# GENERATING BRICKS

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## ABSTRACT

A brick is a 3-connected graph such that the graph obtained from it by deleting any two distinct vertices has a perfect matching. The importance of bricks stems from the fact that they are building blocks of the matching decomposition procedure of Kotzig, and Lovász and Plummer. We prove a "splitter theorem" for bricks. More precisely, we show that if a brick $H$ is a "matching minor" of a brick $G$, then, except for a few well-described exceptions, a graph isomorphic to $H$ can be obtained from $G$ by repeatedly applying a certain operation in such a way that all the intermediate graphs are bricks and have no parallel edges. The operation is as follows: first delete an edge, and for every vertex of degree two that results contract both edges incident with it. This strengthens a recent result of de Carvalho, Lucchesi and Murty.

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## 1. INTRODUCTION

All graphs in this paper are finite and simple; that is, may not have loops or multiple edges. The following well-known theorem of Tutte [15] describes how to generate all 3-connected graphs, but first a definition. Let $v$ be a vertex of a graph $H$, and let $N_1, N_2$ be a partition of the neighbors of $v$ into two disjoint sets, each of size at least two. Let $G$ be obtained from $H\backslash v$ (we use $\backslash$ for deletion and $-$ for set-theoretic difference) by adding two vertices $v_1$ and $v_2$, where $v_i$ has neighbors $N_i \cup \{v_{3-i}\}$. We say that $G$ was obtained from $H$ by *splitting a vertex*. Thus for 3-connected graphs splitting a vertex is the inverse of contracting an edge that belongs to no triangle. A *wheel* is a graph obtained from a cycle by adding a vertex joined to every vertex of the cycle.

**(1.1)** *Every 3-connected graph can be obtained from a wheel by repeatedly applying the operations of adding an edge between two nonadjacent vertices and splitting a vertex.*

A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. Seymour [14] extended (1.1) as follows.

**(1.2)** *Let $H$ be a 3-connected minor of a 3-connected graph $G$ such that $H$ is not isomorphic to $K_4$ and $G$ is not a wheel. Then a graph isomorphic to $G$ can be obtained from $H$ by repeatedly applying the operations of adding an edge between two nonadjacent vertices and splitting a vertex.*

Our objective is to prove an analogous theorem for bricks, where a *brick* is a 3-connected bicritical graph, and a graph $G$ is *bicritical* if $G\backslash u\backslash v$ has a perfect matching for every two distinct vertices $u, v \in V(G)$. A related notion is that of a *brace*, by which we mean a connected bipartite graph such that every matching of size at most two is contained in a perfect matching. Bricks and braces are important, because they are the building blocks of the matching decomposition procedure of Kotzig, and Lovász and Plummer [7], which we now briefly review.

Let $G$ be a graph, and let $X \subseteq V(G)$. We use $\delta(X)$ to denote the set of edges with one end in $X$ and the other in $V(G) - X$. A *cut* in $G$ is any set of the form $\delta(X)$ for some $X \subseteq V(G)$. A cut $C$ is *tight* if $|C \cap M| = 1$ for every perfect matching $M$ in $G$. Every

cut of the form $\delta(\{v\})$ is tight; those are called *trivial*, and all other tight cuts are called *nontrivial*. Let $\delta(X)$ be a nontrivial tight cut in a graph $G$, let $G_1$ be obtained from $G$ by identifying all vertices in $X$ into a single vertex and deleting all resulting parallel edges, and let $G_2$ be defined analogously by identifying all vertices in $V(G) - X$. Then many matching-related problems can be solved for $G$ if we are given the correspoding solutions for $G_1$ and $G_2$. As an example, consider $\mathrm{lat}(G)$, the *matching lattice* of a graph $G$, defined as the set of all integer linear combinations of characteristic vectors of perfect matchings of $G$. It is not hard to see that a description of $\mathrm{lat}(G)$ can be read off from descriptions of $\mathrm{lat}(G_1)$ and $\mathrm{lat}(G_2)$. We will return to the matching lattice shortly.

The above decomposition process can be iterated, until we arrive at graphs with no nontrivial tight cuts. Lovász [8] proved that the list of indecomposable graphs obtained at the end of the procedure does not depend on the choice of tight cuts made during the process. These indecomposable graphs were characterized by Edmonds, Lovász and Pulleyblank [2, 3]:

**(1.3)** *Let $G$ be a connected graph such that every edge of $G$ belongs to a perfect matching. Then $G$ has no nontrivial tight cut if and only if $G$ is a brick or a brace.*

Coming back to the matching lattice, Lovász [6] proved that if $G$ is a brace, then $\mathrm{lat}(G)$ consists of all integral vectors $\mathbf{w} \in \mathbf{Z}^{E(G)}$ such that $\mathbf{w}(\delta(v)) = \mathbf{w}(\delta(v'))$ for every two vertices $v, v' \in V(G)$. This is not true for bricks, because the Petersen graph is a counterexample. However, Lovász [8] proved the following deep result.

**(1.4)** *Let $G$ be a brick other than the Petersen graph. Then $\mathrm{lat}(G)$ consists precisely of all vectors $\mathbf{w} \in \mathbf{Z}^{E(G)}$ such that $\mathbf{w}(\delta(v)) = \mathbf{w}(\delta(v'))$ for every two vertices $v, v' \in V(G)$.*

Our motivation for generating bricks came from Pfaffian orientations [4]. An orientation $D$ of a graph $G$ is *Pfaffian* if every even cycle $C$ such that $G \backslash V(C)$ has a perfect matching has an odd number of edges directed in either direction of the cycle. A graph is *Pfaffian* if it has a Pfaffian orientation. This is an important concept, because the number of perfect matchings in a Pfaffian graph can be computed efficiently [4]. No polynomial-time algorithm to recognize Pfaffian graphs is known, even though there is one for bipartite

graphs [11, 13], using a structure theorem obtained in [9, 13]. The above-mentioned tight cut decomposition procedure can be used to reduce the Pfaffian graph decision problem to bricks and braces [5, 16]. Thus it remains to understand which bricks have a Pfaffian orientation, but that seems to be a much harder problem than the corresponding question for braces. Using the main theorem of this paper we managed to shed some light on this perplexing question, but the structure of Pfaffian graphs remains a mystery. We will report on these findings elsewhere. A characterization of Pfaffian graphs in terms of drawings in the plane (with crossings) has been recently obtained by the first author [12].

Let us now describe our theorem. We need a few definitions first. Let $G$ be a graph, and let $v_0$ be a vertex of $G$ of degree two incident with the edges $e_1 = v_0v_1$ and $e_2 = v_0v_2$. Let $H$ be obtained from $G$ by contracting both $e_1$ and $e_2$ and deleting all resulting parallel edges. We say that $H$ was obtained from $G$ by *bicontracting* or *bicontracting the vertex* $v_0$, and write $H = G/v_0$. Let us say that a graph $H$ is a *reduction* of a graph $G$ if $H$ can be obtained from $G$ by deleting an edge and bicontracting all resulting vertices of degree two. By a *prism* we mean the unique 3-regular planar graph on six vertices. The following is a generation theorem of de Carvalho, Lucchesi and Murty [1].

