An introduction to Yantra magic squares and Agrippa–type magic matrices

George P. H. Styan[2]

January 18, 2012

[2]This beamer file is for an invited talk presented as a video on Thursday, 5 January 2012, at the International Workshop and Conference on Combinatorial Matrix Theory and Generalized Inverses of Matrices, Manipal University, Manipal (Karnataka), India, 2–11 January 2012. This talk is based on joint research with Ka Lok Chu & Götz Trenkler and, in part, on Reports 2011-07 and 2012-01 (lecture notes) from the Department of Mathematics and Statistics, McGill University, Montréal.

This beamer file is for an invited talk presented as a video on Thursday, 5 January 2012, at the International Workshop and Conference on Combinatorial Matrix Theory and Generalized Inverses of Matrices, Manipal University, Manipal (Karnataka), India, 2–11 January 2012.

This talk is based on joint research with Ka Lok Chu & Götz Trenkler and, in part, on Reports 2011-07 and 2012-01 (lecture notes) from the Department of Mathematics and Statistics, McGill University, Montréal.

Special thanks go to Christian Boyer for constructing the 22×22 Fermat–Boyer magic square and to Daniel J. H. Rosenthal for drawing the Jupiter Planet Yantra. In addition, we are grateful to Nicolas C. Ammerlaan and Thomas W. Ammerlaan for making the video, and to Oskar Maria Baksalary, S. W. Drury, and Evelyn Matheson Styan for their help. This research was supported, in part, by the Natural Sciences and Engineering Research Council of Canada.

The handwritten image of the 14×14 Fermat magic square by Félix Vicq d'Azyr (1746–1794) is from MS #10556, Département de manuscrits, Bibliothèque nationale de France, Paris.
My sincere thanks to Christian Boyer for the image and to the Bibliothèque nationale de France (BnF) for allowing us to reproduce it here.

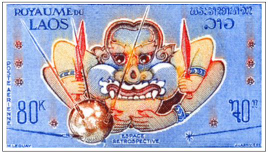
I want to tell you about

Yantra magic squares

and

Agrippa-type magic matrices

And to illustrate our findings with postage stamps



(xv) (xvi)

Magic Squares.

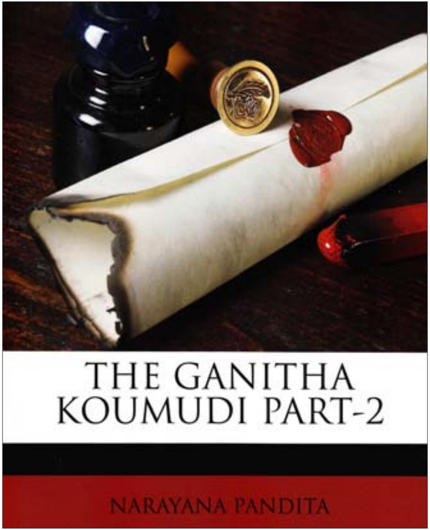
Magic Squares are figures resembling a chess-board in which the terms of an arithmetical progression are so arranged that their sum, whether taken diagonally or by rows or columns, is always the same.

The construction of such magic squares containing an odd or even number of cells had been known to the Hindus for

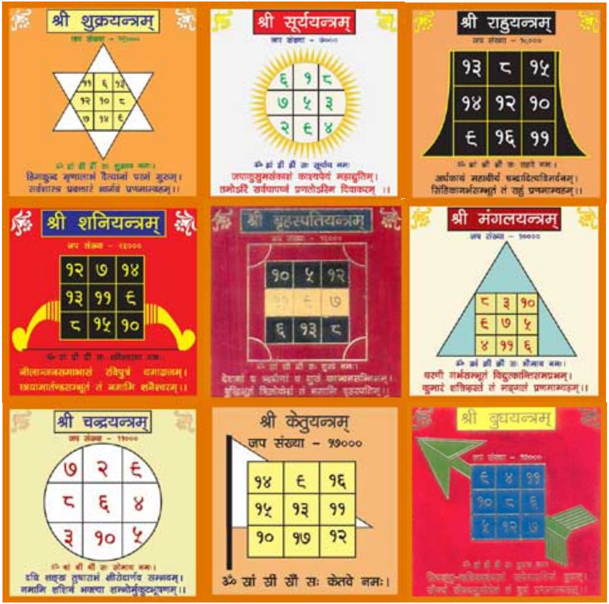
a long time. In Tantra Śāstra they are called Yantras. As they were supposed to possess mystical properties, they were kept secret and were not dealt with in Arithmetic by Indian mathematicians. But Nārāyaṇa, defying this superstitious belief, touched upon the subject of magic squares under the heading of Bhadragaṇita and gave defintie rules for the construction of them

KHAJURI,
Benares Cantt.

PADMAKARA DVIVEDI



THE GANITHA KOUMUDI PART-2
NARAYANA PANDITA


श्री शुक्रयन्त्रम्
श्री सूर्ययन्त्रम्
श्री राहुयन्त्रम्
श्री शनियन्त्रम्
श्री बृहस्पतियन्त्रम्
श्री मंगलयन्त्रम्
श्री चन्द्रयन्त्रम्
श्री केतुयन्त्रम्
जप संख्या – १७०००
ॐ स्रां स्रीं स्रौं सः केतवे नमः।
श्री बुधयन्त्रम्

We study various properties of Yantra magic squares:
"Yantra" is a "geometrical diagram used like an icon usually in meditation".

A classic $n \times n$ magic square (magic matrix) is an arrangement of the numbers $1, 2, \ldots, n^2$ in an $n \times n$ array so that the numbers in every row, every column and in the two main diagonals add up to the same magic sum

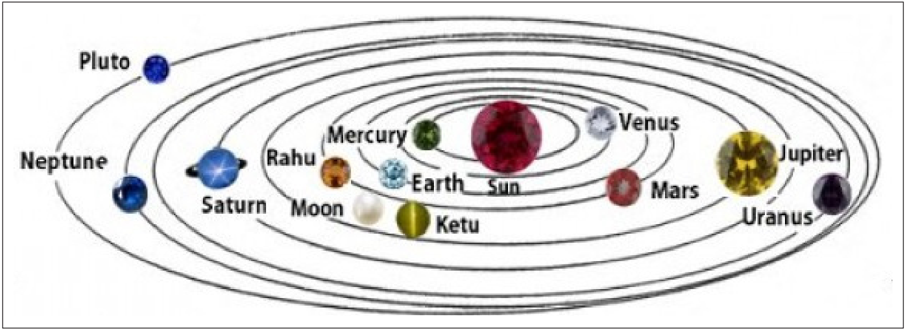
$$m = \frac{n(n^2+1)}{2}.$$

When $n = 3$ the 3×3 classic Luoshu magic matrix $\begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix}$ with $m = 15$.

We begin with 3×3 magic squares depicted on "Planet Yantras":

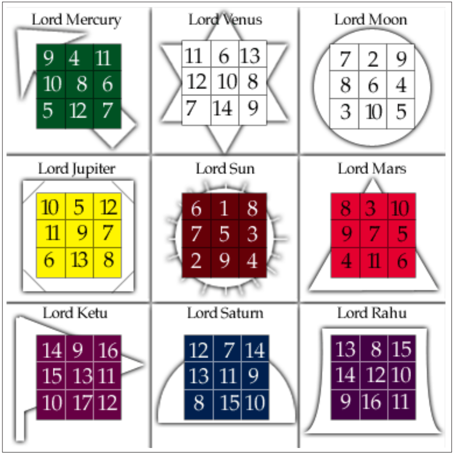
1. Sun/Surya,
2. Moon/Chandra,
3. Mars/Mangala,
4. Mercury/Budha,
5. Jupiter/Brihaspati–Guru
6. Venus/Shukra
7. Saturn/Shani,
8. Rahu,
9. Ketu.

The lunar nodes Rahu and Ketu are the points
where the moon's path in the sky crosses the sun's path.

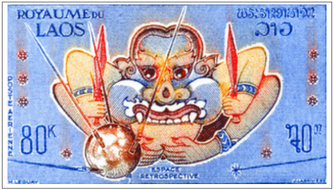


We also consider "Navagraha Yantras": 3×3 arrangements of the nine Planet Yantras, usually with a Sun/Surya Planet Yantra in the centre.

Placing a Sun/Surya Planet Yantra in the centre, we construct
8 Navagraha Yantras, each in the form of a 9×9 composite fully-magic square.



Philatelic $\mathbf{L}_1$ of the 9 planets with Jupiter in the centre



$$\mathbf{L}_1 = \begin{pmatrix} 6 & 1 & 8 \\ 7 & 5 & 3 \\ 2 & 9 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} \text{Venus} & \text{Sun} & \text{Rahu} \\ \text{Saturn} & \text{Jupiter} & \text{Mars} \\ \text{Moon} & \text{Ketu} & \text{Mercury} \end{pmatrix}$$

$$= \begin{pmatrix} \text{Dahomey} & \text{Jaipur} & \text{Laos} \\ \text{Ciskei} & \text{Ciskei} & \text{Ciskei} \\ \text{Italy} & \textit{Ketu} & \text{Austria} \end{pmatrix}.$$

In the matrix $\mathbf{L}_1$ the numbers in the rows, columns, and two main diagonals all add up to same *magic sum* 15.

The only images of the 9 Planet Yantras depicting magic squares that we have found each depict a 3×3 magic square, respectively

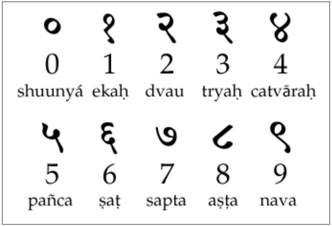
$$\mathbf{L}_k = \mathbf{L}_1 + (k-1)\mathbf{E}, \quad k = 1, 2, ..., 9,$$

where $\mathbf{L}_1$ is the 3×3 double-flipped Luoshu fully-magic matrix

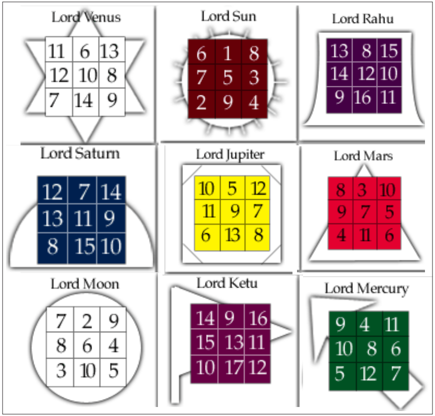
$$\mathbf{L}_1 = \begin{pmatrix} 6 & 1 & 8 \\ 7 & 5 & 3 \\ 2 & 9 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \mathbf{F}\mathbf{L}_0\mathbf{F}.$$

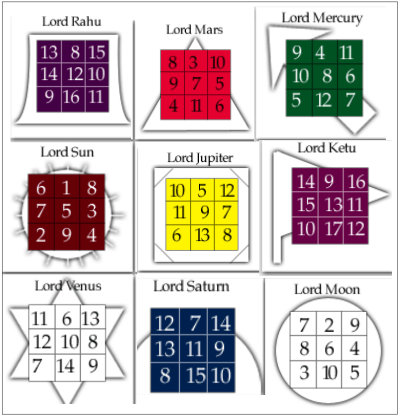
Here $\mathbf{F}$ is the 3×3 "flip matrix" and $\mathbf{L}_0$ the "standard" Luoshu magic matrix.

