M}^2$ is symmetric. We identify and study 46080 Caïssan beauties—Caïssan magic squares which are also CSP3-magic; a CSP3-path is made by a special knight that leaps over 3 instead of 2 squares. We find that just 192 of these Caïssan beauties are EP. We generalize an algorithm given by Cavendish [2:(1894)] for generating Caïssan beauties and find these are all EP. We also study the n-queens problem first posed with $n = 8$ by Bezzel [1:(1848)] and the Firth–Zukertort “magic chess board” due to Firth [3:(1887)].

An extensive annotated and illustrated bibliography of over 300 items, many with hyperlinks, ends our report. We give special attention to items by (or connected with) “Ursus”: Henry James Kesson (b. c. 1844), Andrew Hollingworth Frost (1819–1907), Charles Planck (1856–1935), and Pavle Bidev (1912–1988). We have tried to illustrate our findings as much as possible, and whenever feasible with images of postage stamps or other philatelic items.

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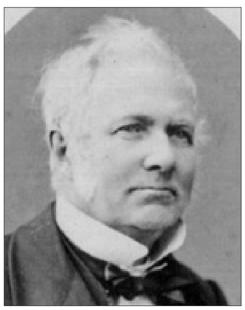
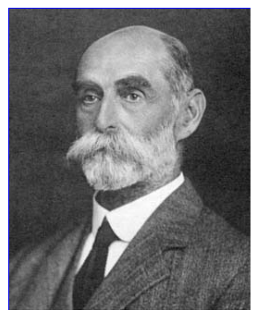
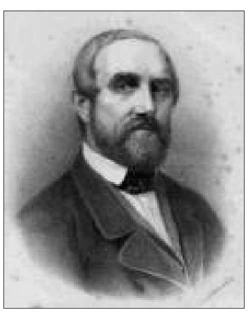
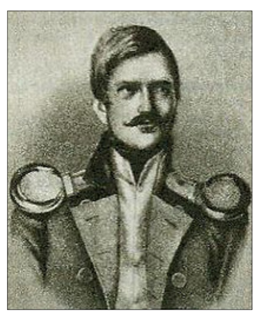
9.6 Portrait Gallery: Agrippa–Bondi







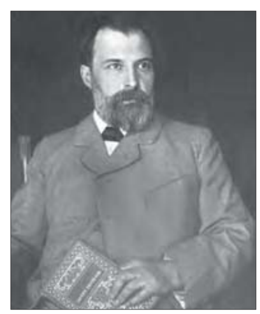

Agrippa [73], Andrews [135], Ball [129, 145, 152, 164]
Bellavitis [98], Ben-Israel [206], Bennett [127], Benson [165, **?**],
Bezzel [93, 204], Bilguer [133], Bondi [174].

Portrait Gallery: Boyer–Carus



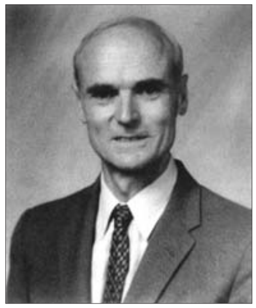


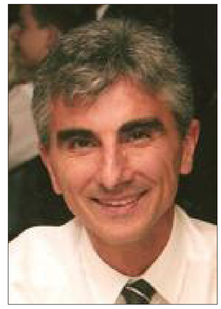


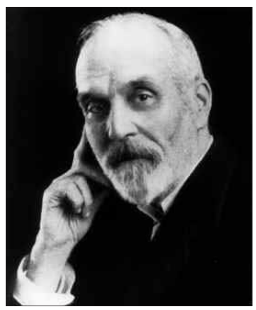


Boyer [51, 32], Brée [196, 197, 198], Brewster [89], Browne [**?**],
Caïssa (3) [7, 79, 80, 257],
Calvo [186, 207], Cardano [74], Carus [**?**].

Portrait Gallery: "Cavendish"–Euclid

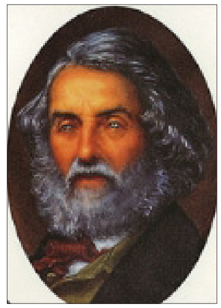


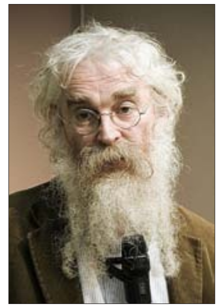


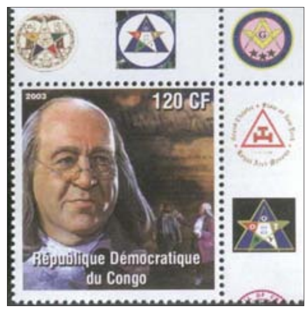


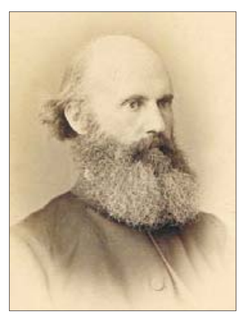
"Cavendish" [Henry Jones] [11, 115, 211], Coxeter [161, 164], Demirörs [182, 183], Dickins [162], Donkersteeg [261], Dudeney [159], Edison [143]
Euclid (2) [153].

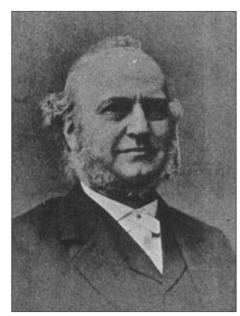
Portrait Gallery: Finkel–Percival Frost





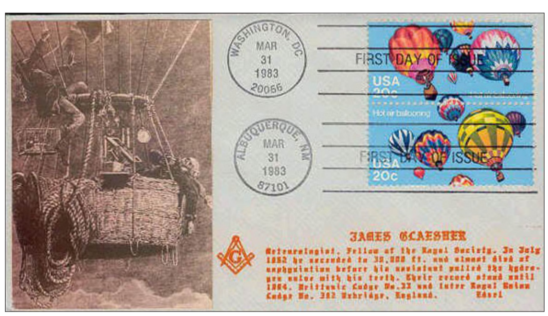






Euler [§9.7.1], Everest [TBC], Finkel [231, 10], Fox [154], Frankenstein [36], Franklin (2a) [147, 241], Franklin (2b), Andrew Frost [§9.2], Percival Frost [30].