**(1.5)** *If $G$ is a brick other than $K_4$, the prism, and the Petersen graph, then some reduction of $G$ is a brick other than the Petersen graph.*

Thus if a brick $G$ is not the Petersen graph, then the reduction operation can be repeated until we reach $K_4$ or the prism. By reversing the process (1.5) can be viewed as a generation theorem. It is routine to verify that (1.5) implies (1.4), and that demonstrates the usefulness of (1.5). Our main theorem strengthens (1.5) in two respects. (We have obtained our result independently of [1], but later. We are indebted to the authors of [1] for bringing their work to our attention.) The first strengthening is that the generation procedure can start at graphs other than $K_4$ or the prism, as we explain next. Let a graph $J$ be a subgraph of a graph $G$. We say that $J$ is a *central subgraph* of $G$ if $G\backslash V(J)$ has a perfect matching. We say that a graph $H$ is a *matching minor* of $G$ if $H$ can be obtained from a central subgraph of $G$ by repeatedly bicontracting vertices of degree two. Thus if $H$ can be obtained from $G$ by repeatedly taking reductions, then $H$ is isomorphic to a

matching minor of $G$. We will denote the fact that $G$ has a matching minor isomorphic to $H$ by writing $H \hookrightarrow G$. This is consistent with our notation for embeddings, to be introduced in Section 4. Since every brick has a matching minor isomorphic to $K_4$ or the prism by [7, Theorem 5.4.11], the following implies (1.5).

**(1.6)** *Let $G$ be a brick other than the Petersen graph, and let $H$ be a brick that is a matching minor of $G$. Then a graph isomorphic to $H$ can be obtained from $G$ by repeatedly taking reductions in such a way that all the intermediate graphs are bricks not isomorphic to the Petersen graph.*

We say that a graph $H$ is a *proper reduction* of a graph $G$ if it is a reduction in such a way that the bicontractions involved do not produce parallel edges. We would like to further strengthen (1.6) by replacing reductions by proper reductions; such an improvement is worthwhile, because in applications it reduces the number of cases that need to be examined. Unfortunately, (1.6) does not hold for proper reductions, but all the exceptions can be conveniently described. Let us do that now. We refer to Figure 1(a)-(e).

Let $C_1$ and $C_2$ be two vertex-disjoint cycles of length $n \geq 3$ with vertex-sets $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ (in order), respectively, and let $G_1$ be the graph obtained from the union of $C_1$ and $C_2$ by adding an edge joining $u_i$ and $v_i$ for each $i = 1, 2, \ldots, n$. We say that $G_1$ is a *planar ladder*. Let $G_2$ be the graph consisting of a cycle $C$ with vertex-set $\{u_1, u_2, \ldots, u_{2n}\}$ (in order), where $n \geq 2$ is an integer, and $n$ edges with ends $u_i$ and $u_{n+i}$ for $i = 1, 2, \ldots, n$. We say that $G_2$ is a *Möbius ladder*. A *ladder* is a planar ladder or a Möbius ladder. Let $G_1$ be a planar ladder as above on at least six vertices, and let $G_3$ be obtained from $G_1$ by deleting the edge $u_1u_2$ and contracting the edges $u_1v_1$ and $u_2v_2$. We say that $G_3$ is a *staircase*. Let $t \geq 2$ be an integer, and let $P$ be a path with vertices $v_1, v_2, \ldots, v_t$ in order. Let $G_4$ be obtained from $P$ by adding two distinct vertices $x, y$ and edges $xv_i$ and $yv_j$ for $i = 1, t$ and all even $i \in \{1, 2, \ldots, t\}$ and $j = 1, t$ and all odd $j \in \{1, 2, \ldots, t\}$. Let $G_5$ be obtained from $G_4$ by adding the edge $xy$. We say that $G_5$ is an *upper prismoid*, and if $t \geq 4$, then we say that $G_4$ is a *lower prismoid*. A *prismoid* is a lower prismoid or an upper prismoid. We are now ready to state our main theorem.

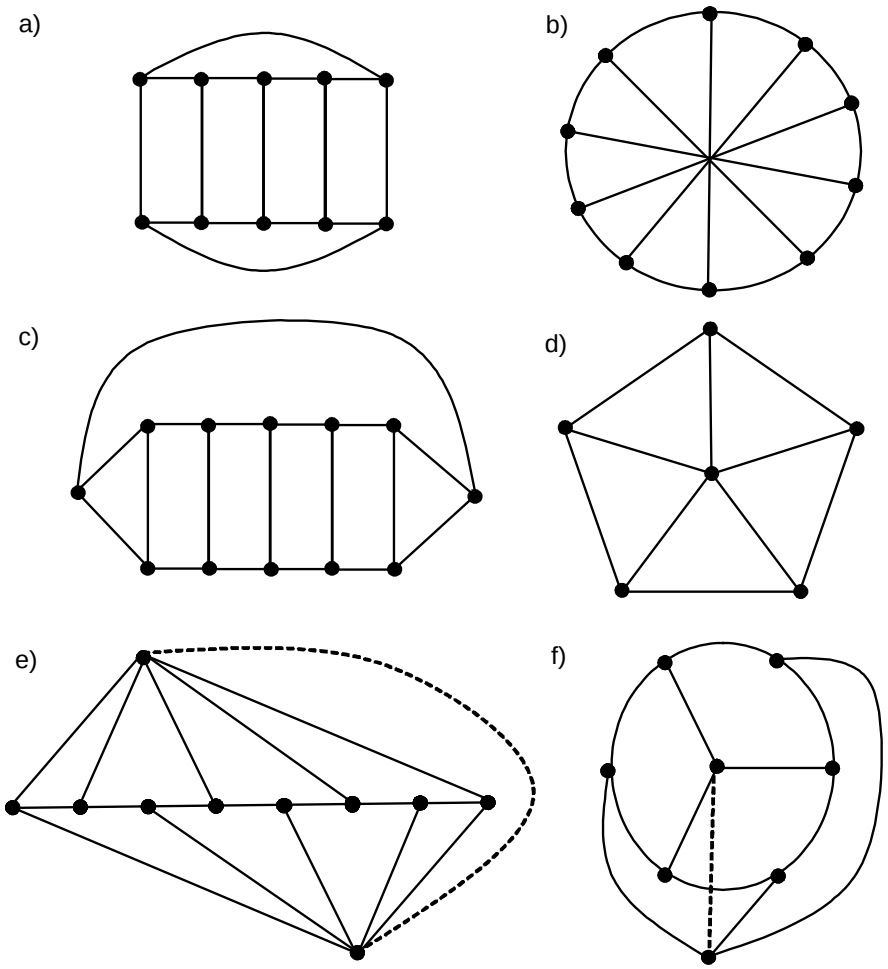



**Figure 1. Exceptional families..**

**(1.7)** *Let $H, G$ be bricks, where $H$ is isomorphic to a matching minor of $G$. Assume that $H$ is not isomorphic to $K_4$ or the prism, and $G$ is not a ladder, wheel, staircase or prismoid. Then a graph isomorphic to $H$ can be obtained from $G$ by repeatedly taking proper reductions in such a way that all the intermediate graphs are bricks not isomorphic to the Petersen graph.*

If $H$ is a brick isomorphic to a matching minor of a brick $G$ and $G$ is a ladder, wheel, staircase or prismoid, then $H$ itself is a ladder, wheel, staircase or prismoid, and can be obtained from a graph isomorphic to $G$ by taking (possibly improper) reductions in such a way that all intermediate graphs are bricks. Thus (1.7) implies (1.6). (Well, this is not immediately clear if the graph $H$ from (1.6) is a $K_4$ or a prism, but in those cases the

implication follows with the aid of the next theorem.)