Most of the Planet Yantras with 3×3 magic squares that we have found show the numerals in Sanskrit, with the number “9” depicted in two different ways.



Navagraha Planet Yantras arranged as 9×9 composite magic squares





Our "Navagraha (nine-planets) Yantra magic square"
is defined by the eight composite 9×9 fully-magic matrices

$$\begin{aligned} \mathbf{N}_j &= \mathbf{E} \otimes \mathbf{L}_1 + \mathbf{M}_j \otimes \mathbf{E} - \mathbf{E} \otimes \mathbf{E} \\ &= \mathbf{E} \otimes \mathbf{L}_1 + (\mathbf{M}_j - \mathbf{E}) \otimes \mathbf{E}; \quad j = 1, 2, \ldots, 8, \end{aligned}$$

where the eight classic 3×3 magic matrices

$$\mathbf{M}_1 = \mathbf{L}_1 = \begin{pmatrix} 6 & 1 & 8 \\ 7 & 5 & 3 \\ 2 & 9 & 4 \end{pmatrix}, \quad \mathbf{M}_2 = \mathbf{F}\mathbf{L}_1 = \begin{pmatrix} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{pmatrix}, \quad \mathbf{M}_3 = \mathbf{L}_1\mathbf{F} = \begin{pmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{pmatrix},$$

$$\mathbf{M}_4 = \mathbf{L}_1' = \begin{pmatrix} 6 & 7 & 2 \\ 1 & 5 & 9 \\ 8 & 3 & 4 \end{pmatrix}, \quad \mathbf{M}_5 = \mathbf{F}\mathbf{L}_1' = \begin{pmatrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{pmatrix}, \quad \mathbf{M}_6 = \mathbf{L}_1'\mathbf{F} = \begin{pmatrix} 2 & 7 & 6 \\ 9 & 5 & 1 \\ 4 & 3 & 8 \end{pmatrix},$$

$$\mathbf{M}_7 = \mathbf{F}\mathbf{L}_1\mathbf{F} = \begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix}, \quad \mathbf{M}_8 = \mathbf{F}\mathbf{L}_1'\mathbf{F} = \begin{pmatrix} 4 & 3 & 8 \\ 9 & 5 & 1 \\ 2 & 7 & 6 \end{pmatrix}.$$

The magic square defined by

$$\mathbf{M}_7 = \mathbf{F}\mathbf{L}_1\mathbf{F} = \begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix}$$

was chosen as "The Table of Saturn in his Compass" by Heinrich Cornelius Agrippa von Nettesheim (1486–1535).

Gerolamo Cardano (1501–1576) named this magic square "Luna" after the moon (rather than Saturn).

In the magic matrix $\mathbf{M}_7$ the numbers in the rows, columns, and two main diagonals all add up to same *magic sum* 15.

Paracelsus (1493–1541)
[born Philippus Aureolus Theophrastus Bombastus von Hohenheim]
named the magic square defined by

$$\mathbf{M}_2 = \mathbf{FL}_1 = \begin{pmatrix} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{pmatrix},$$

Talisman de Saturne, ou Samedi (Saturday).

$$\mathbf{N}_1 = \begin{pmatrix} \mathbf{L}_6 & \mathbf{L}_1 & \mathbf{L}_8 \\ \mathbf{L}_7 & \mathbf{L}_5 & \mathbf{L}_3 \\ \mathbf{L}_2 & \mathbf{L}_9 & \mathbf{L}_4 \end{pmatrix}, \quad \mathbf{N}_2 = \begin{pmatrix} \mathbf{L}_2 & \mathbf{L}_9 & \mathbf{L}_4 \\ \mathbf{L}_7 & \mathbf{L}_5 & \mathbf{L}_3 \\ \mathbf{L}_6 & \mathbf{L}_1 & \mathbf{L}_8 \end{pmatrix}, \quad \mathbf{N}_3 = \begin{pmatrix} \mathbf{L}_8 & \mathbf{L}_1 & \mathbf{L}_6 \\ \mathbf{L}_3 & \mathbf{L}_5 & \mathbf{L}_7 \\ \mathbf{L}_4 & \mathbf{L}_9 & \mathbf{L}_2 \end{pmatrix},$$

$$\mathbf{N}_4 = \begin{pmatrix} \mathbf{L}_6 & \mathbf{L}_7 & \mathbf{L}_2 \\ \mathbf{L}_1 & \mathbf{L}_5 & \mathbf{L}_9 \\ \mathbf{L}_8 & \mathbf{L}_3 & \mathbf{L}_4 \end{pmatrix}, \quad \mathbf{N}_5 = \begin{pmatrix} \mathbf{L}_8 & \mathbf{L}_3 & \mathbf{L}_4 \\ \mathbf{L}_1 & \mathbf{L}_5 & \mathbf{L}_9 \\ \mathbf{L}_6 & \mathbf{L}_7 & \mathbf{L}_2 \end{pmatrix}, \quad \mathbf{N}_6 = \begin{pmatrix} \mathbf{L}_2 & \mathbf{L}_7 & \mathbf{L}_6 \\ \mathbf{L}_9 & \mathbf{L}_5 & \mathbf{L}_1 \\ \mathbf{L}_4 & \mathbf{L}_3 & \mathbf{L}_8 \end{pmatrix},$$

$$\mathbf{N}_7 = \begin{pmatrix} \mathbf{L}_4 & \mathbf{L}_9 & \mathbf{L}_2 \\ \mathbf{L}_3 & \mathbf{L}_5 & \mathbf{L}_7 \\ \mathbf{L}_8 & \mathbf{L}_1 & \mathbf{L}_6 \end{pmatrix}, \quad \mathbf{N}_8 = \begin{pmatrix} \mathbf{L}_4 & \mathbf{L}_3 & \mathbf{L}_8 \\ \mathbf{L}_9 & \mathbf{L}_5 & \mathbf{L}_1 \\ \mathbf{L}_2 & \mathbf{L}_7 & \mathbf{L}_6 \end{pmatrix}.$$

$$\mathbf{N}_1 = \begin{pmatrix} \mathbf{L}_6 & \mathbf{L}_1 & \mathbf{L}_8 \\ \mathbf{L}_7 & \mathbf{L}_5 & \mathbf{L}_3 \\ \mathbf{L}_2 & \mathbf{L}_9 & \mathbf{L}_4 \end{pmatrix} = \begin{pmatrix} 11 & 6 & 13 & 6 & 1 & 8 & 13 & 8 & 15 \\ 12 & 10 & 8 & 7 & 5 & 3 & 14 & 12 & 10 \\ 7 & 14 & 9 & 2 & 9 & 4 & 9 & 16 & 11 \\ 12 & 7 & 14 & 10 & 5 & 12 & 8 & 3 & 10 \\ 13 & 11 & 9 & 11 & 9 & 7 & 9 & 7 & 5 \\ 8 & 15 & 10 & 6 & 13 & 8 & 4 & 11 & 6 \\ 7 & 2 & 9 & 14 & 9 & 16 & 9 & 4 & 11 \\ 8 & 6 & 4 & 15 & 13 & 11 & 10 & 8 & 6 \\ 3 & 10 & 5 & 10 & 17 & 12 & 5 & 12 & 7 \end{pmatrix},$$

The $n \times n$ magic matrix $\mathbf{M}$ with magic sum m is $\mathbf{F}$-associated whenever

$$\mathbf{M} + \mathbf{FMF} = 2m\bar{\mathbf{E}},$$

where $\bar{\mathbf{E}}$ is the $n \times n$ matrix with every entry equal to $1/n$ and $\mathbf{F} = \mathbf{F}_n$ is the $n \times n$ "flip matrix"

$$\mathbf{F} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

The magic matrices $\mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_8$ all have magic sum 81 and rank 5, and all are $\mathbf{F}$-associated, since $\mathbf{F}_9 = \mathbf{F}_3 \otimes \mathbf{F}_3$ and $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_8$ are all $\mathbf{F}$-associated.

In an $n \times n$ **F**-associated magic matrix with magic sum m the sums of pairs of entries diametrically equidistant from the centre are all equal to $2m/n$.

An **F**-associated magic square is often called (just) "associated" (with no qualifier to the word "associated") or "regular" or "symmetrical".

Magic-square planetary talismans (MSPTs) were created from 1531–1653 by

- Heinrich Cornelius Agrippa von Nettesheim (1486–1535),
- Gerolamo Cardano (1501–1576),
- Paracelsus (1493–1541)
  [born Philippus Aureolus Theophrastus Bombastus von Hohenheim] ,
- Athanasius Kircher (1602–1680).

for the seven planets:
Saturn, Jupiter, Mars, Sun, Venus, Mercury, and the Moon.

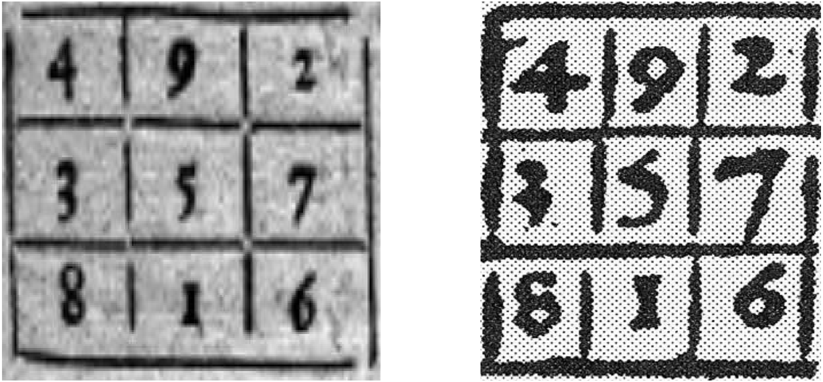
(from left to right) Agrippa [*Wikipedia*]; Cardano: Altai Republic 2011 (External Airmail); Paracelsus: Switzerland 1993 (*Scott* 928); Kircher: Malta 2002 (*Scott* 1094).

