A short course on matching theory, ECNU Shanghai, July 2011.

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LECTURE 1
Fundamental definitions and theorems.

1.1. Outline of Lecture

- Definitions
- Hall's theorem
- Tutte's Matching theorem

1.2. Basic definitions.

A *matching* in a graph G is a set of edges M such that no two edges share a common end. A vertex v is said to be *saturated* or *matched* by a matching M if v is an end of an edge in M. Otherwise, v is *unsaturated* or *unmatched*. The *matching number* $\nu(G)$ of G is the maximum number of edges in a matching in G.

A matching M is *perfect* if every vertex of G is saturated. We will be primarily interested in perfect matchings. We will denote by $\mathcal{M}(G)$ the set of all perfect matchings of a graph G and by $m(G) := |\mathcal{M}(G)|$ the number of perfect matchings. The main goal of this course is to demonstrate classical and new results related to computing or estimating the quantity $m(G)$.

Example 1. The graph K_4 is the complete graph on 4 vertices. Let $V(K_4) = \{1,2,3,4\}$. Then $\{12,34\},\{13,24\},\{14,23\}$ are all perfect matchings of K_4, i.e. all the elements of the set $\mathcal{M}(G)$. We have $|m(G)| = 3$. Every edge of K_4 belongs to exactly one perfect matching.

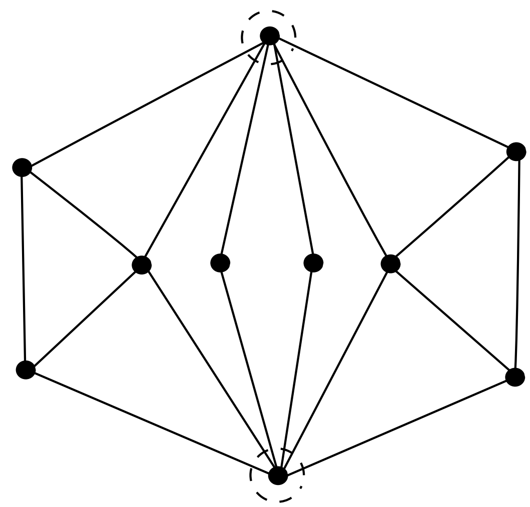

Figure 1. A graph with no perfect matching.

Computation of $m(G)$ is of interest for the following reasons. If G is a graph representing connections between the atoms in a molecule, then $m(G)$ encodes some stability and thermodynamic properties of the molecule.

Let G be a bipartite graph with bipartition (R, C). If $R = \{r_1, r_2, \dots, r_n\}$, $C = \{c_1, c_2, \dots, c_n\}$ then the *biadjacency matrix* $B = (b_{ij})_{i,j=1}^{n}$ of the graph G is the $(0,1)$-matrix defined by $b_{ij} = 1$ if $r_i c_j \in E(G)$, and $b_{ij} = 0$, otherwise. The number of perfect matchings is equal to the permanent of the biadjacency matrix:

$$m(G) = \mathrm{perm}(B) := \sum_{\sigma \in S_n} \prod_{i=1}^{n} b_{i\sigma(i)}.$$

Valiant has shown that computing the permanent is a $\sharp P$-complete problem, even for $(0,1)$-matrices, and so can not be done efficiently in general. It can however be computed in polynomial time for certain classes of graphs, e.g. for planar graphs. We will see this later.

The *perfect matching problem* is the problem of determining whether a graph has a perfect matching. This will be the first problem we examine. Clearly if $|V(G)|$ is odd then G has no perfect matching. A graph, containing two vertices of degree one sharing a neighbor does not have a perfect matching, and neither does a graph on Figure 1.

Given a matching M an *M-alternating path* (or *cycle*) in a graph G is a path(or cycle) which contains alternately edges from M and $E(G)-M$. An M-alternating path is *M-augmenting* if it starts and ends with a vertex unsaturated in M. Alternating paths cycles and augmenting paths are considered in proofs of many theorems in matching theory.

