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LECTURE 6
Number of perfect matchings in regular graphs

6.1. Outline of Lecture

- Perfect matchings in cubic bipartite graphs
- Van der Waerden's conjecture
- Exponentially many perfect matchings in general regular graphs

6.2. Voorhoeve's bound on the number of perfect matchings in cubic bipartite graphs

The goal of this lecture is to show that every k-regular bipartite graph, as well as every k-regular $(k-1)$-edge-connected graph, has exponentially many perfect matchings in terms of the number of vertices.

We start by proving a tight exponential lower bound on $m(G)$ for bipartite cubic graphs. First, some definitions. For a path $v_1v_2v_3v_4$, the graph obtained from G by *splitting along the path* $v_1v_2v_3v_4$ is the cubic graph G' obtained as follows: remove the vertices v_2 and v_3 and add the edges v_1v_4 and $v'_1v'_4$ where v'_1 is the neighbor of v_2 different from v_1 and v_3 and v'_4 is the neighbor of v_3 different from v_2 and v_4. We say that a perfect matching M of G is a *canonical extension* of a perfect matching M' of G' if $M \triangle M' \subseteq E(G) \triangle E(G')$, i.e. M and M' agree on the edges shared by G and G'.

Theorem 1 (Voorhoeve)**.** *Let G be a bipartite cubic graph on $2n$ vertices. Then $m(G) \geq \frac{3}{2} \cdot \left(\frac{4}{3}\right)^n$.*

Proof. We will show by induction on n that $m(G-e) \geq \left(\frac{4}{3}\right)^n$ for every $e \in E(G)$. This implies the theorem. The base case $n = 1$ is trivial.

For the induction step, fix an edge $e = uv \in E(G)$. Let w_1 and w_2 be the two other neighbors of u. (As we allow parallel edges we think of the neighborhood of u as a multiset $\{u, w_1, w_2\}$.) Let $\{u, x_i, y_i\}$ be the neighbors of w_i for $i = 1, 2$. Let $G_1, \ldots, G_4$ be the graphs obtained from G by splitting along the paths vuw_1x_1, vuw_1y_1, vuw_2x_2 and vuw_2y_2.

Every perfect matching avoidng e in G_i canonically extends to a perfect matching avoiding e in G. Let S be the sum of the number of perfect matchings of G_i avoiding e, for $i = 1, \ldots, 4$. By induction hypothesis $S \geq 4\left(\frac{4}{3}\right)^{n-3}$. On the other hand, a perfect matching M of G avoiding e is the canonical extension of a perfect matching avoiding e in precisely three of the graphs G_i, $i \in \{1, 2, 3, 4\}$. For instance if $w_1y_1, uw_2 \in M$, then G_2 is the only graph (among the four) that does not have a perfect matching M' that canonically extends to M. As a consequence, there are precisely $S/3$ perfect matchings containing e in G, implying the required bound. □

6.3. Van der Waerden's conjecture.

We have noted before that if B is the biadjacency matrix of a bipartite graph G then $m(G) = \mathrm{perm}(B)$. A matrix is *doubly stochastic* if it is non-negative and each row and column sum is equal to 1. If G is k-regular then $\frac{1}{k}B$ is doubly stochastic. Thus lower bounds on permanents of doubly stochastic matrices imply lower bounds on the number of perfect matchings in regular bipartite graphs. Van der Waerden has conjectured that the permanent of $n \times n$ doubly stochastic matrix is lower bounded by $n!/n^n$. The bound is achieved when every entry of the matrix is equal to $1/n$. This conjecture has been proved more than fifty years after it was stated by Falikman and Egorychev.

Note that van der Waerden's conjecture implies that

$$m(G) \geq 3^n \frac{n!}{n^n} \geq \left(\frac{3}{e}\right)^n$$

for a bipartite cubic graph G on $2n$ vertices. This is a weaker bound than that provided by Theorem 1, but is still exponential in n.

For bipartite k-regular graphs Schrijver has proved the following generalization of Theorem 1. His proof extends Voorhoeve's ideas, but is considerably more technical.

Theorem 2. *Let G be a k-regular bipartite grap then*

$$m(G) \geq \left(\frac{(k-1)^{k-1}}{k^{k-2}}\right)^n.$$

A common generalization of Falikman and Egorychev's theorem and of Schrijver's theorem was recently obtained by Gurvits. His proof is short, although somewhat technical. We will briefly outline it, following the exposition in the paper "On Leonid Gurvits's proof for permanents" by Laurent and Schrijver.