- Agrippa was a German magician, occult writer, theologian, astrologer, and alchemist. His magic-square planetary talismans (MSPTs) were first published in 1531 in his *De occulta philosophia libri tres,*

- Cardano was an Italian Renaissance mathematician, physician, astrologer and gambler. His MSPTs were first published in 1539 in his *Practica arithmetice et mensurandi singularis.*

- Paracelsus was a Swiss Renaissance physician, botanist, alchemist, astrologer, and general occultist. His MSPTs were first published in 1567 in *Archidoxa Magica.*

- Kircher was a German Jesuit scholar who published in the fields of oriental studies, geology, and medicine. His MSPTs were first published in 1653 in *Oedipi Aegyptiaci, Tomi Secundi.*



(left panel) by Agrippa for Saturn; (right panel) by Cardano for the Moon.

The Agrippa–Cardano magic square

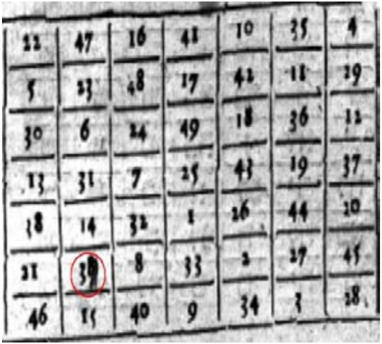
$$\mathbf{L} = \begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix}$$

is known as Luoshu or Lo Shu (Luo River Writing). Legend has it that the very first magic square was discovered in China about 4000 years ago on the back of a turtle in the River Luo, a tributary of the Yellow River in China.

11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15

11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15

(left panel) by Agrippa for Mars: (right panel) by Cardano for Venus.



22	47	16	41	10	35	4
5	23	48	17	42	11	29
30	6	24	49	18	36	12
13	31	7	25	43	19	37
38	14	32	1	26	44	20
21	[illegible]	8	33	2	27	45
46	15	40	9	34	3	28

22	47	16	41	10	35	4
5	23	48	17	42	11	29
30	6	24	49	18	36	12
13	31	7	25	43	19	37
38	14	32	1	26	44	20
21	39	8	33	2	27	45
46	15	40	9	34	3	28

(left panel[32]) by Agrippa for Venus; (right panel) by Cardano for Mars.

[32] The entry in position (6,2) should be "39".

37	75	29	70	21	62	13	45	5
6	35	79	30	71	22	63	14	46
47	7	39	80	31	72	23	55	15
16	45	8	40	81	32	64	24	56
57	17	49	9	41	73	33	68	25
26	55	18	50	1	42	74	34	66
67	27	59	10	51	2	43	78	35
36	68	19	60	11	52	3	44	76
77	28	69	20	61	12	53	4	45

37	78	29	70	21	62	3	45	5
6	38	79	30	71	22	63	14	46
47	7	39	80	31	72	23	55	15
16	48	8	40	81	32	64	24	56
57	17	49	9	41	73	33	65	25
26	58	18	50	1	42	74	34	66
67	27	59	10	51	2	43	75	35
36	68	19	60	11	52	3	44	76
77	28	69	20	61	12	53	4	45

(left panel) by Agrippa for the Moon: (right panel) by Cardano for Saturn[34].

[34] In both panels "45" in position (1, 8) should be "54"; in the left panel the "35" in position (2, 2) should be "38".

We may generate odd-order "Agrippa-type" (Agrippa–Cardano) magic matrices using a "magic matrix algorithm", which we believe to be new and which we now describe.

Our "magic matrix algorithm" generates classic nonsingular $n \times n$ magic matrices $\mathbf{A}_n$ for any odd number $n \geq 3$.

The Matlab algorithm magic(n) by Cleve Moler (1993) also generates classic nonsingular $n \times n$ magic matrices with n odd but they (all for $n \geq 5$?) essentially differ from our Agrippa-type magic matrices $\mathbf{A}_n$.

We begin the description of our magic matrix algorithm with the $n \times n$ "one-step forward shift matrix" $\mathbf{S}_n$ $(n = 2, 3, 4, \ldots)$, which has $+1$ in all $n-1$ positions on the leading principal super-diagonal and in the bottom-left corner and so is a circulant. When $n = 5$ we have

$$\mathbf{S}_5 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We now define the $n \times n$ matrix

$$\mathbf{T}_n = \mathbf{S}_n - \mathbf{S}'_n = \mathbf{S}_n - \mathbf{S}_n^{-1}; \quad n = 2, 3, 4, \ldots$$

with $+1$ in all $n-1$ positions on the leading principal super-diagonal and in the bottom-left corner and with -1 in all $n-1$ positions on the leading principal sub-diagonal and in the top-right corner.

We note that $\mathbf{S}'_n = \mathbf{S}_n^{-1}$ since $\mathbf{S}_n$ is orthogonal. When $n = 5$ we have

$$\mathbf{T}_5 = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

We now define the $n \times n$ matrix with $n = 2, 3, 4, \ldots$

$$\mathbf{U}_n = \mathbf{F}_n\mathbf{T}_n - n\mathbf{T}_n = (\mathbf{F}_n - n\mathbf{I}_n)\mathbf{T}_n = (\mathbf{F}_n - n\mathbf{I}_n)(\mathbf{S}_n - \mathbf{S}_n')$$

where $\mathbf{F}_n$ is the $n \times n$ flip matrix.

When $n = 5$ we have

$$\mathbf{T}_5 = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \end{pmatrix}; \quad \mathbf{U}_5 = \begin{pmatrix} 1 & -5 & 0 & -1 & 5 \\ 5 & 0 & -6 & 0 & 1 \\ 0 & 4 & 0 & -4 & 0 \\ -1 & 0 & 6 & 0 & -5 \\ -5 & 1 & 0 & 5 & -1 \end{pmatrix}.$$

Three equivalent formulas for our "magic matrix algorithm":

$$\mathbf{A}_n = n(n^2-1)(\mathbf{U}_n + \bar{\mathbf{E}}_n)^{-1} - \tfrac{1}{2}n(n^2-3)\bar{\mathbf{E}}_n \tag{1}$$

$$= n(n^2-1)\Big(\mathbf{U}_n + \frac{2(n^2-1)}{n^2+1}\bar{\mathbf{E}}_n\Big)^{-1} \tag{2}$$

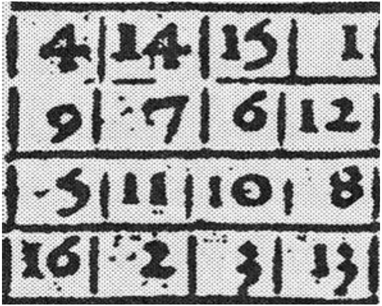
$$= n(n^2-1)\mathbf{U}_n^{+} + \tfrac{1}{2}n(n^2+1)\bar{\mathbf{E}}_n. \tag{3}$$

The matrix $\bar{\mathbf{E}}_n$ has every entry equal to $\frac{1}{n}$, and $\mathbf{U}_n = (\mathbf{F}_n - n\mathbf{I}_n)(\mathbf{S}_n - \mathbf{S}_n')$.

$$\mathbf{S}_5 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad \mathbf{U}_5 = \begin{pmatrix} 1 & -5 & 0 & -1 & 5 \\ 5 & 0 & -6 & 0 & 1 \\ 0 & 4 & 0 & -4 & 0 \\ -1 & 0 & 6 & 0 & -5 \\ -5 & 1 & 0 & 5 & -1 \end{pmatrix};$$

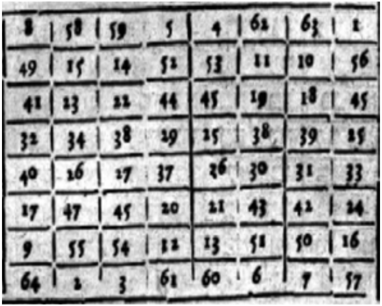
$$\mathbf{A}_5 = 120\left(\mathbf{U}_5 + \frac{24}{13}\bar{\mathbf{E}}_5\right)^{-1} = \begin{pmatrix} 11 & 24 & 7 & 20 & 3 \\ 4 & 12 & 25 & 8 & 16 \\ 17 & 5 & 13 & 21 & 9 \\ 10 & 18 & 1 & 14 & 22 \\ 23 & 6 & 19 & 2 & 15 \end{pmatrix}.$$

4	14	15	1
9	7	6	12
5	11	10	8
16	2	3	13



4	14	15	1
9	7	6	12
5	11	10	8
16	2	3	13

(left panel) by Agrippa for Jupiter; (right panel) by Cardano for Mercury.



8	58	59	5	4	62	63	1
49	15	14	52	53	11	10	56
41	23	22	44	45	19	18	45
32	34	38	29	25	38	39	25
40	26	27	37	26	30	31	33
17	47	45	20	21	43	42	24
9	55	54	12	13	51	50	16
64	2	3	61	60	6	7	57

8	58	59	5	4	62	63	1
49	15	14	52	53	11	10	56
41	23	22	44	45	19	18	48
32	34	35	29	28	38	39	25
40	26	27	37	36	30	31	33
17	47	46	20	21	43	42	24
9	55	54	12	13	51	50	16
64	2	3	61	60	6	7	57

(left panel) by Agrippa for Mercury; (right panel) by Cardano for Jupiter.

The matrix algorithm used by Matlab to generate magic matrices $\mathbf{M}_n$ for doubly-even $n = 4k$ has been described in some detail by Kirkland & Neumann (1995); see also Moler (1993, 2004).