Exercise 1. A matching M in a graph G is maximum ($|M| = \nu(G)$) if and only if G contains no M-augmenting path.

1.3. Perfect matchings in bipartite graphs. Hall's theorem

Let G be a bipartite graph with bipartition (A, B). Let $N(S)$ denote *the neighborhood* of S, the set of vertices adjacent to vertices in S. Note that every perfect matching "chooses" a neighbor in B for every vertex in A. If G has a perfect matching then $|N(S)| \geq |S|$ for every $S \subseteq A$ or $S \subseteq B$. The converse also holds:

Theorem 1 (Hall's theorem)**.** *A bipartite graph G with biparitition (A, B) contains a matching saturating A if and only if*

$$|N(S)| \geq |S| \tag{1}$$

for every $S \subseteq A$.

First proof. By induction on $|A|$. If $|A| = 1$ then the theorem clearly holds. For the induction step, we first notice that if $|N(S)| > |S|$ for every non-empty $S \subset A$, $S \neq A$ then we can choose any $v \in A$ match it to any neighbor $u \in B$ and then match vertices in $G \setminus \{u, v\}$ by the induction hypothesis. Therefore we suppose that for some $S \subsetneq A$ we have $|N(S)| = |S|$. Let $G_1 = G[S \cup N(S)]$ and let $G_2 = G \setminus (S \cup N(S))$. (We use $G[X]$ to denote the restriction of G to the vertex set X.) We claim that G_1 and G_2 satisfy the conditions of the theorem and therefore contain matching saturating S and $A - S$, respectively. The claim trivially holds for G_1. Consider now $S' \subseteq V(G_2) \cap A$ and let $S'' = S' \cup S$. Then

$$|N(S')| \geq |N(S'')| - |N(S)| \geq |S''| - |N(S)| = |S''| - |S| \geq |S'|,$$

and the claim holds for G_2. $\square$

Second proof. The next proof we give has an advantage of providing an algorithm for testing whether a bipartite graph has a perfect matching or not. Let M be a matching in G. If M is not perfect we will either construct an M-augmenting path or find a set S violating the Hall's condition (1). By the easy implication of Exercise 1 this would prove the theorem. Let $a_0 \in A$ be a vertex unmatched by M. Choose a neighbor b_1 of a_0 in B. If b_1 is unmatched then $a_0 b_1$ is an augmenting path, consisting of one edge. Otherwise, we choose a_1 so that $a_1b_1 \in M$. Note that there exists an M-alternating path starting at a_0 and ending in a_1.

We repeat the construction procedure, as follows. Suppose $a_1, a_2, \ldots, a_k$, $b_1, \ldots, b_k$ have been constructed, so that each a_i is an end of an M-alternating path starting in a_0. Let $S = \{a_0, a_1, \ldots, a_k\}$. Either S violates the Hall's condition or there exists $b_{k+1} \in N(S) - \{b_1, b_2, \ldots, b_k\}$. If b_{k+1} is unmatched we have an augmenting path and, otherwise, we select a_{k+1} with $a_{k+1}b_{k+1} \in M$.

As claimed, our process of numbering the vertices terminates either with an M-augmenting path or a set violating the Hall's condition. □

Exercise 2. A graph is called *k-regular* if every vertex in it has degree k.

a) Show that every k-regular bipartite graph has a perfect matching.

b) Deduce that the edges of any k-regular bipartite graph can be colored in k colors so that every vertex is incident with an edge of every color.

A vertex cover in a graph G is a set C such that every edge has an end in C. A concept of the vertex cover is dual to that of a matching. If M is a matching and C a vertex cover then $|M| \leq |C|$. Therefore if $|M| = |C|$ then M is the maximum matching and C is the minimum vertex cover. Let $\tau(G)$ denote the size of the minimum vertex cover in G.