As a counterpart to (1.7) we should describe the starting graphs for the generation process of (1.7). Notice that $K_4$ is a wheel, a Möbius ladder, a staircase and an upper prismoid, and that the prism is a planar ladder, a staircase and a lower prismoid. Later in this section we show

**(1.8)** *Let $G$ be a brick not isomorphic to $K_4$, the prism or the Petersen graph. Then $G$ has a matching minor isomorphic to one of the following seven graphs: the graph obtained from the prism by adding an edge, the lower prismoid on eight vertices, the staircase on eight vertices, the staircase on ten vertices, the planar ladder on ten vertices, the wheel on six vertices, and the Möbius ladder on eight vertices.*

McCuaig [10] proved an analogue of (1.7) for braces. To state his result we need another exceptional class of graphs, depicted in Figure 1(f). Let $C$ be an even cycle with vertex-set $v_1, v_2, \ldots, v_{2t}$ in order, where $t \geq 2$ is an integer and let $G_6$ be obtained from $C$ by adding vertices $v_{2t+1}$ and $v_{2t+2}$ and edges joining $v_{2t+1}$ to the vertices of $C$ with odd indices and $v_{2t+2}$ to the vertices of $C$ with even indices. Let $G_7$ be obtained from $G_6$ by adding an edge $v_{2t+1}v_{2t+2}$. We say that $G_7$ is an *upper biwheel*, and if $t \geq 3$ we say that $G_6$ is a *lower biwheel*. A *biwheel* is a lower biwheel or an upper biwheel. McCuaig's result is as follows.

**(1.9)** *Let $H, G$ be braces, where $H$ is isomorphic to a matching minor of $G$. Assume that if $H$ is a planar ladder, then it is the largest planar ladder matching minor of $G$, and similarly for Möbius ladders, lower biwheels and upper biwheels. Then a graph isomorphic to $H$ can be obtained from $G$ by repeatedly taking proper reductions in such a way that all the intermediate graphs are braces.*

Actually, (1.9) follows from a version of our theorem stated in Section 11.

Let us now introduce terminology that we will be using in the rest of the paper. Let $H, G, v_0, v_1, v_2, e_1, e_2$ be as in the definition of bicontraction. Assume that both $v_1$ and $v_2$ have degree at least three and that they have no common neighbors except $v_0$; then no parallel edges are produced during the contraction of $e_1$ and $e_2$. Let $v$ be the new vertex

that resulted from the contraction. If both $v_1$ and $v_2$ have degree at least three, then we say that $G$ was obtained from $H$ by *bisplitting the vertex* $v$. We call $v_0$ the *new inner vertex* and $v_1$ and $v_2$ the *new outer vertices*.

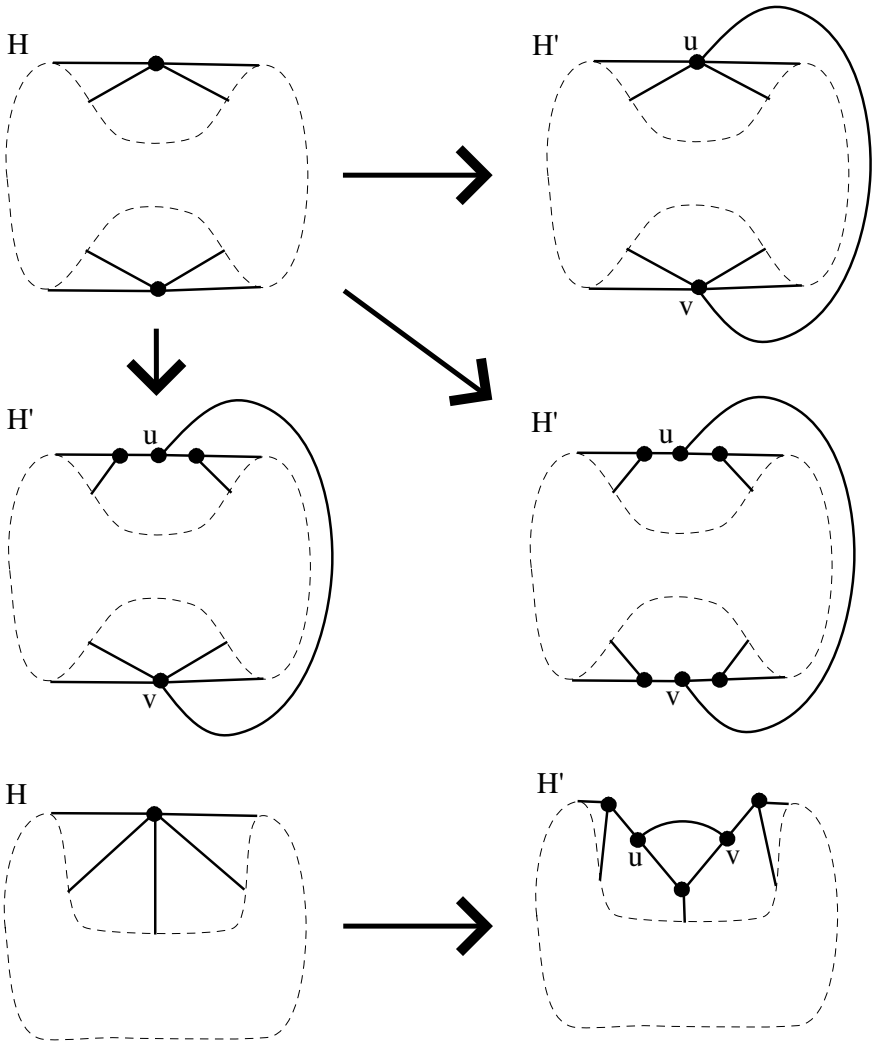



**Figure 2. Linear extensions of $H$.**

Let $H$ be a graph. We wish to define a new graph $H''$ and two vertices of $H''$. Either $H'' = H$ and $u, v$ are two nonadjacent vertices of $H$, or $H''$ is obtained from $H$ by bisplitting a vertex, $u$ is the new inner vertex of $H''$ and $v \in V(H'')$ is not adjacent to $u$, or $H''$ is obtained by bisplitting a vertex of a graph obtained from $H$ by bisplitting a

vertex, and $u$ and $v$ are the two new inner vertices of $H''$. Finally, let $H' = H'' + (u, v)$. We say that $H'$ is a *linear extension* of $H$ (see Figure 2). Thus $H'$ is a linear extension of $H$ if and only if $H$ is a proper reduction of $H'$. By *the cube* we mean the graph of the 1-skeleton of the 3-dimensional cube. Notice that the cube and $K_{3,3}$ are bipartite, and hence are not bricks. Using this terminology (1.7) can be restated in a mildly stronger form. It is easy to check that if $G'$ is obtained from a brick $G$ by bisplitting a vertex into new outer vertices $v_1$ and $v_2$, then $\{v_1, v_2\}$ is the only set $X \subseteq V(G')$ such that $|X| \geq 2$ and $G' \backslash X$ has at least $|X|$ odd components. Thus a linear extension of a brick is a brick, and hence (1.10) implies (1.7).

