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## LECTURE 2
## The perfect matching polytope.

### 2.1. Outline of Lecture

- Polytopes and linear programming.
- The perfect matching polytope in bipartite graphs.
- The perfect matching polytope in general graphs.
- An application: Union of perfect matchings in cubic graphs.

### 2.2. Linear programming. A reminder.

Linear programming is a very powerful technique, applicable not only in optimization, but in combinatorial theory. For example, we will see that König's theorem (Lecture 1,Theorem 2) is an instance of the principle of (integral) linear programming duality. In this section, we remind ourselves of the basic relevant concepts.

Given a collection of vectors $v_1, v_2, \ldots .v_m \in \mathbb{R}^n$ their *linear combination* is $\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_m v_m \in \mathbb{R}^n$, where $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$. The linear combination is *affine* if $\lambda_1 + \ldots + \lambda_m = 1$, it is *convex* if additionally all $\lambda_i \geq 0$. The *convex hull* of a family of vectors is the set of all convex combinations of vectors in this family. The convex hull of a finite family of vectors is called a *polytope*. A classical theorem of Weil states that every polytope is an intersection of a finite collection of half spaces. Thus every polytope can be characterized either by a (minimal) set of vectors whose convex combinations generate this polytope or by a family of half spaces. The *dimension* of a polytope is the dimension of the affine subspace of $\mathbb{R}^n$ generated by it." A *vertex* of a polytope $P$ is a vector which can not be expressed by a convex combination of other vectors in $P$. If a polytope is given by a system of linear inequalities (i.e. as an intersection of half spaces), then every vertex is the unique solution of a system of equations, obtained by taking a subsystem of inequalities defining $P$ and replacing them by equalities.

The *fundamental problem of linear programming* is to find maximum or minimum of an objective linear function subject to a set of linear constraints. It can be written as

$$\begin{array}{llr} \text{maximize} & & \mathbf{c}\cdot\mathbf{x} \\ \text{subject} & \text{to} & \mathbf{x}\geq 0 \\ & & \mathbf{A}\mathbf{x}\leq\mathbf{b} \end{array} \tag{1}$$

A *dual program* is defined by

$$\begin{array}{llr} \text{minimize} & & \mathbf{b}\cdot\mathbf{y} \\ \text{subject} & \text{to} & \mathbf{y}\geq 0 \\ & & \mathbf{A}^{\mathbf{t}}\mathbf{y}\geq\mathbf{c} \end{array} \tag{2}$$

**Theorem 1** (The duality theorem for linear programming)**.** *If either one of the linear programs (1) and (2) has a solution and a finite optimum then so does the other, and the optima are equal.*

## 2.3. Fractional matchings in bipartite graphs.

We will identify perfect matchings with vectors in a Euclidean space. This will allow us to exploit linear algebraic and linear programming techniques. Let $\mathbb{R}^{E(G)}$ be the set of vectors with components indexed by the edges of the graph $G$. For a vector $\mathbf{w} = (w(e) \mid e \in E(G)) \in \mathbb{R}^{E(G)}$ and $F \subseteq E(G)$ define $\mathbf{w}(F) := \sum_{e\in F} w(e)$. For a set $F \subseteq E(G)$ define the *characteristic vector* $\chi_{\mathbf{F}} \in \mathbb{R}^{E(G)}$ of $F$ by $\chi_F(e) = 1$ if $e \in F$, and $\chi_F(e) = 0$ otherwise. In particular, we have $\mathbf{w}(F) = \mathbf{w} \cdot \chi_{\mathbf{F}}$ for all $F \subseteq E(G)$ and all $\mathbf{w} \in \mathbb{R}^{E(G)}$. As one additional bit of notation, we denote $\nabla(X)$ for a set $X \subseteq V(G)$ to be the set of edges with one end in $X$ and another in $V(G) - X$, that is $\nabla(X)$ is the cut separating $X$ and $V(G) - X$. For brevity we write $\nabla(v)$ instead of $\nabla(\{v\})$ to denote the set of all edges incident to $v$.

