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LECTURE 4
Pfaffian orientations.

4.1. Outline of Lecture

- Determinants and perfect matchings.
- Pfaffian orientations.

4.2. Determinants and perfect matchings.

In this lecture we examine the possibility of solving the perfect matching problem and evaluating $m(G)$ efficiently using linear algebra, specifically by evaluating determinants. The material presented in the first section is due to Jim Geelen.

Let G be a bipartite graph with bipartition (R, C) and let $(z_e \mid e \in E(G))$ be algebraically independent variables. We define a variant of the biadjacency matrix introduced in Section 1.2, an $R \times C$-matrix $B = (b_{ij})$, where $b_{uv} = z_e$ if $e = uv$, and $b_{uv} = 0$, otherwise. We call B the *bipartite matching matrix* of G. For example, let G be obtained from the complete bipartite graph $K_{3,3}$ with bipartition (R, C), where $R = \{a, b, c\}$ and $C = \{1, 2, 3\}$, by deleting the edge $c1$. Then the bipartite matching matrix of G is

$$B = \begin{pmatrix} z_{a1} & z_{a2} & z_{a3} \\ z_{b1} & z_{b2} & z_{b3} \\ 0 & z_{c2} & z_{c3} \end{pmatrix}, \tag{1}$$

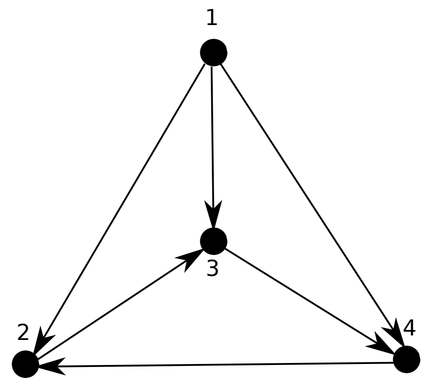

Figure 1. A Pfaffian orientation of K_4.

and

$$\det B = z_{a1}z_{b2}z_{c3} - z_{a1}z_{b3}z_{c2} - z_{a2}z_{b1}z_{c3} + z_{a3}z_{b1}z_{c2}.$$

More generally, if B is the bipartite matching matrix of a balanced bipartite graph G then

$$\det B = \sum_{M \in \mathcal{M}(G)} \operatorname{sgn}(M) \prod_{e \in M} z_e,$$

where $\operatorname{sgn}(M)$ is the sign of the permutation associated with M. In particular, $\det M \neq 0$ if and only if G has a perfect matching.

For general graphs there also exists a relation between determinants and perfect matchings. Let G be a simple graph, let D be some orientation of its edges, and let $(z_e \mid e \in D)$ be once again a collection of algebraically independent variables. A $V(G) \times V(G)$ skew-symmetric matrix $T = (t_{uv})$, called the *Tutte matrix of* D and introduced by Tutte in 1947 is defined as follows: $t_{uv} = z_e$ and $t_{vu} = -z_e$ if $e = uv \in D$, and $z_{uv} = 0$ if $uv \notin E(G)$. For example, if $G = K_4$ with $V(G) = \{1, 2, 3, 4\}$ and D is the orientation shown on Figure 1, then

$$T = \begin{pmatrix} 0 & z_{12} & z_{13} & z_{14} \\ -z_{12} & 0 & z_{23} & -z_{42} \\ -z_{13} & -z_{23} & 0 & z_{34} \\ -z_{14} & z_{42} & -z_{34} & 0 \end{pmatrix}. \tag{2}$$

One can check that

$$\det T = (z_{12}z_{34} + z_{13}z_{42} + z_{14}z_{23})^2.$$

In general, let G be a graph on $2n$ vertices with $V(G) = \{1, 2, \ldots, 2n\}$, let D be an orientation of G and let $M = \{u_1v_1, u_2v_2, \ldots, u_nv_n\}$ be a perfect matching of G with $u_iv_i \in D$ for $1 \leq i \leq n$. Define $\operatorname{sgn}_D(M)$, the *sign* of M, to be the sign of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2n-1 & 2n \\ u_1 & v_1 & u_2 & v_2 & \dots & u_n & v_n \end{pmatrix}.$$

Note that the sign of a perfect matching is well-defined as it does not depend on the order in which the edges of M are listed. The *Pfaffian* of the Tutte matrix of D is defined as

$$\mathrm{Pf}(T) = \sum_{M \in \mathcal{M}(G)} \mathrm{sgn}_D(M) \prod_{e \in M} z_e. \tag{3}$$

Exercise 1. Show that $\det T = (\mathrm{Pf}(T))^2$.

It follows from Exercise 1 that G has a perfect matching if and only if T is nonsingular.

Can we develop an efficient algorithm for the perfect matching problem based on the observations above? One can not efficiently perform operations on a matrix with indeterminate entries. Instead, we attempt replacing the indeterminates (z_e) by particular values. However, the resulting *evaluation* of the matrix can become singular. For example, if we replace all the variables by 1 then the matrix B in (1) becomes singular. On the other hand, for the matrix T in (2), we have $\mathrm{Pf}(T) = m(K_4)$. Fortunately, it follows from the following theorem of Zippel and Schwarz that if G has a perfect matching then a random evaluation of its bipartite matching matrix (or its Tutte matrix) is non-zero with high probability.