**(1.10)** *Let $G$ be a brick other than the Petersen graph, and let $H$ be a 3-connected matching minor of $G$ not isomorphic to $K_4$, the prism, the cube, or $K_{3,3}$. If $G$ is not isomorphic to $H$ and $G$ is not a ladder, wheel, biwheel, staircase or prismoid, then a linear extension of $H$ is isomorphic to a matching minor of $G$.*

The main step in the proof of (1.10) is the following.

**(1.11)** *Let $G$ be a brick other than the Petersen graph, and let $H$ be a 3-connected matching minor of $G$. Assume that if $H$ is a planar ladder, then there is no strictly larger planar ladder $L$ with $H \hookrightarrow L \hookrightarrow G$, and similarly for Möbius ladders, wheels, lower biwheels, upper biwheels, staircases, lower prismoids and upper prismoids. If $H$ is not isomorphic to $G$, then some matching minor of $G$ is isomorphic to a linear extension of $H$.*

It is routine to verify that if $G$ is a ladder, wheel, biwheel, staircase or prismoid, $G'$ is a linear extension of $G$, and $H$ is a 3-connected matching minor of $G$ not isomorphic to $K_4$, the prism, the cube, or $K_{3,3}$, then $G'$ has a matching minor isomorphic to a linear extension of $H$. Thus (1.11) implies (1.10), and we omit the details. The proof of (1.11) will occupy the rest of the paper. However, assuming (1.11) we can now deduce (1.8).

*Proof of* (1.8), *assuming* (1.11). Let $G$ be a brick not isomorphic to $K_4$, the prism or the Petersen graph. By [7, Theorem 5.4.11], $G$ has a matching minor $M$ isomorphic to $K_4$ or

the prism. Since $M$ is not bipartite, it is not a biwheel, a planar ladder on $4k$ vertices, or a Möbius ladder on $4k+2$ vertices. Thus if a prismoid, wheel, ladder or staircase larger than $M$ is isomorphic to a matching minor of $G$, then $G$ has a matching minor as required for (1.8). Thus we may assume that the hypothesis of (1.11) is satisfied, and hence a matching minor of $G$ is isomorphic to a linear extension of $M$. But $K_4$ does not have any linear extensions, and the prism has, up to isomorphism, exactly one, namely the graph obtained from it by adding an edge. This proves (1.8). □

Here is an outline of the paper. First we need to develop some machinery; that is done in Sections 2, 3, and 4. In Section 5 we prove a first major step toward (1.11), namely that the theorem holds provided a graph obtained from $H$ by bisplitting a vertex is isomorphic to a matching minor of $G$. Then in Section 6 we reformulate our key lemma in a form that is easier to apply, and introduce several different types of extensions. In Section 7 we use the 3-connectivity of $G$ to show that at least one of those extensions of $H$ is isomorphic to a matching minor of $G$, and in Sections 8–10 we gradually eliminate all the additional extensions. Theorem (1.11) is proved in Section 10. Finally, in Section 11 we state a strengthening of (1.11) that can be obtained by following the proof of (1.11) with minimal changes. We delegate the stregthening to the last section, because the statement is somewhat cumbersome and perhaps of lesser interest. Its applications include (1.11), (1.9) and a generation theorem for a subclass of factor-critical graphs.

A word about notation. If $H$ is a graph, and $u, v \in V(H)$ are distinct nonadjacent vertices, then $H + (u, v)$ or $H + uv$ denotes the graph obtained from $H$ by adding an edge with ends $u$ and $v$. Now let $u, v \in V(H)$ be adjacent. By *bisubdividing* the edge $uv$ we mean replacing the edge by a path of length three, say a path with vertices $u, x, y, v$, in order. Let $H'$ be obtained from $H$ by this operation. We say that $x, y$ (in that order) are the *new vertices*. Thus $y, x$ are the new vertices resulting from subdividing the edge $vu$ (we are conveniently exploiting the notational asymmetry for edges). Now if $w \in V(H) - \{u\}$, then by $H + (w, uv)$ we mean the graph $H' + (w, x)$. Notice that the graphs $H + (w, uv)$ and $H + (w, vu)$ are different. In the same spirit, if $a, b \in V(H)$ are adjacent vertices of $H$ with $\{u, v\} \neq \{a, b\}$, then we define $H + (uv, ab)$ to be the graph $H' + (x, ab)$.

## 2. OCTOPI AND FRAMES

Let $H$ be a graph with a perfect matching, and let $X \subseteq V(H)$ be a set of size $k$. If $H \backslash X$ has at least $k$ odd components, then $X$ is called a *barrier* in $H$. The following is easy and well-known.

**(2.1)** *A brick has no barrier of size at least two.*

Now if $H$ and $X$ are as above and $H$ is a subgraph of a brick $G$, then $X$ cannot be a barrier in $G$. If $H$ is a central subgraph of $G$, then we get the following useful outcome. In the application $R_1, R_2, \ldots, R_k$ will be the components of $G \backslash X$.

**(2.2)** *Let $G$ be a brick and let $H$ be a subgraph of $G$. Let $M$ be a perfect matching of $G \backslash V(H)$ and let $V(H)$ be a disjoint union of $X$, $R_1, R_2, \ldots, R_k$, where $k \geq 2$, $|X| \leq k$ and $|R_i|$ is odd for every $i \in \{1, 2, \ldots, k\}$. Then there exist distinct integers $i, j \in \{1, 2, \ldots, k\}$ and an $M$-alternating path joining a vertex in $R_i$ to a vertex in $R_j$.*

*Proof.* Suppose for a contradiction that the lemma is false, and let $H$ be a maximal subgraph of $G$ that satisfies the hypothesis of the lemma, but not the conclusion.

By (2.1) there exists an edge $e_1 \in E(G)$ with one end $v \in R_i$ for some $i \in \{1, 2, \ldots, k\}$ and the other end $u \in V(G) - R_i - X$. Without loss of generality we may assume that $i = 1$. If $u \in V(H)$ then the path with edge-set $\{e_1\}$ is as required. Thus $u \notin V(H)$, and hence $u$ is incident with an edge $e_2 \in M$. Let $w$ be the other end of $e_2$; then clearly $w \notin V(H)$. Let $X' = X \cup \{u\}$, $R_{k+1} = \{w\}$, $M' = M - \{e_2\}$ and construct $H'$ by adding the vertices $u$ and $w$ and edges $e_1$ and $e_2$ to $H$. By the maximality of $H$ the graph $H'$, matching $M'$ and sets $X', R_1, R_2, \ldots, R_{k+1}$ satisfy the conclusion of the lemma. Thus for some distinct integers $i, j \in \{1, 2, \ldots, k+1\}$ there exists an $M'$-alternating path $P$ joining a vertex in $R_i$ to a vertex in $R_j$. Since $H$ does not satisfy the conclusion of the lemma we may assume that $j = k + 1$. Let $P'$ be the graph obtained from $P$ by adding the edges $e_1$ and $e_2$. If $i > 1$, then $P'$ is a path and satisfies the conclusion of the lemma.