We define the *perfect matching polytope* $\mathcal{PM}(G)$ of a graph $G$ as the convex hull of the set $\{\chi_M \mid M \in \mathcal{M}(G)\}$ of characteristic vectors of perfect matchings of $G$. Our goal is to describe $\mathcal{PM}(G)$ as an intersection of a family of subspaces, preferably in such a way that the inequalities defining these subspaces are natural. What linear inequalities are satisfied by characteristic vectors of perfect matchings (and, therefore, by all vectors in $\mathcal{PM}(G)$)? Clearly, for every $\mathbf{x} \in \mathcal{PM}(G)$ we have $x(e) \geq 0$ for every $e \in E(G)$, and $x(\nabla(v)) = 1$ for every $v \in V(G)$. These two conditions can be written in a more compact form as follows. The *incidence matrix* $\mathbf{A} = (a_{ve})_{v \in V(G), e \in E(G)}$ of a graph $G$ is a matrix with rows indexed by vertices of $G$ and columns indexed by edges of $G$, where $a_{ve} = 1$, if $v$ is an end of $e$, and $a_{ve} = 0$, otherwise. Define the *fractional perfect matching polytope* of a graph $G$, as

$$\mathcal{FPM}(G) := \{\mathbf{x} \in \mathbb{R}^{E(G)} \mid \mathbf{x} \geq \mathbf{0},\ \mathbf{Ax} = \mathbf{1}\}. \tag{3}$$

As mentioned above, we have $\mathcal{PM}(G) \subseteq \mathcal{FPM}(G)$ for every graph $G$.

The exposition of the results in this section follows Lovász and Plummer's book "Matching theory"(1986).

**Theorem 2.** *Let $A$ be the incidence matrix of a bipartite graph $G$. Then the determinant of every square submatrix of $A$ is equal to* 0 *or* $\pm 1$*. (Such a matrix $A$ is called* totally unimodular*.)*

**Proof.** Let $B$ be a $k \times k$ submatrix of $A$. We proceed by induction on $k$. If some column of $B$ contains at most one non-zero entry, then we can decompose $\det B$ along this column and proceed by induction. Thus we may assume that every column of $B$ contains exactly two non-zero entries (equal to 1) in rows corresponding to different parts of the bipartition of $G$. It follows that the sum of rows of $B$ corresponding to the vertices in one class of the bipartition is equal to the sum of rows of $B$ corresponding to the vertices in the other class. Thus the rows of $B$ are linearly dependent and $\det B = 0$. □

**Theorem 3.** *We have*

$$\mathcal{PM}(G) = \mathcal{FPM}(G),$$

*for every bipartite graph $G$.*

**Proof.** By Theorem 2, Kramer's rule and our discussion of vertices of a polytope above it follows that every vertex of $\mathcal{FPM}(G)$ is integral. It is easy to see that each integral point in $\mathcal{FPM}(G)$ is a characteristic vector of some perfect matching. Thus $\mathcal{FPM}(G) \subseteq \mathcal{PM}(G)$ and we have already seen that the reverse inclusion also holds. □

It is known that an optimum of any linear program can be found in time polynomial in the size of the program. Theorem 3 shows that the *weighted perfect matching problem* is solvable in polynomial time for bipartite graphs. More precisely, given a vector $\mathbf{w} \in \mathbb{R}^{E(G)}$ one can find in polynomial time

$$\max_{M \in \mathcal{M}(G)} \sum_{e \in M} w(e).$$

While we concetrate on perfect matchings in these notes a useful analogue of Theorem 3 holds for general matchings. Define the *matching polytope* $\mathcal{MP}(G)$ as the convex hull of characteristic vectors of matchings of $G$. Define by analogue with (3) the *fractional matching polytope* as

$$\mathcal{FM}(G) := \{\mathbf{x} \in \mathbb{R}^{E(G)} \mid \mathbf{x} \geq \mathbf{0},\ \mathbf{Ax} \leq \mathbf{1}\}. \tag{4}$$