We find that our Agrippa-type magic matrices $\mathbf{A}_{4k} = \mathbf{F}\mathbf{M}_{4k}$ for $k = 1, 2$, i.e., the Matlab $\mathbf{M}_{4k}$ are just our $\mathbf{A}_{4k}$ with the rows flipped:

$$\mathbf{A}_4 = \mathbf{F}\mathbf{M}_4 = \begin{pmatrix} 4 & 14 & 15 & 1 \\ 9 & 7 & 6 & 12 \\ 5 & 11 & 10 & 8 \\ 16 & 2 & 3 & 13 \end{pmatrix}; \quad \mathbf{A}_8 = \mathbf{F}\mathbf{M}_8 = \begin{pmatrix} 8 & 58 & 59 & 5 & 4 & 62 & 63 & 1 \\ 49 & 15 & 14 & 52 & 53 & 11 & 10 & 56 \\ 41 & 23 & 22 & 44 & 45 & 19 & 18 & 48 \\ 32 & 34 & 35 & 29 & 28 & 38 & 39 & 25 \\ 40 & 26 & 27 & 37 & 36 & 30 & 31 & 33 \\ 17 & 47 & 46 & 20 & 21 & 43 & 42 & 24 \\ 9 & 55 & 54 & 12 & 13 & 51 & 50 & 16 \\ 64 & 2 & 3 & 61 & 60 & 6 & 7 & 57 \end{pmatrix}.$$

Our procedure for generating Agrippa–type magic matrices $\mathbf{A}_n$ for doubly-even $n = 4, 8, \ldots$ uses the (partner) "magic-basis matrix" or "B-matrix" $\mathbf{B}_n$ defined by

$$\mathbf{A}_n = \frac{1}{2}\Big(n\mathbf{B}_n + \mathbf{F}\mathbf{B}_n'\mathbf{F} + (n^2+1)\mathbf{E}\Big); \quad \mathbf{B}_n = \frac{2}{n^2-1}\Big(n\mathbf{A}_n - \mathbf{F}\mathbf{A}_n'\mathbf{F}\Big) - \frac{n^2+1}{n+1}\mathbf{E}$$

$$\mathbf{B}_4 = \begin{pmatrix} -3 & 3 & 3 & -3 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 3 & -3 & -3 & 3 \end{pmatrix}; \quad \mathbf{B}_8 = \begin{pmatrix} -7 & 7 & 7 & -7 & -7 & 7 & 7 & -7 \\ 5 & -5 & -5 & 5 & 5 & -5 & -5 & 5 \\ 3 & -3 & -3 & 3 & 3 & -3 & -3 & 3 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -3 & 3 & 3 & -3 & -3 & 3 & 3 & -3 \\ -5 & 5 & 5 & -5 & -5 & 5 & 5 & -5 \\ 7 & -7 & -7 & 7 & 7 & -7 & -7 & 7 \end{pmatrix}.$$

The 8×8 matrix $\mathbf{B}_8$ is "bordered" (really 2-bordered) in that the 4×4 "inner-core" matrix $\mathbf{B}_4^{(8)}$ say, formed from $\mathbf{B}_8$ by removing its top 2 rows, bottom 2 rows, first 2 columns and last 2 columns, is also a magic-basis matrix.

$$\mathbf{A}_4 = \begin{pmatrix} 4 & 14 & 15 & 1 \\ 9 & 7 & 6 & 12 \\ 5 & 11 & 10 & 8 \\ 16 & 2 & 3 & 13 \end{pmatrix}; \quad \mathbf{A}_8 = \begin{pmatrix} 8 & 58 & 59 & 5 & 4 & 62 & 63 & 1 \\ 49 & 15 & 14 & 52 & 53 & 11 & 10 & 56 \\ 41 & 23 & 22 & 44 & 45 & 19 & 18 & 48 \\ 32 & 34 & 35 & 29 & 28 & 38 & 39 & 25 \\ 40 & 26 & 27 & 37 & 36 & 30 & 31 & 33 \\ 17 & 47 & 46 & 20 & 21 & 43 & 42 & 24 \\ 9 & 55 & 54 & 12 & 13 & 51 & 50 & 16 \\ 64 & 2 & 3 & 61 & 60 & 6 & 7 & 57 \end{pmatrix},$$

have the "two-diagonals-in-progression" (2-dip) property, with parameters $(n, 1;\ n-1, n+1)$, indicating that the starting numbers are n and 1, respectively, on the main forwards and backwards diagonals, with corresponding step sizes $n-1$ and $n+1$,

---

$$n, \quad n+(n-1), \quad n+2(n-1), \quad \ldots, \quad n+(n-1)(n-1);$$

$$1, \quad (n+1)+1, \quad 2(n+1)+1, \quad \ldots, \quad (n-1)(n+1)+1 = n^2.$$

$$\mathbf{B}_8 = \begin{pmatrix} -7 & 7 & 7 & -7 & -7 & 7 & 7 & -7 \\ 5 & -5 & -5 & 5 & 5 & -5 & -5 & 5 \\ 3 & -3 & -3 & 3 & 3 & -3 & -3 & 3 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -3 & 3 & 3 & -3 & -3 & 3 & 3 & -3 \\ -5 & 5 & 5 & -5 & -5 & 5 & 5 & -5 \\ 7 & -7 & -7 & 7 & 7 & -7 & -7 & 7 \end{pmatrix}.$$

The top row of the 8×8 magic-basis matrix $\mathbf{B}_8$ contains precisely 4 entries equal to -7 and 4 entries equal to 7, and the second row contains precisely 4 entries equal to -5 and 4 entries equal to 5, and so on.

We call $\mathbf{B}_n$ and partner $\mathbf{A}_n$ "smooth" whenever in the 2-dip magic-basis matrix $\mathbf{B}_n$ the numbers in row R add up to 0 and are all equal in absolute value to

$$-2R+n+1 \quad \text{when } R \le \tfrac{1}{2}n \quad \text{and} \quad 2R-n-1 \quad \text{when } R \ge \tfrac{1}{2}n+1.$$

We find that $\mathbf{A}_4$, $\mathbf{A}_8$, $\mathbf{B}_4$, $\mathbf{B}_8$ are all smooth.

We will say that the $n \times n$ magic matrices $\mathbf{A}_n$ and $\mathbf{B}_n$ are "row-balanced" whenever

$$(\mathbf{I}+\mathbf{F})(n\mathbf{A}_n - \mathbf{F}\mathbf{A}_n'\mathbf{F}) = (n^2+1)(n-1)\mathbf{E}$$

or equivalently

$$(\mathbf{I}+\mathbf{F})\mathbf{B}_n = \mathbf{0} \quad \Longleftrightarrow \quad \mathbf{B}_n = -\mathbf{F}\mathbf{B}_n$$

We find that $\mathbf{A}_4$, $\mathbf{A}_8$, $\mathbf{B}_4$, $\mathbf{B}_8$ are all row-balanced.

We will say that the magic matrices $\mathbf{A}_n$ and $\mathbf{B}_n$ are "column-balanced" whenever

$$\mathbf{B}_n(\mathbf{I}-\mathbf{F}) = \mathbf{0} \quad \Longleftrightarrow \quad \mathbf{B}_n = \mathbf{B}_n\mathbf{F} \quad \Longleftrightarrow \quad (n\mathbf{A}_n - \mathbf{F}\mathbf{A}_n'\mathbf{F})(\mathbf{I}-\mathbf{F}) = \mathbf{0}.$$

We will say that the $n \times n$ magic matrices $\mathbf{A}_n$ and $\mathbf{B}_n$ with doubly-even $n \geq 4$ are "doubly-balanced" whenever they are both row- and column-balanced, and then

$$\mathbf{B}_n = \mathbf{B}_n\mathbf{F} = -\mathbf{F}\mathbf{B}_n = -\mathbf{F}\mathbf{B}_n\mathbf{F}.$$

We find that $\mathbf{A}_4$, $\mathbf{A}_8$, $\mathbf{B}_4$, $\mathbf{B}_8$ are all doubly-balanced.

From the special sign pattern in each of the rows $\mathbf{B}_8$ together with $\mathbf{B}_8$ being smooth and doubly-balanced, we find the following procedure for generating $n \times n$ doubly-balanced Agrippa–Matlab magic-basis matrices for doubly-even $n = 4, 8, 12, \ldots$

Let $n = 2k$ and so $k = n/2$,

$$\mathbf{A}_n = \tfrac{1}{2}\Big(n\mathbf{B}_n - \mathbf{B}_n' + (n^2+1)\mathbf{E}_n\Big); \quad \mathbf{B}_n = \begin{pmatrix} -\mathbf{I}_k \\ \mathbf{F}_k \end{pmatrix} \mathbf{D}_k \mathbf{b}_k \mathbf{b}_k' \begin{pmatrix} \mathbf{I}_k & \mathbf{F}_k \end{pmatrix},$$

where the $k \times k$ diagonal matrix

$$\mathbf{D}_k = \mathrm{diag}(n-1,\ n-3,\ n-5,\ \ldots,\ 5,\ 3,\ 1); \quad n = 2k.$$

Then the $k \times 1$ "magic-basis vector" $\mathbf{b}_k =$

$$\mathbf{b}_k^{(1)} = (\; +1 \quad -1 \quad -1 \quad +1 \quad +1 \quad \cdots \quad -1 \quad -1 \quad +1 \quad +1 \quad -1 \;)'; \quad n = 2k,$$

generates the Agrippa–Cardano matrices $\mathbf{A}_4$ and $\mathbf{A}_8$.

Drury (2011) created an algorithm for generating classic 2-dip smooth row-balanced $\mathbf{A}_n$ with magic-basis vector

$$\mathbf{b}_k^{(2)} = (\; +1 \quad +1 \quad +1 \quad +1 \quad +1 \quad \cdots \quad -1 \quad -1 \quad -1 \quad -1 \quad -1 \;)'; \quad n = 2k,$$

the sign change being between the entries in positions $\frac{n}{4}$ and $\frac{n}{4} + 1$.