Thus we may assume that $i = 1$. Let $H'' = H \cup P'$, $M'' = M - E(P')$ and $R_1' = R_1 \cup V(P')$. Then the graph $H''$, matching $M''$ and sets $X, R_1', R_2, \ldots, R_k$ also satisfy the conclusion of the lemma by the maximality of $H$. Thus we may assume that there is an

$M''$-alternating path $Q$ joining a vertex in $R_1'$ to a vertex in $R_j$ for some $j \in \{2, 3, \ldots, k\}$. If neither of the ends of $Q$ lies in $V(P')$ then $Q$ is a required path for $H$. If one of them, say $x$, is in $V(P')$, we add to $Q$ one of the subpaths of $P'$ with end $x$ to obtain a required path. □

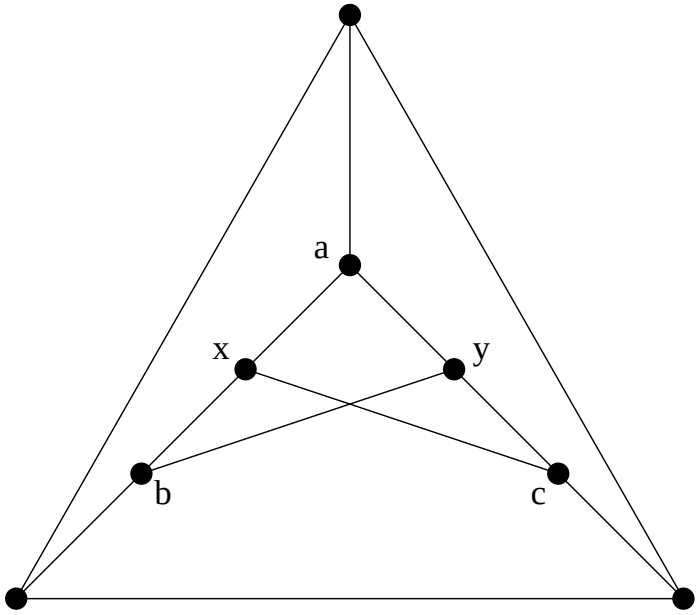



**Figure 3. A graph $H$, containing $K_4$ as a matching minor.**

To motivate the next definitions, let us consider the following example. Let $G$ be a brick that has a matching minor isomorphic to $K_4$. Later in the proof there will come a step when one will be able to deduce that $G$ has a matching minor isomorphic to the graph $H$ depicted in Figure 3. Unfortunately, $H$ is not a brick, because the set $\{a, b, c\}$ is a barrier. So we try to apply (2.2). More precisely, $G$ has a central subgraph $J$ isomorphic to a graph obtained from $H$ by repeatedly bisubdividing edges of $H$; let $a', b', c', x', y' \in V(J)$ correspond to $a, b, c, x, y$, respectively. By applying (2.2) we deduce that one of a number of outcomes holds, including a possibility that $G$ has a matching minor isomorphic to $J + (x', y'a')$. The latter graph, however, is not a brick and the only brick matching minor it contains is $K_4$. Thus we need a strengthening of (2.2) in the case when each $R_i$ has a restricted structure, what we call an octopus. Let us introduce the necessary definitions.

Let $H$ be a graph, let $C$ be a subgraph of $H$ with an odd number of vertices, and let $P_1, P_2, ..., P_k$ be odd paths in $H$. For $i = 1, 2, ..., k$ let $u_i$ and $v_i$ be the ends of $P_i$. If for $i = 1, 2, ..., k$ we have $u_i \in V(C)$ and $V(P_i) \cap (V(C) \cup \bigcup_{j \neq i} V(P_j)) = \{u_i\}$ we say that $\Omega = (C, P_1, P_2, \ldots, P_k)$ is an *octopus* in $H$. We say that the paths $P_1, P_2, ..., P_k$ are the *tentacles* of $\Omega$, $C$ is the *head* of $\Omega$ and $v_i$ are the *ends* of $\Omega$. We define the *graph of* $\Omega$ to be $C \cup P_1 \cup P_2 \cup \cdots \cup P_k$, and by abusing notation slightly we will denote this graph also by $\Omega$. We say that a matching $M$ in $G$ is $\Omega$-*compatible* if every tentacle is $M$-alternating and no vertex of $C$ is incident to an edge of $M$. See Figure 4. Then in the example above each component of $J \backslash \{a', b', c'\}$ can be turned into an octopus $\Omega$, and the perfect matching in $G \backslash V(J)$ can be extended to an $\Omega$-compatible matching in $G$.

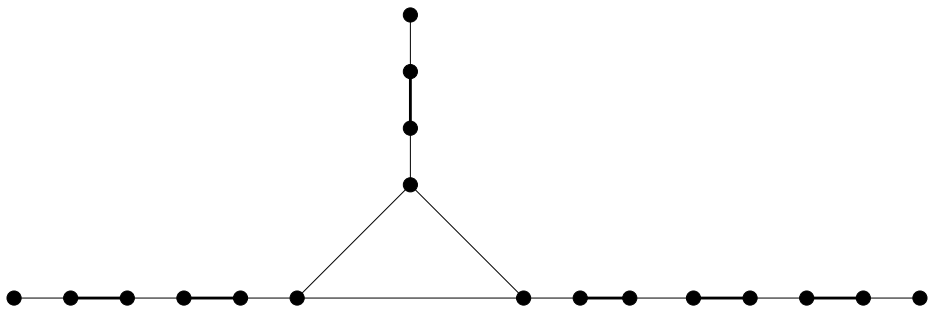


**Figure 4. An octopus $\Omega$ and an $\Omega$-compatible matching.**

Let $G$ be a graph, and let $k \geq 1$ be an integer. We say that the pair $(\mathcal{F}, X)$ is a *frame* in $G$ if $X \subseteq V(G)$ and $\mathcal{F} = \{\Omega_1, \Omega_2, \ldots, \Omega_k\}$ satisfy

(1) $\Omega_1, \Omega_2, \ldots, \Omega_k$ are octopi,

(2) for $i = 1, 2, \ldots, k$ the ends and only the ends of $\Omega_i$ belong to $X$,

(3) for distinct $i, j \in \{1, 2, \ldots, k\}$, $V(\Omega_i) \cap V(\Omega_j) \subseteq X$,

(4) $|X| \leq k$.