6.4. Exponentially many perfect matchings in general regular graphs.

We will review the paper "Exponentially many perfect matchings in cubic graphs" by Esperet, Kardoš, King, Král' and Norin. It proves the conjecture of Lovász and Plummer from mid-1970's showing the following.

Theorem 3. *There exists a constant $\epsilon > 0$ such that $m(G) \geq 2^{\epsilon|V(G)|}$ for every cubic bridgeless graph G. ($\epsilon = 1/3656$.)*

Problem 1. Does there exist a constant $\epsilon > 0$ such that in every cubic graph G with no non-trivial cuts of size at most 3 every edge belongs to at least $2^{\epsilon|V(G)|}$ perfect matchings.

Problem 2. What is the optimal value of ϵ in Theorem 3?

More generally, Lovász and Plummer conjectured the following.

Conjecture 1. For $k \geq 3$ there exist constants $c_1(k), c_2(k) > 0$ such that every k-regular $(k-1)$-edge connected graph contains at least $c_2(k)c_1(k)^{|V(G)|}$ perfect matchings. Furthermore, $c_1(k) \to \infty$ as $k \to \infty$.

A weaker version of Conjecture 1 is implied by Theorem 3.

Theorem 4. *There exists a constant $\epsilon > 0$ such that $m(G) \geq 2^{\epsilon|V(G)|}$ for every integer $k \geq 3$ and every k-regular $(k-1)$-edge connected graph G.*

Can we relax the regularity condition somewhat and still obtain exponential lower bounds on the number of perfect matchings if we require the graph to be matching-covered? Conjecture 1 was stated with $(k-1)$-edge connectivity condition relaxed to a requirement that G is matching covered. Unfortunately, this strengthening is false, as a consequence of the following result of Geelen and Norin.

Lemma 1. *For every integer $k \geq 4$ there exist constants c_1 and c_2 so that for every N there exists a bipartite graph G with $|V(G)| = 2n > N$, at most $c_1\sqrt{n}$ vertices of degree $k-2$, all other vertices of degree k and $m(G) \leq 2^{c_2\sqrt{n}}$.*

Proof. Fix an integer $l \geq k$. Let G be a graph with bipartition (A, B), where $A = A_1 \cup A_2 \cup \ldots \cup A_l$, $B = B_1 \cup B_2 \cup \ldots \cup B_l$ and both of the unions are disjoint. We further require that $|A_1| = |B_l| = (k-1)l$, $|B_1| = |A_l| = (k-1)l + 1$, and $|A_i| = |B_i| = l$ for $1 < i < l$. It is not very difficult to construct a graph G satisfying the following list of conditions

- The only vertices of G of degree $k-2$ belong to A_1 and B_l;
- $G[A_i, B_i]$ is 1-regular for every $1 < i < l$;
- All the edges between $\cup_{1 \leq j \leq i}(A_j \cup B_j)$ and the rest of G have one end in B_i and another in A_{i+1};
- G is matching-covered.

Then $G[A_1, B_1]$ has at most $k^{(k-1)l}$ matchings saturating A_1. These matchings can be extended to a matching saturating $A_1 \cup B_1$ in at most k ways each. As $G[A_2, B_2]$ is 1-regular we can extend each matching uniquely to a matching of $G[A_1 \cup A_2, B_1 \cup B_2]$ saturating $A_1 \cup A_2$. Repeating the argument, we obtain

$$m(G) \leq \left(k^{(k-1)l}\right)^2 \cdot k^{l-1},$$

with the exponent in the bound linear in l. We have

$$|V(G)| = 2l(l-2) + 4(k-1)l + 2$$

, quadratic in l, and lemma follows. □

Exercise 1. Use Lemma 1 to disprove a strengthening of Conjecture 1 to matching-covered graphs. (Hint: replace vertices of degree $k-2$ by bounded size subgraphs.)

Problem 3. For a fixed integer k consider the family of matching-covered bipartite graphs on n vertices with minimum degree 3 and maximum degree k. Improve lower and upper bounds on minimum value of $m(G)$ among graphs in this family, provided by the dimension of matching polytope and Lemma 1 respectively.