From an algorithm given by Cavendish (1894) we find that choosing $\mathbf{b}_k =$

$$\mathbf{b}_k^{(3)} = (\; +1 \quad -1 \quad +1 \quad -1 \quad +1 \quad \cdots \quad -1 \quad +1 \quad -1 \quad +1 \quad -1 \;)'; \quad n = 2k,$$

also generates classic 2-dip smooth row-balanced $\mathbf{A}_n$ for doubly-even $n = 4, 8, \ldots$

When $n = 4$ our three choices $\mathbf{b}_k^{(1)}, \mathbf{b}_k^{(2)}, \mathbf{b}_k^{(3)}$ of the magic-basis vector coincide and we have

$$\mathbf{b}_2 = \begin{pmatrix} +1 & -1 \end{pmatrix}'; \qquad \mathbf{A}_4 = \begin{pmatrix} 4 & 14 & 15 & 1 \\ 9 & 7 & 6 & 12 \\ 5 & 11 & 10 & 8 \\ 16 & 2 & 3 & 13 \end{pmatrix}.$$

We believe that the Agrippa matrix $\mathbf{A}_4$ here is the only 4×4 classic 2-dip smooth doubly-balanced rank-3 fully-magic matrix.

$$\mathbf{b}_4^{(1)} = \begin{pmatrix} +1 & -1 & -1 & +1 \end{pmatrix}'; \qquad \mathbf{b}_4^{(2)} = \begin{pmatrix} +1 & +1 & -1 - 1 \end{pmatrix}';$$

$$\mathbf{A}_8^{(1)} = \begin{pmatrix} 8 & 58 & 59 & 5 & 4 & 62 & 63 & 1 \\ 49 & 15 & 14 & 52 & 53 & 11 & 10 & 56 \\ 41 & 23 & 22 & 44 & 45 & 19 & 18 & 48 \\ 32 & 34 & 35 & 29 & 28 & 38 & 39 & 25 \\ 40 & 26 & 27 & 37 & 36 & 30 & 31 & 33 \\ 17 & 47 & 46 & 20 & 21 & 43 & 42 & 24 \\ 9 & 55 & 54 & 12 & 13 & 51 & 50 & 16 \\ 64 & 2 & 3 & 61 & 60 & 6 & 7 & 57 \end{pmatrix}; \quad \mathbf{A}_8^{(2)} = \begin{pmatrix} 8 & 7 & 59 & 60 & 61 & 62 & 2 & 1 \\ 16 & 15 & 51 & 52 & 53 & 54 & 10 & 9 \\ 41 & 42 & 22 & 21 & 20 & 19 & 47 & 48 \\ 33 & 34 & 30 & 29 & 28 & 27 & 39 & 40 \\ 25 & 26 & 38 & 37 & 36 & 35 & 31 & 32 \\ 17 & 18 & 46 & 45 & 44 & 43 & 23 & 24 \\ 56 & 55 & 11 & 12 & 13 & 14 & 50 & 49 \\ 64 & 63 & 3 & 4 & 5 & 6 & 58 & 57 \end{pmatrix}.$$

$$\mathbf{b}_4^{(3)} = \begin{pmatrix} +1 & -1 & +1 & -1 \end{pmatrix}'; \qquad \mathbf{A}_8^{(3)} = \begin{pmatrix} 8 & 58 & 6 & 60 & 61 & 3 & 63 & 1 \\ 49 & 15 & 51 & 13 & 12 & 54 & 10 & 56 \\ 24 & 42 & 22 & 44 & 45 & 19 & 47 & 17 \\ 33 & 31 & 35 & 29 & 28 & 38 & 26 & 40 \\ 25 & 39 & 27 & 37 & 36 & 30 & 34 & 32 \\ 48 & 18 & 46 & 20 & 21 & 43 & 23 & 41 \\ 9 & 55 & 11 & 53 & 52 & 14 & 50 & 16 \\ 64 & 2 & 62 & 4 & 5 & 59 & 7 & 57 \end{pmatrix}.$$

Our objective here is to obtain a sequence of classic fully-magic matrices $\mathbf{A}_6$, $\mathbf{A}_{10}$, ..., starting with, and having many of the properties of the (original) Agrippa–Cardano magic matrix

$$\mathbf{A}_6 = \begin{pmatrix} 6 & 32 & 3 & 34 & 35 & 1 \\ 7 & 11 & 27 & 28 & 8 & 30 \\ 19 & 14 & 16 & 15 & 23 & 24 \\ 18 & 20 & 22 & 21 & 17 & 13 \\ 25 & 29 & 10 & 9 & 26 & 12 \\ 36 & 5 & 33 & 4 & 2 & 31 \end{pmatrix}.$$

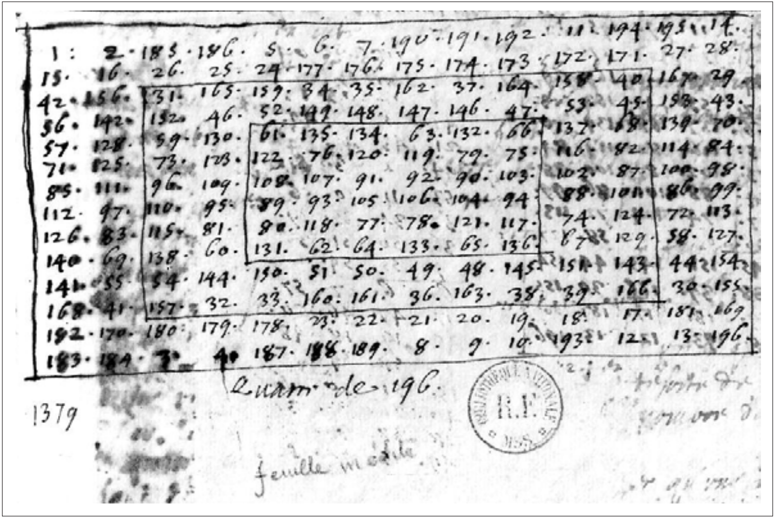
This matrix $\mathbf{A}_6$ is the only $n \times n$ Agrippa magic matrix with n singly-even.

We now seek $n \times n$ fully-magic matrices $\mathbf{A}_n$ and B-bar partners $\bar{\mathbf{B}}_{n/2,n}$, for singly-even $n = 4k + 2 \geq 10$ which, like $\mathbf{A}_6$, satisfy our 7 desiderata in Table 6.1 below.

We will use a 14×14 "bordered" magic square due to Pierre de Fermat (1601/1607–1665) and so we will call our $\mathbf{A}_6$, $\mathbf{A}_{10}$, $\mathbf{A}_{14}$, $\ldots$, "Agrippa–Fermat magic matrices".

We also use a 22×22 magic square specially constructed for us *à la Fermat* by Christian Boyer in Paris.

Fermat's 14×14 magic square was given in a letter by Fermat to Père Marin Mersenne (1588–1648). The original copy of this letter has been lost but a hand-written copy by Félix Vicq d'Azyr (1746–1794) is in the Bibliothèque nationale de France in Paris.



1	2	185	186	5	6	7	190	191	192	11	194	195	14
15	16	26	25	24	177	176	175	174	173	172	171	27	28
42	156	31	165	159	34	35	162	37	164	158	40	167	29
56	142	152	46	52	149	148	147	146	47	53	45	153	43
57	128	59	130	61	135	134	63	132	66	137	68	139	70
71	125	73	123	122	76	120	119	79	75	116	82	114	84
85	111	96	109	108	107	91	92	90	103	102	87	100	98
112	97	110	95	89	93	105	106	104	94	88	101	86	99
126	83	115	81	80	118	77	78	121	117	74	124	72	113
140	69	138	60	131	62	64	133	65	136	67	129	58	127
141	55	54	144	150	51	50	49	48	145	151	143	44	154
168	41	157	32	33	160	161	36	163	38	39	166	30	155
182	170	180	179	178	23	22	21	20	19	18	17	181	169
183	184	3	4	187	188	189	8	9	10	193	12	13	196

Quarré de 196.

1379

feuille mobile

From *Copies de lettres et opuscules de Fermat, Descartes, Galilée, etc., Fol. 1, Copies de lettres de Fermat à Roberval, au P. Mersenne, à Frenicle, etc.*, by Félix Vicq d'Azyr (1746–1794), MS #10556, Département de manuscrits, Bibliothéque nationale de France, Paris. Via Christian Boyer.

(left panel) Pierre de Fermat (1601/1607–1665) and Fermat's Last Theorem: France 2001;
(right panel) The Mersenne prime $2^{13466917} - 1$: Liechtenstein 2004, *Scott* 1297.

Table 6.1: Desiderata for our Agrippa–Fermat magic matrices $\mathbf{A}_n$; $n = 4k + 2 \geq 6$.

1. classic, with entries 1, 2, $\ldots n^2$ in some order, magic sum $\frac{n(n^2+1)}{2}$
2. 2-dip with parameters $(n, 1;\ n-1,\ n+1)$
3. nonsingular, with partner B-bar matrix $\bar{\mathbf{B}}_{n/2,n}$ of full row rank $n/2$
4. row-balanced, with $\mathbf{B}_n = -\mathbf{F}\mathbf{B}_n$
5. smooth, with all entries of $\bar{\mathbf{B}}_{n/2,n}$ equal to ± 1
6. bordered, and with
7. characteristic polynomial: $\det(\lambda \mathbf{I}_n - \mathbf{B}_n) = \lambda^{n-3}(\lambda^3 + 120)$.

When $n = 6$ we have

$$\mathbf{B}_6 \;=\; \frac{2}{35}\big(6\mathbf{A}_6 - \mathbf{F}\mathbf{A}_6'\mathbf{F}\big) - \frac{37}{7}\mathbf{E} \;=\; \begin{pmatrix} -5 & 5 & -5 & 5 & 5 & -5 \\ -3 & -3 & 3 & 3 & -3 & 3 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 & -1 & -1 \\ 3 & 3 & -3 & -3 & 3 & -3 \\ 5 & -5 & 5 & -5 & -5 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{I}_3 \\ -\mathbf{F}_3 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \end{pmatrix}.$$

The magic-basis matrix $\mathbf{B}_6$ is row-balanced in that its last 3 rows flipped coincide with its first three rows with a sign change, i.e.,

$$\mathbf{B}_6 = -\mathbf{F}\mathbf{B}_6$$

which we write as

$$\mathbf{B}_6 = \mathbf{J}_{6,3}\mathbf{D}_3\bar{\mathbf{B}}_{3,6}$$

where

$$\mathbf{J}_{6,3} = \begin{pmatrix} \mathbf{I}_3 \\ -\mathbf{F}_3 \end{pmatrix}; \quad \mathbf{D}_3 = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \bar{\mathbf{B}}_{3,6} = \begin{pmatrix} -1 & 1 & -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \end{pmatrix}$$

We call the 3×6 matrix $\bar{\mathbf{B}}_{3,6}$ a "reduced magic-basis matrix" or "B-bar matrix".

Our B-bar matrices $\bar{\mathbf{B}}_{n/2,n}$ for $n = 6, 10, 14, 18, 22, 26$ are shown below.

The full 13×26 matrix is $\bar{\mathbf{B}}_{13,26}$; the B-bar matrix formed by removing the (outer) brown border is the 11×22 matrix $\bar{\mathbf{B}}_{11,22}$. And so on.