We say that $\Omega_1, \Omega_2, \ldots, \Omega_k$ are the *components* of $(\mathcal{F}, X)$. We define the graph of $(\mathcal{F}, X)$ to be $\Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_k$, and denote it by $\mathcal{F}$, again abusing notation. Thus in the above example $G$ has a frame $(\mathcal{F}, \{a', b', c'\})$ with three components. The following is the main result of this section. We say that a graph $H$ is $M$-*covered* if a subset of $M$ is a perfect matching of $H$.

**(2.3)** *Let $G$ be a brick, let $M$ be a matching in $G$, and let $(\mathcal{F}, X)$ be a frame in $G$ such that $G \setminus (V(\mathcal{F}) \cup X)$ is $M$-covered and $M$ is $\Omega$-compatible for each $\Omega \in \mathcal{F}$. Then there exists an $M$-alternating path $P$ joining vertices of the heads of two different components $\Omega_1, \Omega_2$ of $(\mathcal{F}, X)$. Moreover, there is an edge $e \in E(P) - M$ such that the two components of $P \setminus e$ can be numbered $P_1$ and $P_2$ in such a way that $V(P_i) \cap V(\mathcal{F}) \subseteq V(\Omega_i)$ for $i = 1, 2$.*

*Proof.* We say that a subpath $Q$ of a path $P$ is an *$\mathcal{F}$-jump* in $P$ if the ends of $Q$ belong to different components of $\mathcal{F}$ and $Q$ is otherwise disjoint from $\mathcal{F}$. Let $\mathcal{F} = \{\Omega_1, \Omega_2, \ldots, \Omega_k\}$ and let $C_i$ denote the vertex-set of the head of $\Omega_i$. By (2.2) applied to $X, C_1, C_2, \ldots, C_k$ there exists an $M$-alternating path joining vertices of the heads of two different components of $(\mathcal{F}, X)$. Choose such path $P$ with the minimal number of $\mathcal{F}$-jumps in it. We prove that $P$ satisfies the requirements of the theorem.

Let $v_1 \in C_1$ and $v_2 \in C_2$ be the ends of $P$. Since $P$ is $M$-alternating and $M$ is $\Omega_i$-compatible for all $i = 1, 2, \ldots, k$, it follows that no internal vertex of $P$ belongs to $C_i$. Suppose that $P \cap T \neq \emptyset$ for some tentacle $T$ of $\Omega_i$, where $i \geq 3$. Let $\{v_0\} = V(T) \cap C_i$ and let $v \in V(P) \cap V(T)$ be chosen so that $T[v, v_0]$ is minimal. For some $j \in \{1, 2\}$ the path $P[v_j, v] \cup T[v, v_0]$ is $M$-alternating and contradicts the choice of $P$. Thus $V(P) \cap V(\mathcal{F}) \subseteq V(\Omega_1) \cup V(\Omega_2)$.

Define a linear order on $V(P)$ so that $v \succ v'$ if and only if $v' \in P[v_1, v]$. Let $P_0$ be an $\mathcal{F}$-jump in $P$ with ends $u_1 \in V(\Omega_1)$ and $u_2 \in V(\Omega_2)$ chosen so that $u_1 \succ u_2$ and $P[v_1, u_2]$ is minimal. Equivalently we can define $P_0$ as a second $\mathcal{F}$-jump we encounter if we traverse $P$ from $v_1$ to $v_2$. If such an $\mathcal{F}$-jump $P_0$ in $P$ does not exist then $P$ contains a unique $\mathcal{F}$-jump. Let $e \notin M$ be an edge of this unique $\mathcal{F}$-jump; then $P$ and $e$ satisfy the requirements of the theorem. Therefore we may assume the existence of $P_0$.

For $i \in \{1, 2\}$ let $T_i$ be the tentacle of $\Omega_i$ such that $u_i \in V(T_i)$ and let $\{w_i\} = V(T_i) \cap C_i$. Let $s_1 \in V(T_1) \cap V(P)$ be chosen so that $s_1 \succ u_1$ and $T_1[s_1, w_1]$ is minimal. Note that $s_1 \neq w_1$, because the only vertex in $V(P) \cap C_1$ is $v_1$ and $s_1 \succ u_1 \succ v_1$. Let $s_1 t_1$ be the edge of $M$ incident to $s_1$. We have $s_1 t_1 \in E(T_1 \cap P)$ as both $T_1$ and $P$ are $M$-alternating, $s_1 \in T_1[t_1, w_1]$ by the choice of $s_1$ and $s_1 \succ t_1$ as otherwise the path $T_1[w_1, s_1] \cup P[s_1, v_2]$ contradicts the choice of $P$. Let $s_2 \in V(T_2) \cap V(P)$ be chosen so that

$s_2 \prec s_1$ and $T_2[s_2, w_2]$ is minimal. Let $s_2t_2$ be the edge of $M$ incident to $s_2$. We again have $s_2t_2 \in T_2 \cap P$, $s_2 \in T_2[t_2, w_2]$ and $s_2 \prec t_2$, as otherwise the path $P[v_1, s_2] \cup T_2[s_2, w_2]$ contradicts the choice of $P$.

Consider $P' = P[s_2, s_1]$. By the choice of $s_1$ we have $V(P[u_2, s_1]) \cap V(T_1[s_1, w_1]) = \{s_1\}$. Also if $s_2 \prec u_2$ we have $V(P[s_2, u_2] \cap V(\Omega_1) = \emptyset$ by the choice of $P_0$. It follows that $V(P') \cap V(T_1[s_1, w_1]) = \{s_1\}$. By the choice of $s_2$ we have $V(P') \cap V(T_2[s_2, w_2]) = \{s_2\}$. Therefore $T_2[w_2, s_2] \cup P' \cup T_1[w_1, s_1]$ is an $M$-alternating path contradicting the choice of $P$. □

## 3. TWO PATHS MEETING

In this section we study the following problem. Let $G$ be a graph, let $M$ be a matching, and let $P_1$ and $P_2$ be two $M$-alternating paths. In the applications we will be permitted to replace the matching $M$ by a matching $M'$ saturating the same set of vertices, and to replace the paths $P_1$ and $P_2$ by a pair of $M'$-alternating paths with the same ends. Thus we are interested in graphs that are minimal in the sense that there is no replacement as above upon which an edge of $G$ may be deleted. The main result of this section, theorem (3.3) below, asserts that there are exactly four types of minimally intersecting pairs of $M$-alternating paths, three of which are depicted in Figure 5. We start with two auxiliary lemmas.