**Theorem 4.** *We have*

$$\mathcal{MP}(G) = \mathcal{FM}(G),$$

*for every bipartite graph $G$.*

**Exercise 1.** Show that $\mathcal{MP}(G) \subsetneq \mathcal{FM}(G)$ for every non-bipartite graph $G$.

**Exercise 2. a)** Derive König's theorem (Lecture 1, Exercise 2) from Theorem 4 and the duality theorem for linear programming.

**b)** Formulate and prove a weighted generalization of König's theorem.

## 2.4. Edmonds' perfect matching polytope theorem

As might be deduced from Exercise 1, the description of the inequalities defining $\mathcal{PM}(G)$ for non-bipartite graph $G$ is more complex than that of $\mathcal{FPM}(G)$. However, a celebrated theorem of Edmonds(1965) gives a useful description.

**Theorem 5** (Edmonds' perfect matching polytope theorem)**.** *The polytope $\mathcal{PM}(G)$ consist of all vectors $\mathbf{x} \in \mathbb{R}^{E(G)}$ satisfying the following constraints.*

- **(i):** $\mathbf{x} \geq \mathbf{0}$,
- **(ii):** $x(\nabla(v)) = 1$ *for every* $v \in V(G)$,
- **(iii):** $x(\nabla(X)) \geq 1$ *for every odd* $X \subseteq V(G)$.

**Proof.** The proof is by induction on $|V(G)|$. The base case $|V(G)| = 0$ is trivial. For a vector $\mathbf{x} \in \mathbb{R}^{E(G)}$, we will show that if $\mathbf{x}$ satisfies conditions (i), (ii) and (iii) then $\mathbf{x}$ lies in $\mathcal{PM}(G)$. Let us first consider the case that

(*) $x(\nabla(X)) = 1$ for some odd $X \subseteq V(G)$ with $|X|, |V(G) - X| > 1$.

We define graphs $G_1$ and $G_2$ by replacing all the vertices in $X$ and in $V(G) - X$, respectively, by a single vertex. A similar procedure of *cut decomposition* will be employed in other circumstances in the future. Note that $\mathbf{x}$ naturally corresponds to vectors $\mathbf{x}_1 \in \mathbb{R}^{E(G_1)}$ and $\mathbf{x}_2 \in \mathbb{R}^{E(G_2)}$, obtained by restriction. The validity of the theorem for $\mathbf{x}$ follows from the induction hypothesis applied to $G_1$ and $G_2$ and the following exercise.

**Exercise 3.** Let $G$, $G_1$ and $G_2$, $\mathbf{x}$, $\mathbf{x}_1$, $\mathbf{x}_2$ be as above.

a) Show that $\mathbf{x}_i$ satisfies conditions (i),(ii) and (iii) applied to $G_i$ for $i = 1, 2$;
b) Show that if $\mathbf{x}_1 \in \mathcal{PM}(G_1)$, $\mathbf{x}_2 \in \mathcal{PM}(G_2)$ then $\mathbf{x} \in \mathcal{PM}(G)$.

Thus the theorem holds for all $\mathbf{x}$ satisfying condition (*). For a vector $\mathbf{x} \in \mathbb{R}^{E(G)}$ define the *support* of $\mathbf{x}$ by $\operatorname{supp}\mathbf{x} := \{e \in E(G) \mid x(e) \neq 0\}$. We proceed by induction on $|\operatorname{supp}\mathbf{x}|$. Define $G' := G[\operatorname{supp}\mathbf{x}]$ to be the graph induced by edges in $\operatorname{supp}\mathbf{x}$.