And then removing the green border gives the 3×6 matrix $\bar{\mathbf{B}}_{3,6}$ for $\mathbf{A}_6$.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	
1	-1	1	-1	1	1	-1	1	1	-1	-1	1	1	1	-1	-1	-1	1	1	-1	-1	1	1	1	-1	-1	-1	1
2	-1	-1	-1	-1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	1	1	-1	-1	2
3	-1	1	-1	1	1	-1	1	1	-1	-1	1	1	1	-1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	-1	3
4	-1	1	-1	-1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	-1	4
5	-1	1	-1	1	-1	1	1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	5
6	-1	1	-1	1	-1	-1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	1	-1	1	-1	6
7	-1	1	-1	1	-1	1	-1	1	1	-1	1	1	1	-1	-1	-1	1	1	-1	-1	1	-1	1	-1	1	-1	7
8	-1	1	-1	1	-1	1	-1	-1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	1	-1	1	-1	1	-1	8
9	-1	1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1	-1	1	1	-1	1	-1	1	-1	1	-1	1	-1	9
10	-1	1	-1	1	-1	1	-1	1	-1	-1	-1	1	1	1	1	-1	-1	1	1	-1	1	-1	1	-1	1	-1	10
11	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	11
12	-1	1	-1	1	-1	1	-1	1	-1	1	-1	-1	1	1	-1	1	1	-1	1	-1	1	-1	1	-1	1	-1	12
13	-1	1	-1	1	-1	1	-1	1	-1	1	1	-1	-1	-1	1	1	1	-1	1	-1	1	-1	1	-1	1	-1	13

The B-bar matrices $\bar{\mathbf{B}}_{5,10}$ (green border) and $\bar{\mathbf{B}}_{7,14}$ (yellow border) are based on the 14×14 Fermat magic square.

The B-bar matrices $\bar{\mathbf{B}}_{9,18}$ (blue border) and $\bar{\mathbf{B}}_{11,22}$ (pink border) are based on the 22×22 Fermat–Boyer magic square.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	
1	-1	1	-1	1	1	-1	1	1	-1	-1	1	1	1	-1	-1	-1	1	1	-1	-1	1	1	1	-1	-1	-1	1
2	-1	-1	-1	-1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	1	1	-1	-1	2
3	-1	1	-1	1	1	-1	1	1	-1	-1	1	1	1	-1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	-1	3
4	-1	1	-1	-1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	-1	4
5	-1	1	-1	1	-1	1	1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	5
6	-1	1	-1	1	-1	-1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	1	-1	1	-1	6
7	-1	1	-1	1	-1	1	-1	1	1	-1	1	1	1	-1	-1	-1	1	1	-1	-1	1	-1	1	-1	1	-1	7
8	-1	1	-1	1	-1	1	-1	-1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	1	-1	1	-1	1	-1	8
9	-1	1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1	-1	1	1	-1	1	-1	1	-1	1	-1	1	-1	9
10	-1	1	-1	1	-1	1	-1	1	-1	-1	-1	1	1	1	1	-1	-1	1	1	-1	1	-1	1	-1	1	-1	10
11	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	11
12	-1	1	-1	1	-1	1	-1	1	-1	1	-1	-1	1	1	-1	1	1	-1	1	-1	1	-1	1	-1	1	-1	12
13	-1	1	-1	1	-1	1	-1	1	-1	1	1	-1	-1	-1	1	1	1	-1	1	-1	1	-1	1	-1	1	-1	13

1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*	*21*	*22*
1	2	465	466	5	6	469	470	9	10	11	474	475	476	15	16	479	480	19	482	483	22
23	24	42	41	40	39	38	37	36	453	452	451	450	449	448	447	446	445	444	443	43	44
66	420	47	48	423	424	51	52	427	428	429	56	57	58	433	434	61	436	437	64	439	45
88	398	69	70	84	83	82	81	80	409	408	407	406	405	404	403	402	401	85	86	417	67
110	376	108	378	93	94	381	382	97	98	99	386	387	388	103	390	391	106	393	91	395	89
132	354	130	356	115	116	126	125	124	365	364	363	362	361	360	359	127	128	371	113	373	111
154	332	152	334	150	336	139	345	339	142	143	342	145	344	338	148	347	137	349	135	351	133
155	310	174	312	172	314	324	162	168	321	320	319	318	163	169	161	325	159	327	157	329	176
177	288	179	290	181	292	183	294	185	299	298	187	296	190	301	192	303	194	305	196	307	198
199	285	201	283	203	281	205	279	278	208	276	275	211	207	272	214	270	216	268	218	266	220
221	263	223	261	225	259	236	257	256	255	231	232	230	251	250	227	248	238	246	240	244	242
264	241	262	239	260	237	258	235	229	233	253	254	252	234	228	249	226	247	224	245	222	243
286	219	284	217	282	215	271	213	212	274	209	210	277	273	206	280	204	269	202	267	200	265
308	197	306	195	304	193	302	184	295	186	188	297	189	300	191	293	182	291	180	289	178	287
309	175	311	173	313	171	170	316	322	167	166	165	164	317	323	315	160	326	158	328	156	330
352	153	350	151	348	149	337	140	141	340	341	144	343	146	147	346	138	335	136	333	134	331
353	131	355	129	370	358	368	367	366	123	122	121	120	119	118	117	369	357	114	372	112	374
375	109	377	107	379	380	95	96	383	384	385	100	101	102	389	104	105	392	92	394	90	396
397	87	416	400	414	413	412	411	410	79	78	77	76	75	74	73	72	71	415	399	68	418
440	65	421	422	49	50	425	426	53	54	55	430	431	432	59	60	435	62	63	438	46	419
462	442	460	459	458	457	456	455	454	35	34	33	32	31	30	29	28	27	26	25	461	441
463	464	3	4	467	468	7	8	471	472	473	12	13	14	477	478	17	18	481	20	21	484

$$
\left(\begin{array}{rrrrrrrrrrrrrrrrrrrrrr}
-21 & 21 & 21 & -21 & \frac{485}{23} & \frac{485}{23} & -\frac{485}{23} & -21 & 21 & 21 & 21 & -21 & -21 & -21 & \frac{485}{23} & 21 & -\frac{485}{23} & -\frac{485}{23} & 21 & 21 & -21 & -21 \\
-19 & -19 & 19 & 19 & 19 & 19 & 19 & 19 & 19 & 19 & 19 & 19 & 19 & -19 & -19 & -19 & -19 & -19 & -19 & -19 & -19 & -19 \\
-17 & 17 & -17 & 17 & \frac{8245}{483} & -17 & 17 & \frac{8245}{483} & -\frac{8245}{483} & -\frac{8245}{483} & -\frac{8245}{483} & \frac{8245}{483} & \frac{8245}{483} & \frac{8245}{483} & -\frac{8245}{483} & -\frac{8245}{483} & 17 & 17 & -17 & -17 & 17 & -17 \\
-15 & 15 & -15 & -15 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & -15 & -15 & -15 & -15 & -15 & -15 & -15 & 15 & -15 \\
-\frac{343}{23} & 13 & -13 & 13 & -13 & 13 & 13 & -13 & 13 & 13 & 13 & -13 & -13 & -13 & 13 & 13 & -13 & -13 & 13 & -13 & 13 & -\frac{255}{23} \\
-\frac{297}{23} & 11 & -11 & 11 & -11 & -11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & -11 & -11 & -11 & -11 & -11 & 11 & -11 & 11 & -\frac{209}{23} \\
-9 & 9 & -\frac{5095}{483} & 9 & -9 & 9 & -9 & 9 & 9 & -9 & 9 & -\frac{1455}{161} & -9 & \frac{1455}{161} & 9 & -9 & 9 & -9 & 9 & -\frac{3599}{483} & 9 & -9 \\
-\frac{117}{23} & 7 & -\frac{4129}{483} & 7 & -7 & 7 & -7 & -7 & -7 & 7 & 7 & 7 & 7 & -7 & -7 & 7 & 7 & -7 & 7 & -\frac{2633}{483} & 7 & -\frac{205}{23} \\
-5 & 5 & -\frac{1667}{483} & 5 & -5 & 5 & -\frac{673}{161} & 5 & -5 & 5 & -5 & 5 & 5 & -5 & 5 & -\frac{937}{161} & 5 & -5 & 5 & -\frac{3163}{483} & 5 & -5 \\
-3 & 3 & -\frac{701}{483} & 3 & -3 & 3 & -3 & 3 & -3 & -3 & 3 & 3 & -3 & 3 & 3 & -3 & 3 & -3 & 3 & -\frac{2197}{483} & 3 & -3 \\
-1 & 1 & \frac{265}{483} & 1 & -1 & 1 & -\frac{293}{161} & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -\frac{29}{161} & 1 & -1 & 1 & -\frac{1231}{483} & 1 & -1 \\
1 & -1 & -\frac{265}{483} & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & \frac{1231}{483} & -1 & 1 \\
3 & -3 & \frac{701}{483} & -3 & 3 & -3 & 3 & -3 & 3 & 3 & -3 & -3 & 3 & -3 & -3 & 3 & -3 & 3 & -3 & \frac{2197}{483} & -3 & 3 \\
5 & -5 & \frac{1667}{483} & -5 & 5 & -5 & 5 & -5 & 5 & -5 & 5 & -5 & -5 & 5 & -5 & 5 & -5 & 5 & -5 & \frac{3163}{483} & -5 & 5 \\
7 & -7 & \frac{4129}{483} & -7 & 7 & -7 & 7 & 7 & 7 & -7 & -7 & -7 & -7 & 7 & 7 & -7 & -7 & 7 & -7 & \frac{2633}{483} & -7 & 7 \\
\frac{163}{23} & -9 & 9 & -9 & 9 & -9 & 9 & -9 & -9 & 9 & -9 & \frac{1455}{161} & 9 & -\frac{1455}{161} & -9 & 9 & -9 & 9 & -9 & 9 & -9 & \frac{251}{23} \\
\frac{297}{23} & -11 & 11 & -11 & 11 & 11 & -11 & -11 & -11 & -11 & -11 & -11 & -11 & 11 & 11 & 11 & 11 & 11 & -11 & 11 & -11 & \frac{209}{23} \\
\frac{343}{23} & -13 & \frac{7027}{483} & -13 & 13 & -13 & -13 & 13 & -13 & -13 & -13 & 13 & 13 & 13 & -13 & -13 & 13 & 13 & -13 & \frac{5531}{483} & -13 & \frac{255}{23} \\
15 & -15 & 15 & 15 & -15 & -15 & -15 & -15 & -15 & -15 & -15 & -15 & -15 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & -15 & 15 \\
17 & -17 & 17 & -17 & -\frac{8245}{483} & 17 & -17 & -\frac{8245}{483} & \frac{8245}{483} & \frac{8245}{483} & \frac{8245}{483} & -\frac{8245}{483} & -\frac{8245}{483} & -\frac{8245}{483} & \frac{8245}{483} & \frac{8245}{483} & -17 & -17 & 17 & 17 & -17 & 17 \\
19 & 19 & -19 & -19 & -19 & -19 & -19 & -19 & -19 & -19 & -19 & -19 & -19 & 19 & 19 & 19 & 19 & 19 & 19 & 19 & 19 & 19 \\
21 & -21 & -21 & 21 & -\frac{485}{23} & -\frac{485}{23} & \frac{485}{23} & 21 & -21 & -21 & -21 & 21 & 21 & 21 & -\frac{485}{23} & -21 & \frac{485}{23} & \frac{485}{23} & -21 & -21 & 21 & 21
\end{array}\right)
$$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	2	465	466	5	6	469	470	9	10	11	474	475	476	15	16	479	480	19	482	483	22
23	24	42	41	40	39	38	37	36	453	452	451	450	449	448	447	446	445	444	443	43	44
66	420	47	48	423	424	51	52	427	428	429	56	57	58	433	434	61	436	437	64	439	45
88	398	69	70	84	83	82	81	80	409	408	407	406	405	404	403	402	401	85	86	417	67
89	376	108	378	93	94	381	382	97	98	99	386	387	388	103	390	391	106	393	91	395	110
111	354	130	356	115	116	126	125	124	365	364	363	362	361	360	359	127	128	371	113	373	132
154	332	135	334	150	336	139	345	339	142	143	342	145	344	338	148	347	137	349	152	351	133
176	310	157	312	172	314	324	162	168	321	320	319	318	163	169	161	325	159	327	174	329	155
177	288	196	290	181	292	192	294	185	299	298	187	296	190	301	183	303	194	305	179	307	198
199	285	218	283	203	281	205	279	278	208	276	275	211	207	272	214	270	216	268	201	266	220
221	263	240	261	225	259	227	257	256	255	231	232	230	251	250	236	248	238	246	223	244	242
264	241	245	239	260	237	258	235	229	233	253	254	252	234	228	249	226	247	224	262	222	243
286	219	267	217	282	215	271	213	212	274	209	210	277	273	206	280	204	269	202	284	200	265
308	197	289	195	304	193	302	184	295	186	188	297	189	300	191	293	182	291	180	306	178	287
309	175	328	173	313	171	170	316	322	167	166	165	164	317	323	315	160	326	158	311	156	330
331	153	350	151	348	149	337	140	141	340	341	144	343	146	147	346	138	335	136	333	134	352
374	131	355	129	370	358	368	367	366	123	122	121	120	119	118	117	369	357	114	372	112	353
396	109	394	107	379	380	95	96	383	384	385	100	101	102	389	104	105	392	92	377	90	375
397	87	416	400	414	413	412	411	410	79	78	77	76	75	74	73	72	71	415	399	68	418
440	65	421	422	49	50	425	426	53	54	55	430	431	432	59	60	435	62	63	438	46	419
462	442	460	459	458	457	456	455	454	35	34	33	32	31	30	29	28	27	26	25	461	441
463	464	3	4	467	468	7	8	471	472	473	12	13	14	477	478	17	18	481	20	21	484