**(3.1)** *Let $M$ be a matching in $G$, let $P$ be an $M$-alternating path with ends $x$ and $y$, let $C$ be an $M$-alternating cycle such that $x$ and $y$ have degree at most two in $P \cup C$ and let $M' = M \triangle E(C)$. Then there exists an $M'$-alternating path $Q$ with ends $x$ and $y$ satisfying $E(Q) \subseteq E(P) \triangle E(C)$.*

*Proof.* Let $H$ be the subgraph of $G$ with vertex-set $V(G)$ and edge-set $E(P) \triangle E(C)$. Then $x, y$ have degree one in $H$, every other vertex of $H$ has degree zero or two, and if it has degree two, then it is incident with an edge of $M'$. Thus some component of $H$ is an $M'$-alternating path joining $x$ and $y$, as desired. □

**(3.2)** *Let $M$ be a matching in $G$, let $P$ be an $M$-alternating path with ends $w$ and $v$, and let $R$ be a path with ends $v$ and $z$ such that $R \setminus v$ is $M$-covered, $v$ is incident with no edge of $M$, and $w \notin V(R)$. Let $M' = M \triangle E(R)$. Then there exists an $M'$-alternating path $Q$ with ends $w$ and $z$ satisfying $E(Q) \subseteq E(P) \triangle E(R)$.*

*Proof.* This follows similarly as (3.1) by considering the graph with edge-set $E(P) \triangle E(R)$. □

Let $G$ be a graph, let $M$ be a matching in $G$, and let $P$ and $Q$ be two $M$-alternating paths in $G$. For the purpose of this definition let a *segment* be a maximal subpath of $P \cap Q$, and let an *arc* be a maximal subpath of $Q$ with no internal vertex or edge in $P$. We say that $P$ and $Q$ *intersect transversally* if either they are vertex-disjoint, or there exist vertices $q_0, q_1, \ldots, q_7 \in V(Q)$ such

(1) $q_0, q_1, \ldots, q_7$ occur on $Q$ in the order listed, and $q_0$ and $q_7$ are the ends of $Q$,

(2) $q_2, q_1, q_3, q_4, q_6, q_5$ all belong to $P$ and occur on $P$ in the order listed,

(3) if $q_0 \in V(P)$, then $q_0 = q_1 = q_2 = q_3$, and otherwise $Q[q_0, q_1]$ is an arc,

(4) if $q_7 \in V(P)$, then $q_7 = q_6 = q_5 = q_4$, and otherwise $Q[q_6, q_7]$ is an arc,

(5) $Q[q_3, q_4]$ is a segment,

(6) either $q_1 = q_2 = q_3$, or $q_1, q_2, q_3$ are pairwise distinct, $Q[q_1, q_2]$ is a segment, $Q[q_2, q_3]$ is an arc and $q_2$ is not an end of $P$, and

(7) either $q_4 = q_5 = q_6$, or $q_4, q_5, q_6$ are pairwise distinct, $Q[q_5, q_6]$ is a segment, $Q[q_4, q_5]$ is an arc and $q_5$ is not an end of $P$.

It follows that the definition is symmetric in $P$ and $Q$. There are four cases of transversal intersection depending on the number of components of $P \cap Q$; the three cases when $P$ and $Q$ intersect are depicted in Figure 5, where matching edges are drawn thicker. We shall prove the following lemma.

**(3.3)** *Let $M$ be a matching in a graph $G$ and let $P_1$ and $P_2$ be two $M$-alternating paths, where $P_i$ has ends $s_i$ and $t_i$. Assume that $s_1$, $s_2$, $t_1$ and $t_2$ have degree at most two in $P_1 \cup P_2$. Then there exist a matching $M'$ saturating the same set of vertices as $M$ and two*

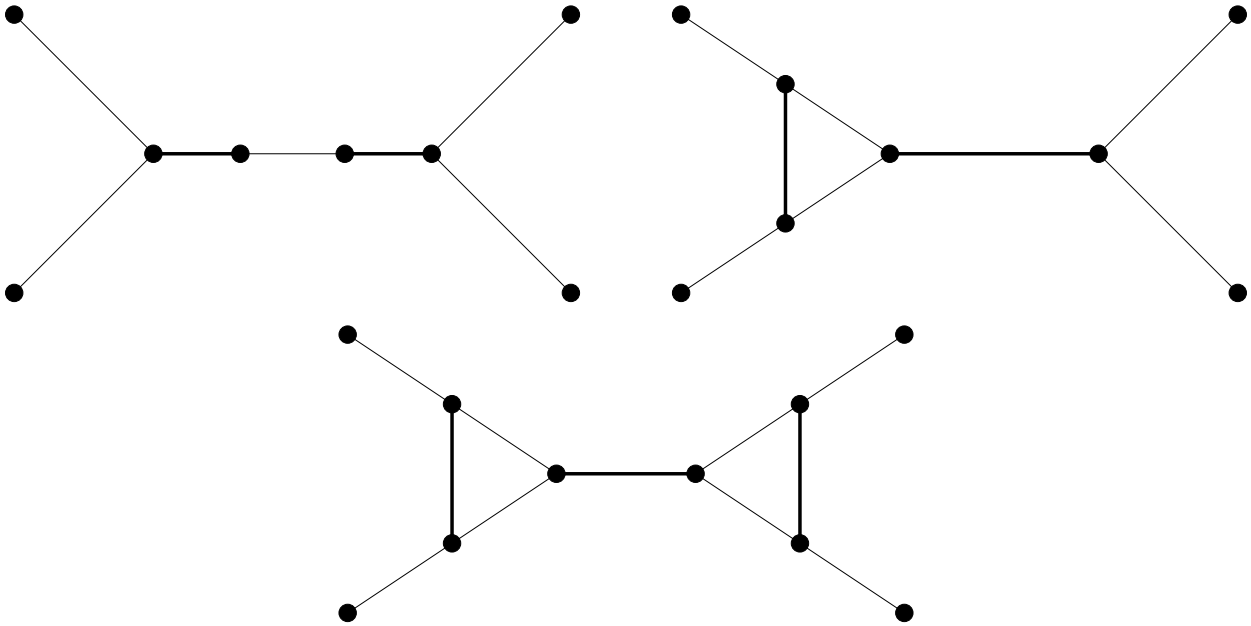


**Figure 5. Three cases of transversal intersection.**

*$M'$-alternating paths $Q_1$ and $Q_2$ such that $M \triangle M' \subseteq E(P_1) \cup E(P_2)$, $Q_i$ has ends $s_i$ and $t_i$ and $Q_1$ and $Q_2$ intersect transversally.*

Unfortunately, for later application we need a more general, but less nice result, the following. Please notice that it immediately implies (3.3) on taking $r = t_2$.

**(3.4)** *Let $M$ be a matching in a graph $G$ and let $P_1$ and $P_2$ be two $M$-alternating paths, where $P_i$ has ends $s_i$ and $t_i$. Assume that $s_1$, $s_2$, $t_1$ and $t_2$ have degree at most two in $P_1 \cup P_2$. Let $r \in V(P_2)$, and let $P_2' = P_2[s_2, r]$. Then one of the following conditions hold:*

(1) *There exist a matching $M'$ saturating the same set of vertices as $M$ and two $M'$-alternating paths $Q_1$ and $Q_2$ such that $Q_i$ has ends $s_i$ and $t_i$, $M \triangle M' \subseteq E(P_1) \cup E(P_2')$, $Q_1 \subseteq P_1 \cup P_2'$, and $Q_1 \cup Q_2$ is a proper subgraph of $P_1 \cup P_2$,*

(2) *$r \neq t_2$, and there exists an $M$-alternating path $R \subseteq P_1 \cup P_2'$ with ends $s_2$ and $t_1$ such that $R$ and $P_1$ intersect transversally,*

(3) *$P_2'$ intersects $P_1$ transversally.*

*Proof.* We may assume that $G = P_1 \cup P_2$ and (1) does not hold. We shall refer to this as the *minimality of $G$*.