Portrait Gallery: Gauß–Golombek



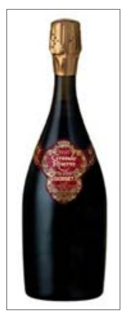
Gauß (4) [96],
Glaisher (2) [104], Golombek

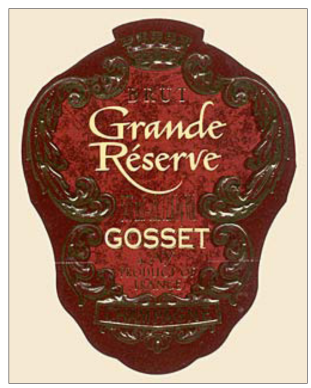
Portrait Gallery: Gosset

[1914-Gosset] "The eight queens problem", by Thorold Gosset,
The Messenger of Mathematics, vol. 44, p. 48 (1914): MRC, GHC Gosset-best.

"Thorold Gosset (1869–1962) was an English lawyer and an amateur mathematician. In mathematics, he is noted for discovering and classifying the semiregular polytopes in dimensions four and higher. According to Coxeter [161] after attaining his law degree in 1895 and having no clients, Thorold Gosset amused himself by attempting to classify the regular polytopes in higher dimensional (greater than three) Euclidean space" [324]

"William Sealy Gosset (1876–1937) is famous as a statistician, best known by his pen name 'Student' and for his work on Student's t-distribution. Born in Canterbury, England to Agnes Sealy Vidal and Colonel Frederic Gosset, he attended Winchester College before reading chemistry and mathematics at New College, Oxford. On graduating in 1899, he joined the Dublin brewery of Arthur Guinness & Son." [324]





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Gosset Grand Rosé Brut Champagne Sparkling wine, Champagne, 375 ML, 11154486	$48.00
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Gosset Celebris Brut rosé Cuvée du Millénaire Champagne 1995 , Code SAQ : 0

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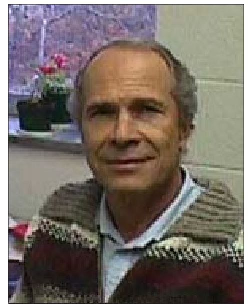
William Sealy Gosset (top left panel) and some
holdings (December 2010) at the SAQ in Montréal of Gosset champagne.

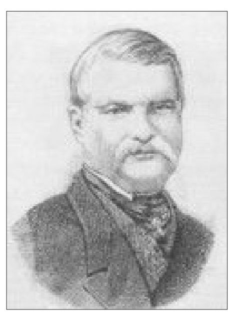
"*Champagne Gosset,* founded in 1584, is one of the oldest champagne houses of the Champagne region in north-eastern France. It was founded when Jean Gosset, a grape grower in Aÿ, left a vineyard to Pierre Gosset who began to export wine under his name. Typical for this era in Champagne, Gosset initially produced still wines, mainly reds. Today's Gosset incorporates a winery in Aÿ which belonged to King Francis I of France, who enjoyed these red Aÿ wines." [324]

Portrait Gallery: Günther–Sir William Jones

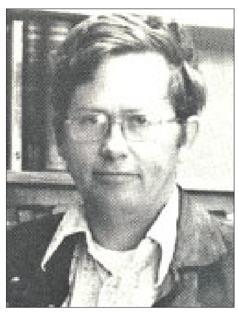






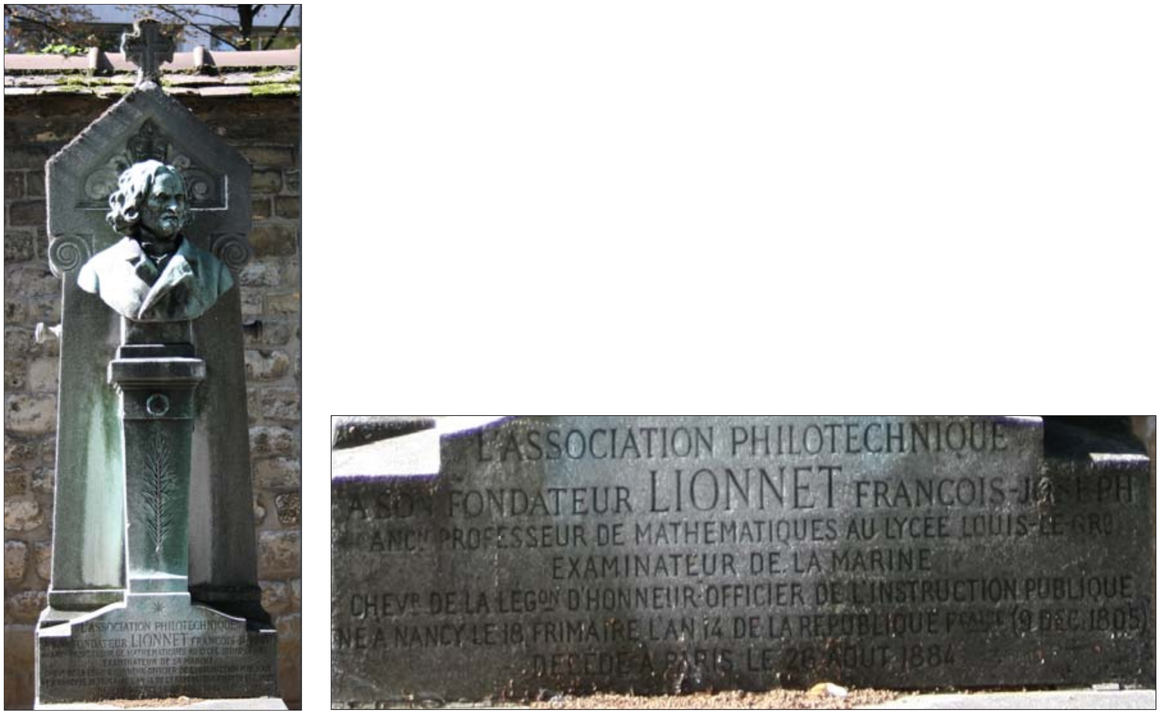
Golombek [168], Günther [105], Hartwig [246],
Heindorff [226, 301, 302], Heinz [199], Hendricks [199], Heydebrand & Lasa [133],
Jacoby [165, **?**], de Jaenisch [99], Jelliss [278, 280], Sir William Jones [79].