Theorem 2 (König's theorem). *If G is bipartite then $\tau(G) = \nu(G)$.*

Note that an analogue of König's theorem does not hold for non-bipartite graphs. A cycle C_{2k+1} on $2k+1$ vertices has $\nu(C_{2k+1}) = k$ and $\tau(C_{2k+1}) = k+1$.

Exercise 3. a) Derive König's theorem from Hall's theorem. **b)** Show that $\tau(G) \leq 2\nu(G)$ for every graph G.

One can determine $\nu(G)$ in polynomial time. By Exercise 3 $\nu(G)$ approximates $\tau(G)$ within a factor of 2. The problem of approximating $\tau(G)$ within a multiplicative factor less than 2 in polynomial time is open.

1.4. Perfect matchings in general graphs

In a graph G let $c_o(G)$ denote the number of odd components of G. Note that $\nu(G) \leq (|V(G)| - c_o(G))/2$. Also, if G has a perfect matching

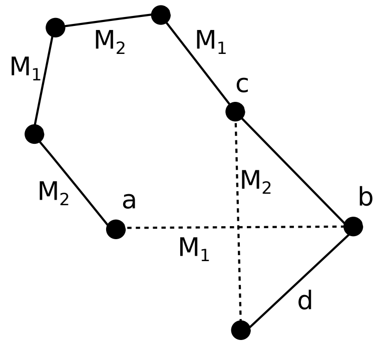

Figure 2. The construction in the proof of Theorem 3.

then $c_o(G-X) \leq |X|$ for every $X \subseteq V(G)$. The pair of encircled vertices on Figure 1 violates this condition. This necessary condition is also sufficient:

Theorem 3 (Tutte's Matching theorem)**.** *A graph G has a perfect matching if and only if*

$$c_o(G-X) \leq |X| \tag{2}$$

for every $X \subseteq V(G)$.

Proof. We assume that G is edge-maximal graph without a perfect matching. It suffices to prove the theorem for such graphs: If G' is a spanning subgraph of G and $X \subseteq V(G)$ violates (2) in G, then X violates (2) in G'.

Suppose we are able to find vertices a, b, c and d in G so that $ab, cd \notin E(G)$, and $bc, bd \in E(G)$. (See Figure 2.) Consider a perfect matching M_1 of $G + ab$ and a perfect matching M_2 of $G + cd$. Consider the component of $G[M_1 \cup M_2]$ containing a. It is a path, which we will denote by P, with one end a and another end in $\{b, c, d\}$. If b is an end of P then P is M_2-augmenting and $M_2 \triangle E(P)$ is a perfect matching in G, contradicting the choice of G. Otherwise, without loss of generality, c is an end of P and $P+cb$ is M_2-augmenting path, once again providing a contradiction.

Therefore no choice of vertices a, b, c and d as above is possible. Let X be the set of those vertices in G which are adjacent to every other vertex. Then every component of $G-X$ is complete. Indeed otherwise, we will be able to find an induced path cdb in one of the components and a non-neighbor a of b. We can match the vertices in even components among themselves, and all but one vertices in the odd components, matching the remaining vertices to X, unless $c_o(G-X) > |X|$, as desired. $\square$

Exercise 4. Given a positive integer n determine minimum $\delta = \delta(n)$, such that every graph G on $2n$ vertices with minimum degree δ has a perfect matching.

Exercise 5. Show that for every bridgeless cubic graph G and every $e \in E(G)$ there exists a perfect matching in G containing e. (A *bridge* in a graph is an edge whose deletion disconnects it. A graph is *bridgeless* if it contains no bridges.)

Exercise 6. (Tutte-Berge formula) The *deficiency* $\mathrm{def}(G)$ of a graph G is defined as the minimum number of vertices avoided by a matching. Clearly $\mathrm{def}(G) = |V(G)| - 2\nu(G)$. Show that

$$\mathrm{def}(G) = \max_{X \subseteq V(G)} (c_o(G - X) - |X|).$$