$$
\begin{pmatrix}
-21 & 21 & 21 & -21 & 21 & 21 & -21 & -21 & 21 & 21 & 21 & -21 & -21 & -21 & 21 & 21 & -21 & -21 & 21 & 21 & -21 & -21 \\
-19 & -19 & 19 & 19 & 19 & 19 & 19 & 19 & 19 & 19 & 19 & 19 & 19 & -19 & -19 & -19 & -19 & -19 & -19 & -19 & -19 & -19 \\
-17 & 17 & -17 & 17 & 17 & -17 & 17 & 17 & -17 & -17 & -17 & 17 & 17 & 17 & -17 & -17 & 17 & 17 & -17 & -17 & 17 & -17 \\
-15 & 15 & -15 & -15 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & -15 & -15 & -15 & -15 & -15 & -15 & -15 & 15 & -15 \\
-13 & 13 & -13 & 13 & -13 & 13 & 13 & -13 & 13 & 13 & 13 & -13 & -13 & -13 & 13 & 13 & -13 & -13 & 13 & -13 & 13 & -13 \\
-11 & 11 & -11 & 11 & -11 & -11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & -11 & -11 & -11 & -11 & -11 & 11 & -11 & 11 & -11 \\
-9 & 9 & -9 & 9 & -9 & 9 & -9 & 9 & 9 & -9 & 9 & -9 & -9 & 9 & 9 & -9 & 9 & -9 & 9 & -9 & 9 & -9 \\
-7 & 7 & -7 & 7 & -7 & 7 & -7 & -7 & -7 & 7 & 7 & 7 & 7 & -7 & -7 & 7 & 7 & -7 & 7 & -7 & 7 & -7 \\
-5 & 5 & -5 & 5 & -5 & 5 & -5 & 5 & -5 & 5 & -5 & 5 & 5 & -5 & 5 & -5 & 5 & -5 & 5 & -5 & 5 & -5 \\
-3 & 3 & -3 & 3 & -3 & 3 & -3 & 3 & -3 & -3 & 3 & 3 & -3 & 3 & 3 & -3 & 3 & -3 & 3 & -3 & 3 & -3 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
3 & -3 & 3 & -3 & 3 & -3 & 3 & -3 & 3 & 3 & -3 & -3 & 3 & -3 & -3 & 3 & -3 & 3 & -3 & 3 & -3 & 3 \\
5 & -5 & 5 & -5 & 5 & -5 & 5 & -5 & 5 & -5 & 5 & -5 & -5 & 5 & -5 & 5 & -5 & 5 & -5 & 5 & -5 & 5 \\
7 & -7 & 7 & -7 & 7 & -7 & 7 & 7 & 7 & -7 & -7 & -7 & -7 & 7 & 7 & -7 & -7 & 7 & -7 & 7 & -7 & 7 \\
9 & -9 & 9 & -9 & 9 & -9 & 9 & -9 & -9 & 9 & -9 & 9 & 9 & -9 & -9 & 9 & -9 & 9 & -9 & 9 & -9 & 9 \\
11 & -11 & 11 & -11 & 11 & 11 & -11 & -11 & -11 & -11 & -11 & -11 & -11 & 11 & 11 & 11 & 11 & 11 & -11 & 11 & -11 & 11 \\
13 & -13 & 13 & -13 & 13 & -13 & -13 & 13 & -13 & -13 & -13 & 13 & 13 & 13 & -13 & -13 & 13 & 13 & -13 & 13 & -13 & 13 \\
15 & -15 & 15 & 15 & -15 & -15 & -15 & -15 & -15 & -15 & -15 & -15 & -15 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & -15 & 15 \\
17 & -17 & 17 & -17 & -17 & 17 & -17 & -17 & 17 & 17 & 17 & -17 & -17 & -17 & 17 & 17 & -17 & -17 & 17 & 17 & -17 & 17 \\
19 & 19 & -19 & -19 & -19 & -19 & -19 & -19 & -19 & -19 & -19 & -19 & -19 & 19 & 19 & 19 & 19 & 19 & 19 & 19 & 19 & 19 \\
21 & -21 & -21 & 21 & -21 & -21 & 21 & 21 & -21 & -21 & -21 & 21 & 21 & 21 & -21 & -21 & 21 & 21 & -21 & -21 & 21 & 21
\end{pmatrix}
$$

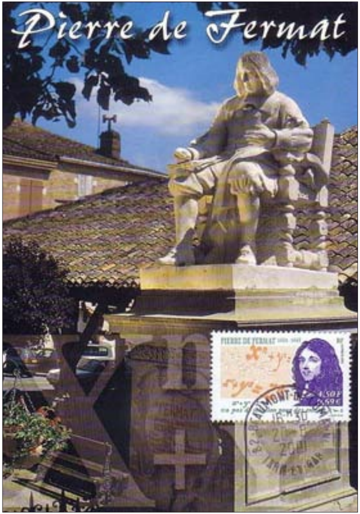
With some trial and error we found this smooth B-bar 13×26 magic-basis matrix

$$\begin{pmatrix}
-1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 \\
-1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\
-1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1
\end{pmatrix}$$