Portrait Gallery: Klarner–Linde

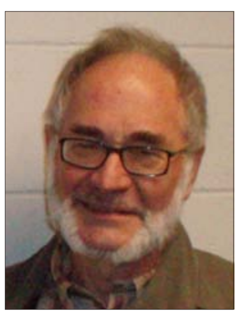


Karpov [§8.5] Kasparov [§8.5], Klarner [177, 174], Kosters [261], Laws [36], Lincoln Imp [16], Linde [106],

Portrait Gallery: Lionnet–Lucas



Gravestone in the Cimetière Montparnasse, Paris: online photograph by G. Freihalter, August 20, 2010.

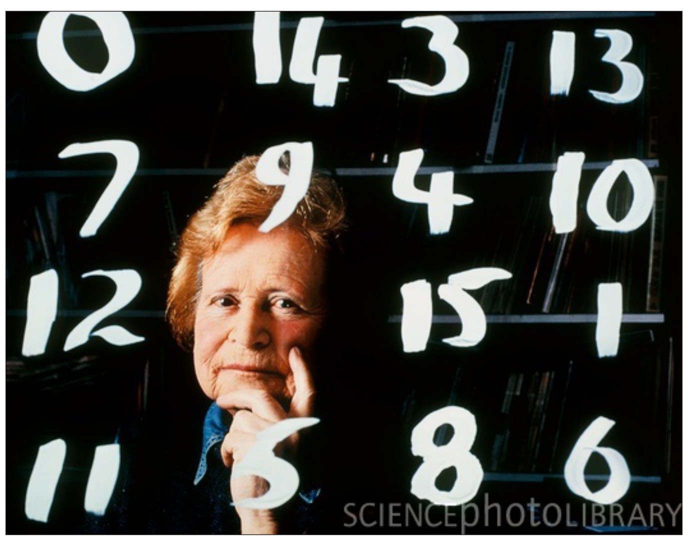


Karpov [§8.5] Kasparov [§8.5], Klarner [177, 174], Kosters [261],
Laws [36], Lincoln Imp [16], Linde [106],
Lionnet [101], Loly [223, 235, 248, 258, 259, 269], Lucas [110, 116].

Portrait Gallery: MacMahon–Ollerenshaw

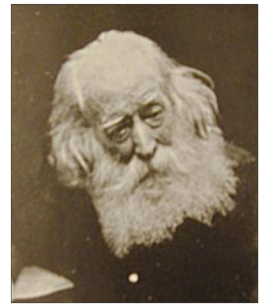


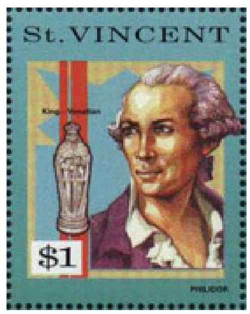


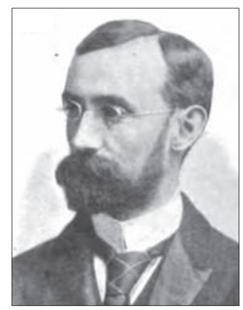


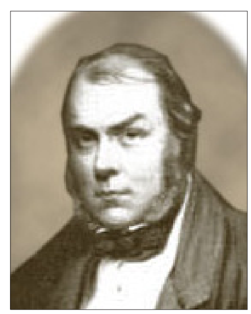
MacMahon [123, 126, 169], Mikami [130], Moler [184, 217],
Mondrian (2) [§9.7.2], Murray [131], Niemeijer [62],
Ollerenshaw [174, 196, 197, 198].

Portrait Gallery: Partridge–Quekett



Paracelsus [$9.7.3], Partridge [124], Pasles [241], Perigal [118, 201], Peters [96], Philidor [87], Pickover [208], Planck [§9.3], Pohle [264], Pólya [141], Quekett [§8.1].

Portrait Gallery: Ray and *The Chess Players*

[1977/ChessPlayers] *Śhatrañj Ke Khilārī' = The Chess Players,* movie, written and directed by Satyajit Ray, based on Munshi Premchand's short story of the same name, 1977. Illustrated flyer online at the Satyajit Ray Society, GHC CPK-25a.

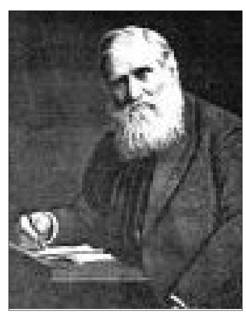
Available on DVD (Kino on Video, New York, 2006) at McGill PN1997 S18787, GHC: "The short-story irony of two nawabs playing interminable games of chess while their domestic domains crumble, and of a king wrapped up in his aesthetic pursuits while his territory is threatened by British expansionism," from flyer online at DVDBeaver, Mississauga, Ontario, GHC CPK-25b.]

Review by Kathleen C. Fennessy: amazon.com DVD US$26.99: Written, composed, and directed by Indian master Satyajit Ray (*Pather Panchali*), The Chess Players presents a stylized world in which the landed gentry lounge about, endlessly pulling on hookahs and engaging in the "king of games." Outside their gilded doors, the order that allows them this luxury–let alone their marriages–is crumbling. They couldn't be more oblivious. As the narrator notes, "Mr. Meer and Mr. Mirza are only playing at warfare. Their armies are pieces of ivory. Their battlefield: a piece of cloth." Set in 1856 Lucknow, the noblemen (Saeed Jaffrey and Sanjeev Kumar) are situated in one of the few Indian territories not ruled by Britain's East India Company.