**Claim:** $G'$ has a perfect matching.

**Proof.** If not then by Tutte's matching theorem (Lecture 1, Theorem 3) there exists a set $X \subseteq V(G)$ such that $c_o(G' - X) > |X|$. Let $Y_1, Y_2, \ldots, Y_k$ be the vertex sets of the odd components of $G' - X$. Then $x(\nabla(Y_i)) \geq 1$ for $1 \leq i \leq k$ by (iii). It follows that

$$|X| \overset{(ii)}{=} \sum_{v \in X} x(\nabla(v)) \overset{(i)}{\geq} \sum_{i=1}^{k} x(\nabla(Y_i)) \geq k > |X|,$$

a contradiction, finishing the proof of the claim. □

Let $M$ be a perfect matching of $G'$. If $\mathbf{x} = \chi_M$, the theorem holds. Otherwise, consider

$$\mathbf{x}' := \frac{\mathbf{x} - \varepsilon\chi_{\mathbf{M}}}{1 - \varepsilon},$$

where $0 < \varepsilon < 1$ is chosen maximum so that $\mathbf{x}' \geq \mathbf{0}$ and $\mathbf{x}'(\nabla(X)) \geq 1$ for every odd $X \subseteq V(G)$. The condition (*) is not satisfied for $\mathbf{x}$, $M$ is contained in $\operatorname{supp}\mathbf{x}$ and $\mathbf{x} \neq \chi_M$, implying that such $\epsilon$ exists. Further, $\mathbf{x}'$ satisfies conditions (i), (ii) and (iii). Either $|\operatorname{supp}\mathbf{x}'| < |\operatorname{supp}\mathbf{x}|$ or (*) holds for $\mathbf{x}'$. It follows from the induction hypothesis that $\mathbf{x}' \in \mathcal{PM}(G)$. Finally, $\mathbf{x} = (1 - \varepsilon)\mathbf{x}' + \varepsilon\chi_{\mathbf{M}}$, and thus $\mathbf{x} \in \mathcal{PM}(G)$. □

**Exercise 4.** Let $G$ be a $(k-1)$-edge connected $k$-regular graph on even number of vertices for some integer $k \geq 1$. Show that $\frac{\mathbf{1}}{\mathbf{k}} \in \mathcal{PM}(\mathbf{G})$.

## 2.5. Unions of perfect matchings in cubic graphs

The material in this section follows the paper with the same title by Tomáš Kaiser, Daniel Král' and the author (2006).

A well-known conjecture of Berge and Fulkerson states that every bridgeless cubic graph contains a family of six perfect matchings covering each edge exactly twice:

**Conjecture 1.** Every cubic bridgeless graph $G$ contains six perfect matching $M_1, \dots, M_6$ such that each edge of $G$ is contained in precisely two of the matchings.

The following version of Conjecture 1 due to Berge has recently been shown to be equivalent to Conjecture 1 by Mazzuocolo.

**Conjecture 2.** Every cubic bridgeless graph $G$ contains five perfect matchings $M_1, \dots, M_5$ such that each edge of $G$ is contained in at least one of the matchings.

Exercise 4 for $k = 3$ can be used to prove a much weaker bound.

**Theorem 6.** *The edges of a cubic bridgeless graph $G$ on $2n$ vertices can be covered by at most $\log_{3/2}(3n) + 1$ perfect matchings.*

**Proof.** By Exercise 4 there exist perfect matchings $M_1, \dots, M_k \in \mathcal{M}(G)$ so that a convex combination $\lambda_1\chi_{\mathbf{M_1}} + \lambda_2\chi_{\mathbf{M_2}} + \dots + \lambda_k\chi_{\mathbf{M_1}}$ is identically $1/3$. Consider a probabilistic distribution on $\mathcal{M}(G)$ so that $\mathbb{P}[\mathbf{M} = M_i] = \lambda_i$ for $1 \le i \le k$. Let $l = \lfloor \log_{3/2}(3n) + 1 \rfloor$ perfect matchings be independently chosen from this probability distribution. Then the probability that a given edge is not in any of the chosen perfect matchings is $(2/3)^l$. It now follows from the union bound that with positive probability every edge is covered. □