and then we constructed its partner Agrippa-type magic matrix $\mathbf{A}_{26}$

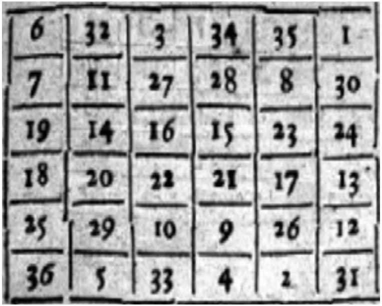
Our *pièce de resistance:*

$$
\begin{pmatrix}
26 & 675 & 24 & 673 & 672 & 21 & 670 & 669 & 18 & 17 & 666 & 665 & 664 & 13 & 12 & 11 & 660 & 659 & 8 & 7 & 656 & 655 & 654 & 3 & 2 & 1 \\
52 & 51 & 29 & 30 & 629 & 630 & 631 & 632 & 633 & 634 & 635 & 636 & 637 & 638 & 639 & 42 & 43 & 44 & 45 & 46 & 47 & 48 & 647 & 648 & 28 & 27 \\
78 & 600 & 76 & 621 & 620 & 73 & 618 & 617 & 70 & 69 & 614 & 613 & 612 & 65 & 64 & 63 & 608 & 607 & 60 & 59 & 604 & 603 & 56 & 55 & 623 & 53 \\
79 & 574 & 102 & 101 & 577 & 578 & 579 & 580 & 581 & 582 & 583 & 584 & 585 & 586 & 587 & 94 & 95 & 96 & 97 & 98 & 99 & 100 & 82 & 81 & 597 & 104 \\
105 & 571 & 107 & 569 & 126 & 567 & 566 & 565 & 122 & 563 & 120 & 561 & 118 & 559 & 116 & 557 & 114 & 555 & 112 & 553 & 110 & 109 & 550 & 128 & 548 & 130 \\
131 & 545 & 133 & 543 & 152 & 151 & 527 & 528 & 529 & 530 & 531 & 532 & 533 & 534 & 535 & 146 & 147 & 148 & 149 & 150 & 136 & 135 & 524 & 154 & 522 & 156 \\
182 & 519 & 180 & 517 & 161 & 515 & 176 & 513 & 512 & 173 & 510 & 509 & 508 & 169 & 168 & 167 & 504 & 503 & 164 & 163 & 500 & 178 & 498 & 159 & 496 & 157 \\
208 & 493 & 206 & 491 & 204 & 489 & 202 & 201 & 477 & 478 & 479 & 480 & 481 & 482 & 483 & 198 & 199 & 200 & 190 & 189 & 474 & 187 & 472 & 185 & 470 & 183 \\
209 & 467 & 211 & 465 & 213 & 463 & 215 & 461 & 226 & 452 & 458 & 223 & 456 & 221 & 220 & 453 & 459 & 217 & 450 & 228 & 448 & 230 & 446 & 232 & 444 & 234 \\
235 & 441 & 237 & 439 & 256 & 437 & 241 & 435 & 243 & 251 & 245 & 428 & 429 & 430 & 431 & 250 & 244 & 434 & 424 & 254 & 422 & 239 & 420 & 258 & 418 & 260 \\
286 & 415 & 284 & 413 & 265 & 411 & 280 & 409 & 269 & 407 & 276 & 402 & 273 & 404 & 405 & 271 & 400 & 278 & 398 & 267 & 396 & 282 & 394 & 263 & 392 & 261 \\
312 & 366 & 310 & 368 & 308 & 370 & 306 & 372 & 304 & 374 & 297 & 301 & 377 & 378 & 298 & 380 & 381 & 295 & 383 & 293 & 385 & 291 & 387 & 289 & 389 & 287 \\
338 & 340 & 336 & 342 & 317 & 344 & 332 & 346 & 330 & 348 & 349 & 324 & 326 & 325 & 353 & 354 & 355 & 321 & 357 & 319 & 359 & 334 & 361 & 315 & 363 & 313 \\
339 & 314 & 341 & 316 & 360 & 318 & 345 & 320 & 347 & 322 & 328 & 350 & 352 & 351 & 327 & 323 & 329 & 356 & 331 & 358 & 333 & 343 & 335 & 362 & 337 & 364 \\
365 & 288 & 367 & 290 & 369 & 292 & 371 & 294 & 382 & 296 & 375 & 379 & 300 & 299 & 376 & 302 & 303 & 373 & 305 & 384 & 307 & 386 & 309 & 388 & 311 & 390 \\
391 & 262 & 393 & 264 & 412 & 266 & 397 & 268 & 399 & 277 & 406 & 275 & 403 & 274 & 272 & 401 & 270 & 408 & 279 & 410 & 281 & 395 & 283 & 414 & 285 & 416 \\
442 & 236 & 440 & 238 & 421 & 240 & 436 & 242 & 425 & 433 & 427 & 246 & 247 & 248 & 249 & 432 & 426 & 252 & 253 & 423 & 255 & 438 & 257 & 419 & 259 & 417 \\
468 & 210 & 466 & 212 & 464 & 214 & 449 & 216 & 460 & 225 & 224 & 457 & 222 & 455 & 454 & 219 & 218 & 451 & 227 & 462 & 229 & 447 & 231 & 445 & 233 & 443 \\
469 & 184 & 471 & 186 & 473 & 188 & 475 & 487 & 191 & 192 & 193 & 194 & 195 & 196 & 197 & 484 & 485 & 486 & 476 & 488 & 203 & 490 & 205 & 492 & 207 & 494 \\
495 & 158 & 497 & 160 & 499 & 162 & 514 & 175 & 174 & 511 & 172 & 171 & 170 & 507 & 506 & 505 & 166 & 165 & 502 & 501 & 177 & 516 & 179 & 518 & 181 & 520 \\
546 & 132 & 544 & 134 & 525 & 541 & 137 & 138 & 139 & 140 & 141 & 142 & 143 & 144 & 145 & 536 & 537 & 538 & 539 & 540 & 526 & 542 & 153 & 523 & 155 & 521 \\
547 & 106 & 549 & 108 & 568 & 125 & 124 & 123 & 564 & 121 & 562 & 119 & 560 & 117 & 558 & 115 & 556 & 113 & 554 & 111 & 552 & 551 & 127 & 570 & 129 & 572 \\
573 & 103 & 575 & 595 & 83 & 84 & 85 & 86 & 87 & 88 & 89 & 90 & 91 & 92 & 93 & 588 & 589 & 590 & 591 & 592 & 593 & 594 & 576 & 596 & 80 & 598 \\
624 & 77 & 622 & 75 & 74 & 619 & 72 & 71 & 616 & 615 & 68 & 67 & 66 & 611 & 610 & 609 & 62 & 61 & 606 & 605 & 58 & 57 & 602 & 601 & 54 & 599 \\
625 & 649 & 627 & 628 & 31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 & 41 & 640 & 641 & 642 & 643 & 644 & 645 & 646 & 49 & 50 & 626 & 650 \\
676 & 25 & 674 & 23 & 22 & 671 & 20 & 19 & 668 & 667 & 16 & 15 & 14 & 663 & 662 & 661 & 10 & 9 & 658 & 657 & 6 & 5 & 4 & 653 & 652 & 651
\end{pmatrix}.
$$

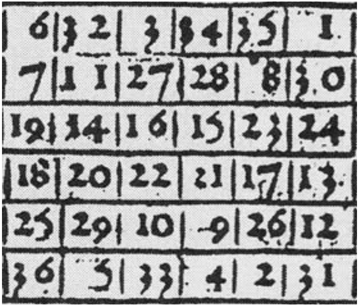


Pierre de Fermat (1601/1607–1665) and Fermat's Last Theorem: (left panel) Germany 2011; (right panel) France 2001, with statue of Fermat in his birthplace Beaumont-de-Lomagne.

Two 6 × 6 magic-square planetary talismans (MSPTs)



6	32	3	34	35	1
7	11	27	28	8	30
19	14	16	15	23	24
18	20	22	21	17	13
25	29	10	9	26	12
36	5	33	4	2	31



6	32	3	34	35	1
7	11	27	28	8	30
19	14	16	15	23	24
18	20	22	21	17	13
25	29	10	9	26	12
36	5	33	4	2	31

(left panel) by Agrippa & (right panel) by Cardano, both for the Sun.

The Agrippa–Cardano magic matrix

$$\mathbf{A}_6 = \begin{pmatrix} 6 & 32 & 3 & 34 & 35 & 1 \\ 7 & 11 & 27 & 28 & 8 & 30 \\ 19 & 14 & 16 & 15 & 23 & 24 \\ 18 & 20 & 22 & 21 & 17 & 13 \\ 25 & 29 & 10 & 9 & 26 & 12 \\ 36 & 5 & 33 & 4 & 2 & 31 \end{pmatrix}$$

is not the only 6×6 classic nonsingular row-balanced smooth 2-dip magic matrix.

$$\bar{\mathbf{B}}_{3,6} = \begin{pmatrix} -1 & 1 & -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \end{pmatrix}$$

and we observe that in each row there are 3 entries equal to -1 with 2 of these 3 entries "fixed" by the 2-dip property.

In the top row the "free" -1 is in column 3,
in the second row the "free" -1 is in column 1,
and in the third row the "free" -1 is in column 2.

We may, therefore, write $\bar{\mathbf{B}}_{3,6} = \bar{\mathbf{B}}_{3,6}^{(312)}$.

In all there are $64 = 4^3$ possible versions of $\bar{\mathbf{B}}_{3,6}$

running from $\bar{\mathbf{B}}_{3,6}^{(211)}$ to $\bar{\mathbf{B}}_{3,6}^{(566)}$,

with just 16 (flavour A) yielding a classic nonsingular fully-magic matrix,

confirming a result of Drury (2011).

The 48 singular magic matrices $\mathbf{A}$ all have rank 5,

with 24 having magic-basis partner $\mathbf{B}$ of rank 3 (flavour B)

and 24 of rank 2 (flavour C).

None of the these 48 singular magic matrices $\mathbf{A}$ are classic.

We want to know in how many ways x, and in which ways, can we create a 5×10 reduced magic-basis matrix $\bar{\mathbf{B}}_{5,10}$ that is partner to an Agrippa-type magic matrix $\mathbf{A}_{10}$ which is

1. classic, with entries 1, 2, $\ldots n^2$ in some order, magic sum $\frac{n(n^2+1)}{2}$
2. 2-dip with parameters $(n, 1;\; n-1,\; n+1)$
3. nonsingular, with partner B-bar matrix $\bar{\mathbf{B}}_{n/2,n}$ of full row rank $n/2$
4. row-balanced, with $\mathbf{B}_n = -\mathbf{F}\mathbf{B}_n$
5. smooth, with all entries of $\bar{\mathbf{B}}_{n/2,n}$ equal to ± 1
6. bordered, and with
7. characteristic polynomial: $\det(\lambda \mathbf{I}_n - \mathbf{B}_n) = \lambda^{n-3}(\lambda^3 + 120)$.

... to the number of ways w_1, and in which ways, we can set up the ± 1 in the green cells below so that the (full) 5×10 B-bar matrix $\bar{\mathbf{B}}_{5,10}$ partners a classic smooth nonsingular 2-dip 10×10 Agrippa-type magic matrix $\mathbf{A}_{10}$?

The ± 1 in the 22 brown cells are fixed.
The row sums must all equal 0 and so $w_1 = 56^2$.

-1	1	1	-1	1	-1	-1	1	1	-1
-1	-1	-1	1	1	1	1	-1	-1	1
-1	1	-1	1	-1	1	1	-1	1	-1
-1	1	-1	-1	1	1	-1	1	1	-1
-1	1	1	-1	-1	-1	1	1	1	-1

Let the 3×2 blue matrices be $\mathbf{Q}_1$, bottom left and $\mathbf{Q}_2$ bottom right.
If, as here we have $\mathbf{Q}_2 = \mathbf{Q}_1\mathbf{F}$, then we may choose the ± 1 in the blue cells in $w_2 = 2^3$ ways. Hence

$$x \leq w_1 \times w_2 = 56^2 \times 2^3 = 7^2 \times 2^9 = 25088.$$

-1	1	1	-1	1	-1	-1	1	1	-1
-1	-1	-1	1	1	1	1	-1	-1	1
-1	1	-1	1	-1	1	1	-1	1	-1
-1	1	-1	-1	1	1	-1	1	1	-1
-1	1	1	-1	-1	-1	1	1	1	-1

I look forward to hearing from you!

My e-mail is styan@math.mcgill.ca