The British, meanwhile, are also playing a game of chess, and equally oblivious Oudh ruler Nawab Wajid Ali Shah (Amjad Khan) is the king they intend to capture. Forthright General Outram (Sir Richard Attenborough, Ghandi), assisted by the more culturally erudite Captain Weston (Tom Alter), is the man charged with the task. It shouldn't be difficult: Like Meer and Mirza, Wajid would prefer to relax—to write poetry, to fly kite–rather than to rule. Along the way, Oudh will fall, but the chess will continue. Based on a story by Munshi Premchand, *The Chess Players* was Ray's most elaborate production. It was also his first in Hindi (with English) and its frames are filled with music, dance, opulent pageantry, and humorous banter–even a lively animated sequence. Behind the attractive façade, however, lies a lament for lost opportunities.

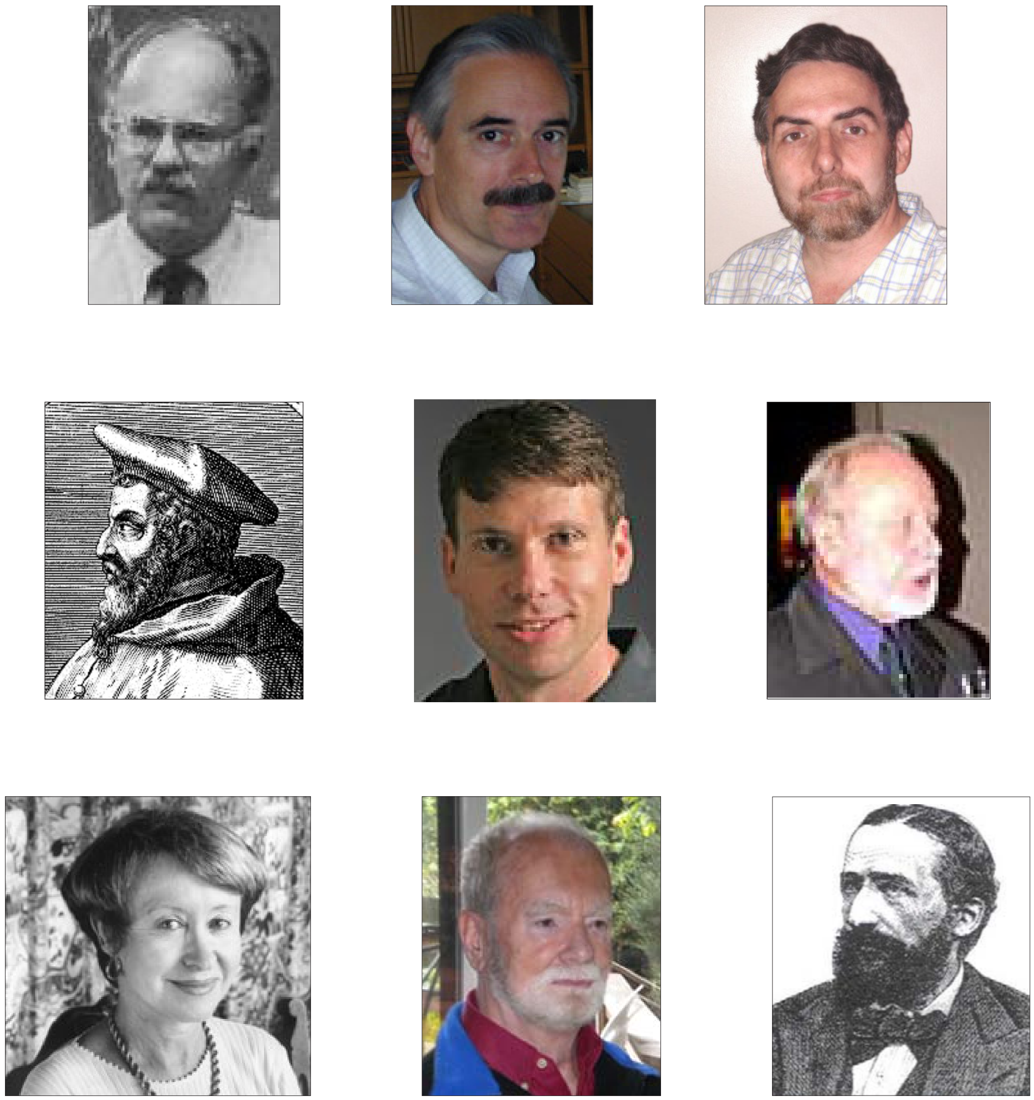
(left panel) Satyajit Ray (1921–1992), India 1994, *Scott* 1476;
(right panel) Saeed Jaffrey as Mir, Sanjeev Kumar as Mirza, in *Śhatrañj Ke Khilārī = The Chess Players* [167].

Portrait Gallery: Schumacher–Sprague



Schumacher [96], Schwerdtfeger [151],
Scott [324], Setsuda [268, 283],
Shenk [225], Sprague [119].

Portrait Gallery: Swetz–Zukertort



Swetz [242], Trump [236], Tyson [185],
Vida [324], Weisstein [266], Whyld [64].
Yalom [218], Zlobec [270], Zukertort [111, 219].

9.6.1: Portraits TBC

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(1) Wilhelm Ernst Martin Georg Ahrens (1872–1927) [121, 139]

(2) Ernest Bergholt (1856–1925) [128]

(3) William Roxby Beverley (c. 1814–1889) [92, 112]

(4) William Beverley [92], William Beverly [112]

(5) Pavle Bidev (1912–1988) [§9.4]

(6) George Edward Carpenter (b. 1844) [103, 120][63]

(7) M. Cashmore [52]

(8) William A. Firth (c. 1815–1890) [111]

(9) Lorraine Screven Frierson (1861–1936) [135]

(10) Thorold Gosset (1869–1962) [132]

(11) H. M. Kingery [135]

(12) Jasper Murdock (fl. 1797) [124]

(13) Franz Nauck [95]

(14) Israel Newton (1763–1856) [124]

(15) E. Pauls [107]

(16) D. F. Savage [135]

(17) Harry A. Sayles [135]

(18) Major-General Robert Shortrede (1800–1866) [90, 100]

(19) "Ursus" [Henry James Kesson (b. c. 1844)] [§§1,4, 9.1]