Theorem 1. *Let $p(z_1, \dots, z_k)$ be a non-zero polynomial of degree at most d, and let S be a finite subset of $\mathbb{R}$. If $\hat{z}_1, \hat{z}_2, \dots, \hat{z}_k$ are chosen from S uniformly and independently at random then $p(\hat{z}_1, \dots, \hat{z}_k) \neq 0$ with probability at least $1 - \frac{d}{|S|}$.*

Exercise 2. Prove Theorem 1 by induction on the number of variables.

The following corollary immediately follow from Theorem 1 and the discussion above. It provides an efficient randomized algorithm for solving the perfect matching problem.

Corollary 1. *Let T be the Tutte matrix corresponding to some orientation of a graph G with a perfect matching. If $\hat{T}$ is an evaluation of T with entries chosen uniformly and independently at random from $\{1, \dots, |V(G)|\}$ then $\hat{T}$ is non-singular with probability at least $\frac{1}{2}$.*

Note that if G is bipartite then one can replace the Tutte matrix in Corollary 1 by the bipartite matching matrix.

4.3. Pfaffian orientations

By (3) and Exercise 1 we can compute $m(G)$ efficiently if we can find an orientation D of G such that the signs of all perfect matchings in D are the same. Such an orientation is called *Pfaffian*. A graph is called *Pfaffian* if it admits a Pfaffian orientation.

Let C be an even cycle in G. We say that C is *evenly oriented* in D if traversing C we encounter an even number of edges of D oriented in the direction of the traversal, and *oddly oriented* otherwise.

Lemma 1. *Let M_1 and M_2 be perfect matchings in a graph G such that $M_1 \triangle M_2$ consists of a single even cycle C. Let D be an orientation of G. Then $\mathrm{sgn}_D(M_1) = \mathrm{sgn}_D(M_2)$ if and only if C is oddly oriented in D.*

Proof. Note that exchanging the numbers of two vertices of G changes the sign of all perfect matchings. Therefore we may assume that the vertices of C are $\{1, 2, \ldots, 2k\}$ in order, for some integer k. Further note that reversing orientation of an edge in C changes C from oddly to evenly oriented and vice versa. This reversal also changes the sign of exactly one of M_1 and M_2. It follows that we may also assume that C is directed. The lemma now follows from the direct computation: C is evenly oriented and $\mathrm{sgn}_D(M_1) \neq \mathrm{sgn}_D(M_2)$, as

$$\mathrm{sgn}\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2k-1 & 2k \\ 2k & 1 & 2 & 3 & \ldots & 2k-2 & 2k-1 \end{pmatrix}$$
$$\neq \mathrm{sgn}\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2k-1 & 2k \\ 1 & 2 & 3 & 4 & \ldots & 2k-1 & 2k \end{pmatrix}.$$

□

A cycle C is said to be *M-alternating* for a matching M if the edges in C alternate between edges of M and $E(G) - M$.

Corollary 2. *For an orientation D of a graph G the following are equivalent.*

(a) *D is Pfaffian,*
(b) *every central cycle of G is oddly oriented in D,*
(c) *every M-alternating cycle of G is oddly oriented in D for some $M \in \mathcal{M}(G)$.*

Exercise 3. Derive Corollary 2 from Lemma 1.

We are now ready to prove the classical theorem of Kasteleyn, which exhibits a wide and natural class of Pfaffian graphs.

Theorem 2. *Every planar graph is Pfaffian.*

Proof. Let G be a planar graph. Fix a drawing of G in the plane. Given an orientation D of G, we say that a cycle C in G is *clockwise even* if traversing C clockwise we encounter an even number of edges of D oriented in the direction of the traversal, and we say that C is *clockwise odd* otherwise. Note that unlike the notion of evenly/oddly oriented cycles introduced earlier this new notion is well-defined for odd cycles.

Let D be an orientation of G so that every face of G, except possibly for the infinite face, is oddly oriented in D. The existence of such an orientation can be derived by induction on $|E(G)|$. For the induction step, we apply induction hypothesis to the graph $G-e$ for some edge e incident to the infinite face of G, and then orient e so that the unique non-infinite face of G incident to e is oddly oriented.

Let C be a cycle in G. We claim that C is oddly oriented in D if and only if the region bounded by C in the plane contains an even number of vertices of G in its strict interior. Note that the theorem follows from this claim by Corollary 2, as every region bounded by a central cycle must contain even number of vertices in its interior.

We verify the claim by induction on the number of edges of G in the interior of the region bounded by C. The base case holds by the choice of D. For the induction step, we partition the region bounded by C into two smaller regions bounded by cycles C_1 and C_2 respectively. Suppose that the region bounded by C_i contains r_i vertices in its interior, for $i = 1, 2$, and that C_1 and C_2 share k vertices. Applying the induction hypothesis, one can routinely verify that traversing C in the clockwise direction one encounters $(r_1+1)+(r_2+1)-(k+1)$ edges in the direction of traversal modulo 2. As the region bounded by C contains $r_1 + r_2$ vertices in its interior, this finishes the proof of the claim. □

Exercise 4. Show that a matching-covered graph G is Pfaffian if and only if every brick and brace in the tight cut decomposition of G is Pfaffian.

Exercise 5. Show that $K_{3,3}$ is not Pfaffian.