In the remainder of this section we investigate the maximum possible size of the union of a given number of perfect matchings in a bridegeless cubic graph. More precisely, we study, for $k \in \{2, 3\}$, the numbers

$$m_k = \inf_G \max_{M_1,\dots,M_k} \frac{|\bigcup_i M_i|}{|E(G)|},$$

where the infimum is taken over all bridgeless cubic graphs $G$, and $M_1, \dots, M_k$ range over all perfect matchings of $G$. Note that Conjecture 2 asserts that $m_5 = 1$.

The Petersen graph $P_{10}$ (see Figure 1) has 15 edges and 6 distinct perfect matchings. It can be checked that any two distinct perfect matchings of $P_{10}$ have precisely one edge in common and that the intersection of any three distinct perfect matchings is empty. Simple

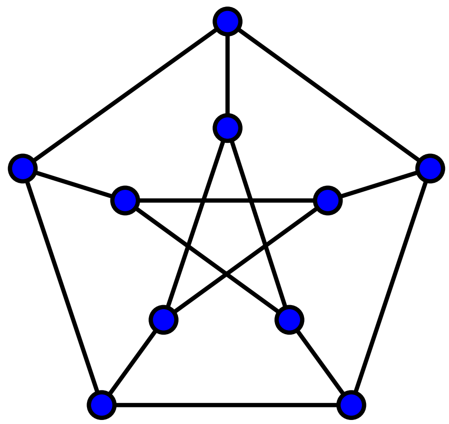

**Figure 1.** The Petersen graph

counting then shows that $m_2 \leq 3/5$ and $m_3 \leq 4/5$. We provide bounds on $m_2$ and $m_3$. Next exercise provides an auxiliary result used in the proof.

**Exercise 5.** Let $G$ be a graph, let $\mathbf{w} \in \mathcal{PM}(G)$ and let $\mathbf{c} \in \mathbb{R}^{E(G)}$. Then $G$ has a perfect matching $M$ such that

$$\mathbf{c} \cdot \chi_{\mathbf{M}} \geq \mathbf{c} \cdot \mathbf{w}.$$

Moreover, there exists such a perfect matching $M$ that contains exactly one edge of each odd cut $C$ with $w(C) = 1$.

**Theorem 7.** *The value of* $m_2$ *is* $3/5$, *and* $0.771 \approx 27/35 \leq m_3 \leq 4/5$.

**Proof.** Fix a cubic bridgeless graph $G$. Define $\mathbf{w}_1 \in \mathbb{R}^{E(G)}$ to have the value $1/3$ on all edges $e \in E(G)$. By Exercise 4, $\mathbf{w}_1 \in \mathcal{PM}(G)$. Note that $w_1(C) = 1$ for each 3-cut $C$ of $G$. Hence, by Exercise 5 there exists $M_1 \in \mathcal{M}(G)$ intersecting each 3-cut in a single edge.

We now use $M_1$ to define the following vector $\mathbf{w}_2 \in \mathbb{R}^{E(G)}$:

$$w_2(e) = \begin{cases} 1/5 & \text{if } e \in M_1, \\ 2/5 & \text{otherwise.} \end{cases}$$

We verify that $\mathbf{w}_2 \in \mathcal{PM}(G)$: the conditions (i) and (ii) of Theorem 5 clearly hold. Let $C$ be an odd cut of $G$. The size of $C$ is odd and it is at least three. If $C$ is a 3-cut, then $w_2(e) = 1/5$ for exactly one of the edges $e$ contained in $C$ and $w_2(e) = 2/5$ for the remaining two edges. If the size of $C$ is five or more, then $w_2(C) \geq 5 \cdot 1/5 = 1$. Hence, the condition (iii) also holds.