(20) Frederic A. Woodruff (b. 1855) [135]

(21) John Worthington [135]

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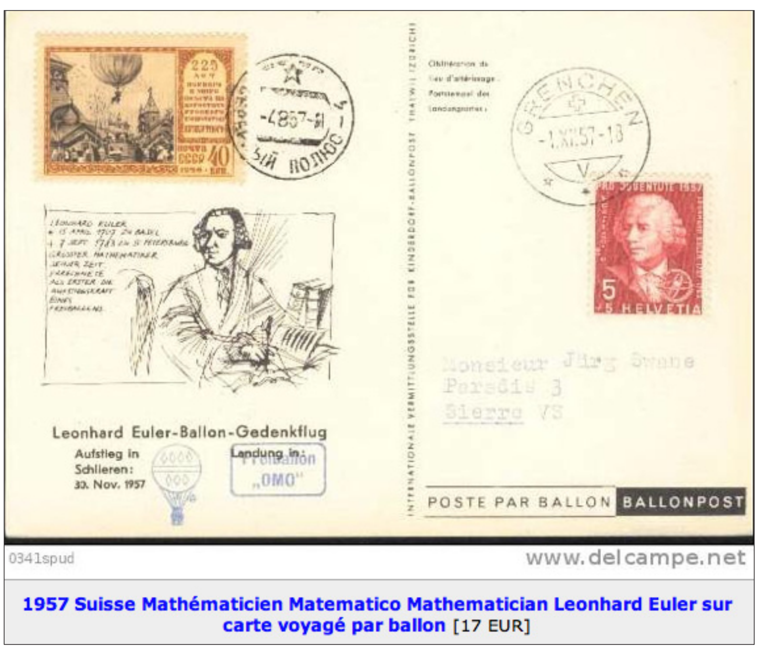
[63] Portrait apparently in *Brentano's Chess Monthly,* c. 1900.

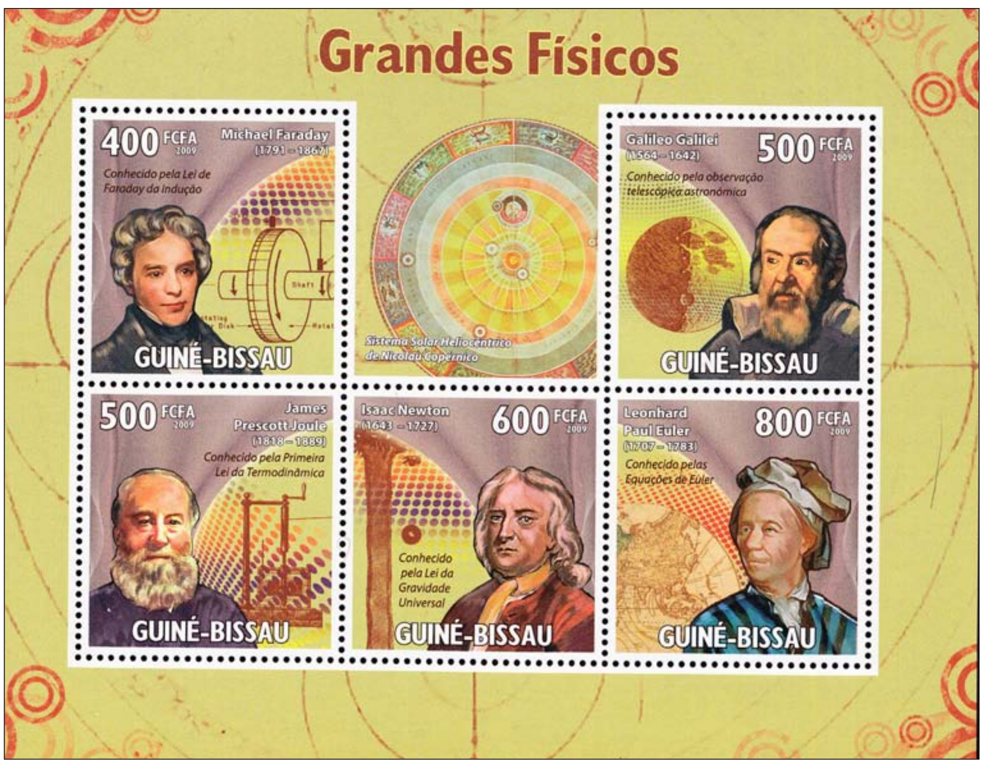
9.7 Philatelic Gallery

9.7.1 Philatelic Euler

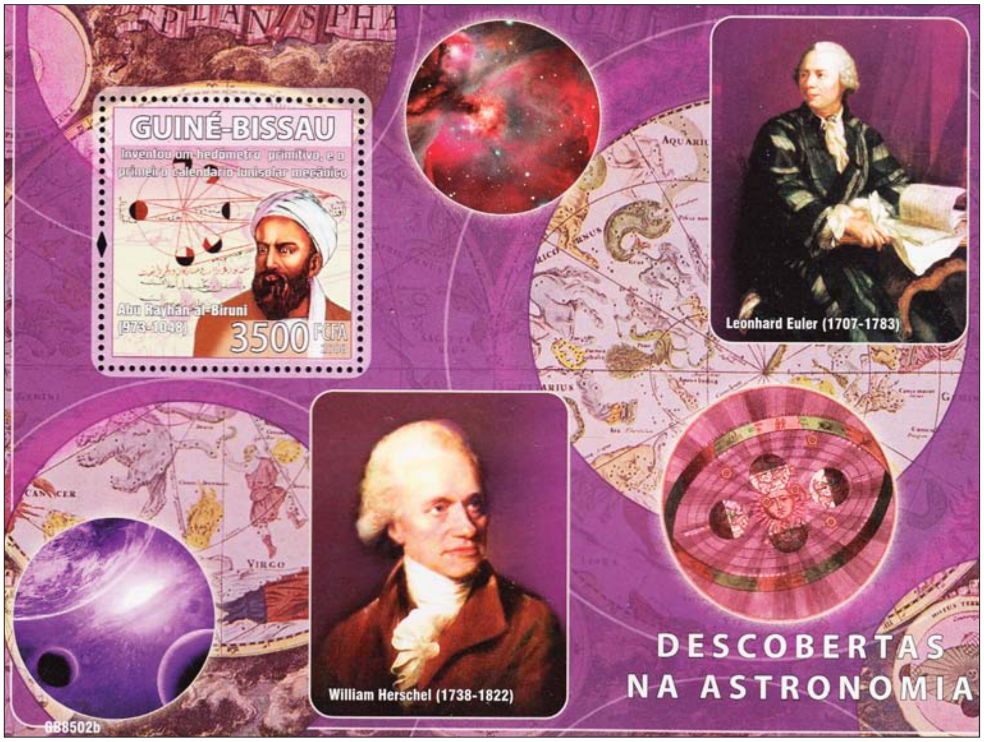


1957 Suisse Mathématicien Matematico Mathematician Leonhard Euler sur carte voyagé par ballon [17 EUR]

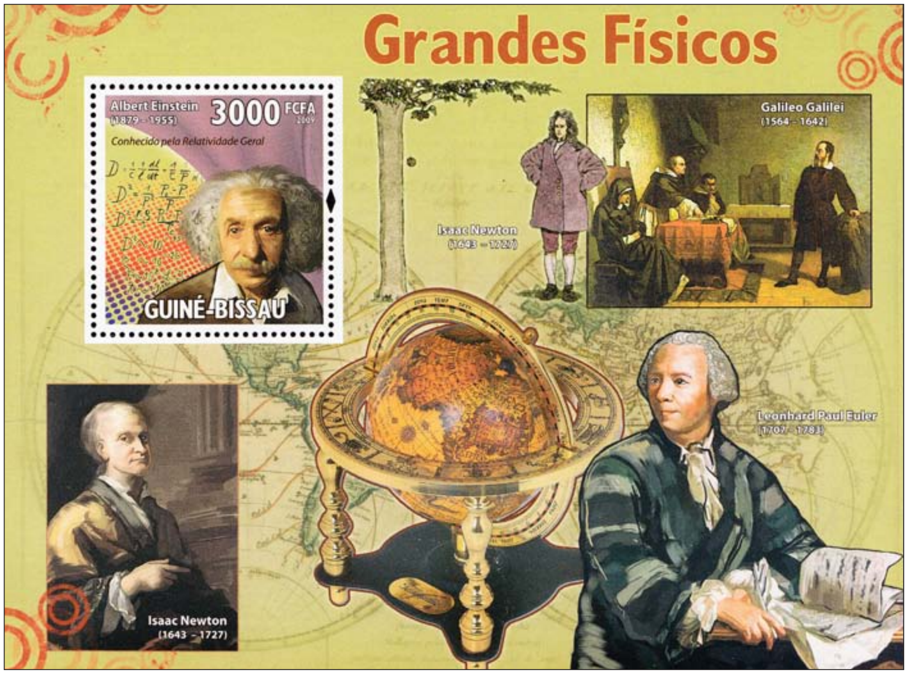
Euler [81, 239]: German Democratic Republic 1950, 1957, 1983 (*Scott* 58, 353, 2371), Switzerland 1957, 2007 (*Scott* B267, 1257), USSR 1957 (*Scott* 1932).

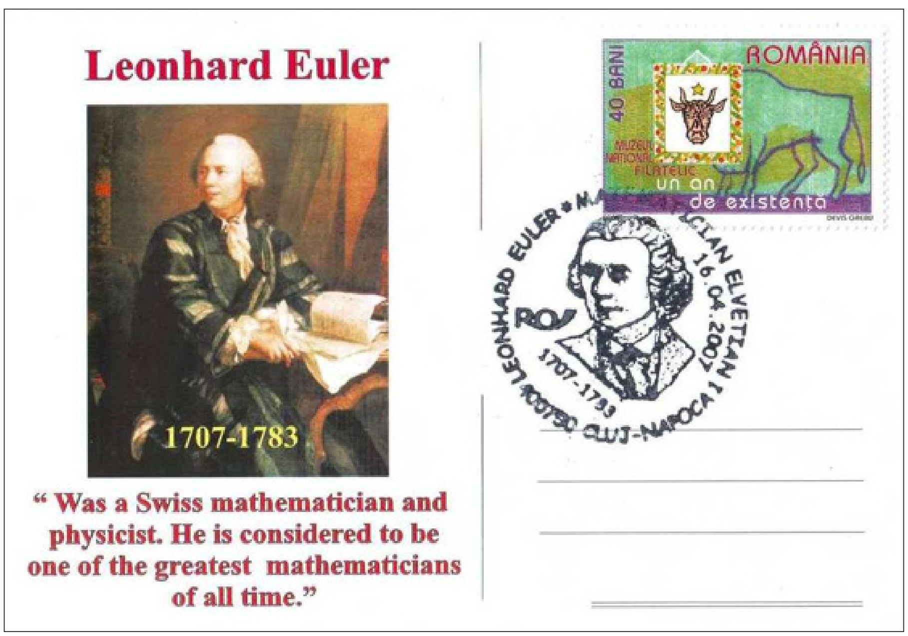


Guinea-Bissau 2009, Euler with "Grandes Físicos":
Michael Faraday, James Prescott Joule, Isaac Newton, Galileo Galilei.



Guinea-Bissau 2008 “Descobertas na astronomia”, Euler with Abu Rayhan al-Biruni, William Herschel; Guinea 2009 “Blaise Pascal”, with Nicolas Copernicus, Galileo Galilei.





Guinea-Bissau 2009, Euler with "Grandes Físicos": Albert Einstein, Isaac Newton, Galileo Galilei; Romania 2007.

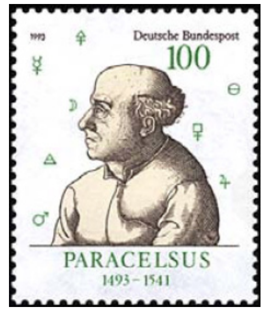
Sakhalin Island 2010 (local mail), Euler with Michael Faraday, Sigmund Freund, Galileo Galilei.