For each $e \in E(G)$, set $c_2(e) = 1 - \chi_{M_1}(e)$. By Lemma 5, there exists a perfect matching $M_2$ such that

$$\mathbf{c}_2 \cdot \chi_{\mathbf{M_2}} \geq \mathbf{c}_2 \cdot \mathbf{w}_2 = \frac{2}{5} \cdot \frac{2}{3} |E(G)| = \frac{4}{15} |E(G)| \ .$$

Since $\mathbf{c}_2 \cdot \chi_{\mathbf{M_2}}$ is just $|M_2 \setminus M_1|$, it follows that

$$|M_1 \cup M_2| = (\frac{1}{3} + \frac{4}{15}) \cdot |E(G)| = \frac{3}{5} |E(G)| .$$

We conclude that $m_2 = 3/5$.

It remains to establish a lower bound on $m_3$. Note that if $C$ is a 5-cut contained in $M_1$, then $w_2(C) = 1$. Hence, by Exercise 5, we may assume that if $C$ is a 5-cut contained in $M_1$, then $|C \cap M_2| = 1$. Similarly, $M_2$ contains exactly one edge of each 3-cut. We now consider the following vector $\mathbf{w}_3 \in \mathbb{R}^{E(G)}$:

$$w_3(e) = \begin{cases} 1/7 & \text{if } e \in M_1 \cap M_2, \\ 2/7 & \text{if } e \in (M_1 \cup M_2) \setminus (M_1 \cap M_2), \\ 3/7 & \text{otherwise.} \end{cases}$$

Again, we verify that $\mathbf{w}_3 \in \mathcal{PM}(G)$. The conditions (i) and (ii) hold trivially. Let us consider an odd cut $C$ of $G$. If $C$ is a 3-cut, then it contains exactly one edge $e_1$ contained in $M_1$ and exactly one edge $e_2$ contained if $M_2$. If $e_1 = e_2$, then $w_3(C) = 1/7 + 2 \cdot 3/7 = 1$. If $e_1 \neq e_2$, then $w_3(C) = 2 \cdot 2/7 + 3/7 = 1$. If $C$ is a 5-cut that is not fully contained in $M_1$, then $|C \cap M_1| \leq 3$ (recall that $C$ is an odd cut). Hence, $w_3(C) \geq 3 \cdot 1/7 + 2 \cdot 2/7 = 1$. If $C \subseteq M_1$, then $|C \cap M_2| = 1$ by the choice of $M_2$. We infer that $w_3(C) \geq 1/7 + 4 \cdot 2/7 > 1$. Finally, if the size of $C$ is seven or more, then $w_3(C) \geq 7 \cdot 1/7 = 1$. We conclude that $w_3$ is a fractional perfect matching of $G$.

Set $c_3(e) = 1 - \chi_{M_1 \cup M_2}(e)$. By Lemma 5, there exists a perfect matching $M_3$ such that

$$\mathbf{c}_3 \cdot \chi_{\mathbf{M_3}} = |M_3 \setminus (M_1 \cup M_2)| \geq \frac{3}{7} \cdot |E \setminus (M_1 \cup M_2)| = \mathbf{c}_3 \cdot \mathbf{w}_3 \ .$$

Consequently,

$$\begin{aligned} |M_1 \cup M_2 \cup M_3| &= |M_1 \cup M_2| + |M_3 \setminus (M_1 \cup M_2)| \\ &\geq \frac{3}{5} |E(G)| + \frac{3}{7} \cdot \frac{2}{5} |E(G)| = \frac{27}{35} |E(G)| . \end{aligned}$$

We infer that $m_3 \geq 27/35$. □

**Problem 1.** Is it possible to refine the proof of Theorem 7 to improve the bound on $m_3$? What lower bound on $m_4$ can be derived using similar methods?