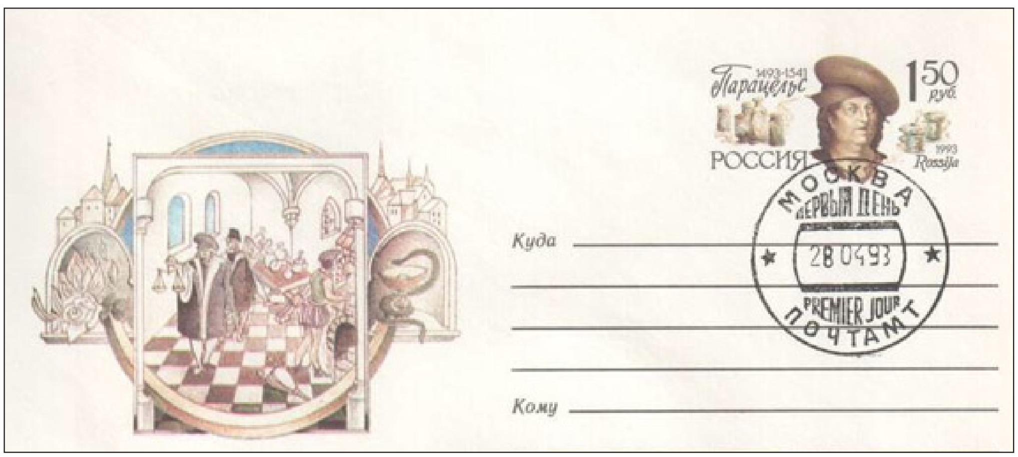
Sakhalin Island 2010 (local mail),
"The Great Scientists on Banknotes" (10 Swiss Francs, Switzerland 1979–1992).

9.7.2 Philatelic Mondrian

Netherlands 1983, 1992 [281], 1994 (*Scott* 652, 807, 850, 851, 852);
Liberia 1997 [310, pp. 100–101] (*Scott* 1276a, 1276b); Togo 1999 (*Scott* 1889f).

9.7.3 Philatelic Paracelsus





Germany 1949, 1993 (*Scott* B311, 1817),
USSR 1993 (envelope),
Austria 1991 (*Scott* 1546), Hungary 1989 (*Scott* 3214), Switzerland 1993 (*Scott* 928), [75].

9.8 Some resources

9.8.1 Some philatelic resources

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[289] [CanadaPost/keyword] online search by keyword in the Canadian Postal Archives Database, Library and Archives Canada, Ottawa.

[290] [CanadaPost/year] online search by year in the Canadian Postal Archives Database, Library and Archives Canada, Ottawa.

[291] [Chemistry/Ramble] *A Philatelic Ramble through Chemistry,* by E. Heilbronner & F. A. Miller, pub. Wiley-VCH, Weinheim, ix + 268 pp. 1998; paperback reprint GHC. [FDC of Max Planck, p. 156.]

[292] [Chess/Collectible] online "The Collectible Chess Gallery," ©2007 The Collectible Stamps Gallery.

[293] [Chess/Gibbons] *Collect Chess on Stamps,* by James S. Young, A Stanley Gibbons Thematic Catalogue, pub. Stanley Gibbons, London, ix + 53 pp., 1999.

[294] [Chess/Rose] online "Chess on Stamps" website by Colin Rose, sponsored by the Theoretical Research Institute.

[295] [CoolStamps] website by Greg & Paulette Caron.

[296] [Delcampe] online auction website for collectibles: Soignies (near Mons), Belgium. 11

[297] [eBay] eBay.com online auction and shopping website.

[298] [Espoo/StPetersburg] Philatelic Service of St.Petersburg Ltd, Kannusillankatu 10, 02770, Espoo, Finland e-mail: info@stspb.ru mobile in Finland: +358-468110755 mobile in Russia: +7-921-9062415 fax in Russia: +7-812-3123245: online.

[299] [France/Phil-Ouest] online *Phil-Ouest : Les timbres de France et les oblitérations de l'Ouest ... et d'ailleurs* website for postage stamps from France and associated postmarks.

[300] [Groth/WWF] online Groth AG: WWF Conservation Stamp Collection website, Gewerbestrasse 19, Unterägeri (near Zürich), Switzerland.

[301] [Heindorff/ArtHistory] "Welcome to Art History on Stamps", by Ann Mette Heindorff [302]: website that describes the development of art history through times as illustrated on postage stamps, giving at the same time an overview of selected artists and their works representative for a given style. 146

[302] [2009-Heindorff/obit] "Internet Philately Leader Ann Mette Heindorff: Born October 19, 1942 – Passed Away April 1, 2009", Obituary written by Toke Nøby and Tony Vella: online at the "De Danske Motivsamlere" [Danish Thematic Association][64]. 146, 163

[303] [Marlen] online Marlen Stamp & Coins Ltd., 156 B Middle Neck Road, Great Neck, NY 11021.

[304] [McLean] online B. McLean Stamps for Collectors, P.O. Box 323, Ellon, Aberdeenshire AB41 7YA, Scotland.

[305] [Miller/Images] online "Images of Mathematicians on Postage Stamps" website by Jeff Miller, Gulf High School, New Port Richey, Florida.

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[64]. ["Before Ann Mette Heindorff died she handed over her web site to Jørgen Jørgensen, the president of the Danish Thematic Association, who will maintain Heindorff's Web pages for the future: online."

[306] [Precursors] online Precursor Contributors to Meteorology (Renaissance [~1400 AD] through World War I) website created and maintained by Garry Toth and Don Hillger. ©2007–2010, Colorado State University.

[307] [Reinhardt/Physics] online "Physics-related stamps" website by Joachim Reinhardt, Institut für Theoretische Physik, Johann Wolfgang Goethe-Universität, Frankfurt am Main.

[308] [*Scott*/Kloetzel] *Scott Standard Postage Stamp Catalogue* (James E. Kloetzel, ed.), published annually (currently in 6 volumes on paper) by Scott Publishing, Sidney, Ohio. 11, 62, 109

[309] [Steurs/WWF] online The most complete site for your WWF stamp collection website, Rijmenam (near Antwerp), Belgium.

[310] [Wilson/*Stamping*] *Stamping Through Mathematics,* by Robin J. Wilson, pub. Springer, New York, 136 pp., 2001. ISBN-13: 978-0387989495 GHC Look inside at amazon.com new US$27.01. 161

9.8.2 Some bio-bibliographic resources

[311] [Amazon.com] online started as an online bookstore, but soon diversified ...[65].

[312] [AMICUS] online Canadian national catalogue shows the published materials held at Library and Archives Canada and also those located in over 1300 libraries across Canada.

[313] [Copac] online is a British academic, and specialist Library Catalogue[66].

[314] [*Current Index to Statistics*] online database published by the Institute of Mathematical Statistics and the American Statistical Association that contains bibliographic data of articles in statistics, probability, and related fields.

[315] [Google books] online Advanced Book Search[67].

[316] [Google Image Search] online search service created by Google that allows users to search the Web for image content.

[317] [HathiTrust] online is a very large scale collaborative repository of digital content from research libraries including content digitized via the Google Books project and Internet Archive digitization initiatives, as well as content digitized locally by libraries[68].

[318] [JSTOR] provides online full-text searches of digitized back issues of several hundred well-known journals, dating back to 1665 with the *Philosophical Transactions of the Royal Society.*

[319] [KBH] online Koninklijke Bibliotheek, 's-Gravenhage [Royal Library, The Hague][69].

[65]Amazon.com is a US-based multinational company, headquartered in Seattle, Washington. It started as an online bookstore, but soon diversified, selling DVDs, CDs, MP3 downloads, computer software, video games, electronics, apparel, furniture, food, and toys.

[66]Union catalogue which provides free access to the merged online catalogues of many major university research libraries in the United Kingdom and Ireland, plus an increasing number of specialist libraries and the British Library, the National Library of Scotland and the National Library of Wales.

[67]Google Books (previously known as Google Book Search and Google Print) is a service from Google that searches the full text of books that Google has scanned, converted to text using optical character recognition, and stored in its digital database.

[68]HathiTrust was founded in October 2008 by the thirteen universities of the Committee on Institutional Cooperation and the University of California. The partnership includes over 50 research libraries across the United States and Europe, and is based on a shared governance structure. Costs are shared by the participating libraries and library consortia. The repository is administered by Indiana University and the University of Michigan. The Executive Director of HathiTrust is John Price Wilkin, who has led large-scale digitization initiatives at the University of Michigan since the mid 1990s. [324]

[69]Found 33 items by or about Pavle Bidev at KBH on 20 March 2011.

[320] [KVK] online: Karlsruher Virtueller Katalog (KVK) is an book search engine administered by the library of the Karlsruhe Institute of Technology (KIT)[70].

[321] [MacTutor] online History of Mathematics archive, Created by John J. O'Connor & Edmund F. Robertson, School of Mathematics and Statistics, University of St Andrews, Scotland.

[322] [MathSciNet] online *Mathematical Reviews:* journal and online database published by the American Mathematical Society that contains brief synopses (and occasionally evaluations) of many articles in mathematics, statistics and theoretical computer science.

[323] [McGill/MUSE] online Classic Library Catalogue MUSE at McGill University, Montréal.

[324] [*Wikipedia*] online: *the free encyclopedia that anyone can edit.* Text available under the Creative Commons Attribution-ShareAlike License; *Wikipedia* is a registered trademark of the Wikimedia Foundation. 7, 8, 9, 10, 11, 12, 24, 61, 62, 65, 83, 89, 91, 97, 98, 105, 109, 114, 144, 145, 152, 153, 164

[325] [WorldCat] online union catalog which itemizes the collections of 71,000 libraries in 112 countries, which participate in the Online Computer Library Center (OCLC) global cooperative.

[326] [Z-MATH/Zbl] online = Zentralblatt MATH is an service providing reviews and abstracts for articles in pure and applied mathematics, published by Springer Science+Business Media[71].

GPHS/October 1, 2011

[70] KVK searches a large number of catalogs of research libraries in Germany, Austria, and Switzerland, as well as several important national libraries in other countries, and some large commercial catalogs.

[71] The service was founded in 1931 by Otto Neugebauer as *Zentralblatt für Mathematik und ihre Grenzgebiete.* The service reviews more than 2,300 journals and serials worldwide, as well as books and conference proceedings. Zentralblatt MATH is now edited by the European Mathematical Society, FIZ Karlsruhe, and the Heidelberg Academy of Sciences. The Zentralblatt database also incorporates the 200,000 entries of the earlier similar publication *Jahrbuch ber die Fortschritte der Mathematik,* 1868–1942, added in 2003.

$$\mathbf{M}^{(\mathrm{p})}_{1(\hat{b}),a=1} = \begin{pmatrix} 1 & 56+b & c & 56+d & 8 & 65-b & 9-c & 65-d \\ 8\,b & 73-9\,b & 8\,b+1-c & 73-8\,b-d & 8\,b-7 & 64-7\,b & 8\,b-8+c & 64-8\,b+d \\ 8\,c-7 & 64-8\,c+b & 9\,c-8 & 64-8\,c+d & 8\,c & 73-8\,c-b & 7\,c+1 & 73-8\,c-d \\ 8\,d & 73-8\,d-b & 8\,d+1-c & 73-9\,d & 8\,d-7 & 64-8\,d+b & 8\,d-8+c & 64-7\,d \\ 57 & b & 56+c & d & 64 & 9-b & 65-c & 9-d \\ 72-8\,b & 7\,b+1 & 73-8\,b-c & 8\,b+1-d & 65-8\,b & 9\,b-8 & 64-8\,b+c & 8\,b-8+d \\ 65-8\,c & 8\,c-8+b & 64-7\,c & 8\,c-8+d & 72-8\,c & 8\,c+1-b & 73-9\,c & 8\,c+1-d \\ 72-8\,d & 8\,d+1-b & 73-8\,d-c & 7\,d+1 & 65-8\,d & 8\,d-8+b & 64-8\,d+c & 9\,d-8 \end{pmatrix}. \tag{9.4.1}$$