We claim that $P_1 \cup P_2'$ contains no $M$-alternating cycles. Suppose for a contradiction there exists an $M$-alternating cycle $C \subseteq P_1 \cup P_2'$. Let $M' = M \triangle E(C)$ and let $Q_1, Q_2$ be the two $M'$-alternating paths obtained by applying (3.1) to $P_1$ and $P_2$, respectively. Since $P_1$ and $P_2$ are $M$-alternating and their union includes $C$, they either share an edge of $M \cap E(C)$, say $e$, or $P_1$ and $P_2$ have the same ends. In the later case replacing $P_2$ by $P_1$ contradicts the minimality of $G$, and so we may assume the former. Now $Q_1 \subseteq P_1 \cup P_2'$ and $Q_1 \cup Q_2$ is a subgraph of $(P_1 \cup P_2) \setminus e$, contradicting the minimality of $G$.

For the purpose of this proof let us define an *arc* as a maximal subpath of $P_2'$ that has at least one edge or contains an end of $P_2'$ and has no internal vertex or edge in $P_1$. Define *segment* as a maximal subpath of $P_1 \cap P_2$. We say that two vertices of $P_1$ have the *same biparity* if their distance on $P_1$ is even, and otherwise we say they have *opposite biparity*. We claim that the ends of every arc have the same biparity. To see that, let $P_2'[s,t]$ be an arc with ends of opposite biparity. There are two cases. Either both end-edges of $P_1[s,t]$ belong to $M$, or both of them do not. If they do, then $P_1[s,t] \cup P_2'[s,t]$ is an $M$-alternating cycle, and if they do not, then $P_1', P_2$ contradict the minimality of $G$, where $P_1'$ is obtained from $P_1$ by replacing the interior of $P_1[s,t]$ by $P_2'[s,t]$. (Notice that the edge of $P_1[s,t]$ incident with $s$ does not belong to $P_1'$ or $P_2$.) This proves our claim that the ends of every arc have the same biparity.

We may assume that there is an arc with both ends on $P_1$, for otherwise (3) holds. Let $P_2'[u_0, v_0]$ be such an arc. Since $u_0, v_0$ have the same biparity, exactly one end-edge of $P_1[u_0, v_0]$ belongs to $M$, say the one incident with $u_0$. Then the unique segment incident with $u_0$, say $P_1[u_0, v_1] = P_2'[u_0, v_1]$ has the property that $v_1$ lies between $u_0$ and $v_0$ on $P_1$. Let $P_2'[v_1, u_1]$ be the unique arc incident with $v_1$. Then either $u_1$ is an end of $P_2'$, or $u_1, v_1$ have the same biparity, opposite to the biparity of $u_0, v_0$.

We claim that either $u_1$ is an end of $P_2'$, or $u_1$ lies between $v_1$ and $v_0$ on $P_1$. To prove this claim we need to prove that neither $u_0$ nor $v_0$ lie between $u_1$ and $v_1$ on $P_1$. To this end suppose first that $u_0$ lies between $u_1$ and $v_1$ on $P_1$. Then $P_1''$ and $P_2$ contradict the minimality of $G$, where $P_1''$ is obtained from $P_1$ by replacing the interior of $P_1[u_1, v_0]$ by $P_2'[u_1, v_0]$ (the edge of $P_1[v_1, v_0]$ incident with $v_1$ does not belong to $P_1'' \cup P_2$). Suppose now that $v_0$ lies between $u_1$ and $v_1$ on $P_1$. Then $P_2'[v_0, u_1] \cup P_1[u_1, v_0]$ is an $M$-alternating

cycle, a contradiction. This proves that either $u_1$ is an end of $P_2'$, or $u_1$ lies between $v_1$ and $v_0$ on $P_1$.

Now assume that $P_2'[u_0, v_0]$ is chosen so that $P_1[u_0, v_0]$ is maximal, and let $u_1, v_1$ be as in the previous paragraph. If $u_1$ is an end of $P_2'$ we stop, and so assume that it is not. Recall that $u_1, v_1$ have opposite biparity from $u_0, v_0$. Thus the unique segment incident with $u_1$, say $P_1[u_1, v_2] = P_2'[u_1, v_2]$ has the property that $v_2$ lies between $v_1$ and $u_1$ on $P_1$. Now let $P_2'[v_2, u_2]$ be the unique arc incident with $v_2$. By the result of the previous paragraph either $u_2$ is an end of $P_2'$, or $u_2$ lies between $v_2$ and $v_1$ on $P_1$. By arguing in this manner we arrive at a sequence of vertices $u_0, v_0, \ldots, u_{k+1}, v_{k+1}$ such that

(i) $u_0, v_1, u_2, v_3, \ldots, v_{k+1}, \ldots, u_3, v_2, u_1, v_0$ occur on $P_1$ in the order listed,

(ii) $u_{k+1}$ is an end of $P_2'$,

(iii) $P_2'[u_i, v_i]$ are arcs for $i = 0, 1, \ldots, k+1$, and

(iv) $P_1[u_i, v_{i+1}]$ are segments for $i = 0, 1, \ldots, k$.

It follows that $u_i, v_i$ have the same biparity and that their biparity depends on the parity of $i$. Let $P_1[v_0, v_0']$ be the unique segment incident with $v_0$. Then $v_0$ lies between $v_0'$ and $u_0$ on $P_1$. Let $P_2'[v_0', u_0']$ be the unique arc incident with $v_0'$. The maximality of $P_1[u_0, v_0]$ and the result of the previous paragraph imply that either $u_0'$ is an end of $P_2'$, or that $u_0, v_0, v_0', u_0'$ occur on $P_1$ in the order listed. In the latter case by an analogous argument there exists a sequence of vertices $u_0', v_0', \ldots, u_{k'+1}', v_{k'+1}'$ such that

(i) $u_0', v_1', u_2', v_3', \ldots, v_{k'+1}', \ldots, u_3', v_2', u_1', v_0'$ occur on $P_1$ in the order listed,

(ii) $u_{k'+1}'$ is an end of $P_2$,

(iii) $P_2'[u_i', v_i']$ are arcs for $i = 0, 1, \ldots, k'+1$, and

(iv) $P_1[u_i', v_{i+1}']$ are segments for $i = 0, 1, \ldots, k'$.

Suppose $r = t_2$. Then $k = 0$, for otherwise $P_1$ and and the path obtained from $P_2$ by replacing the interior of $P_2[v_0, u_1]$ by $P_1[v_0, u_1]$ contradict the minimality of $G$. Similarly, either $u_0'$ is an end of $P_2$ or $k' = 0$. Thus (3) holds.

Therefore we may assume $r \neq t_2$. Suppose $s_2 \neq u_0'$. Then without loss of generality we assume $s_2 = u_{k+1}$. We define $R_1 = P_2'[s_2, u_k] \cup P_1[u_k, t_1]$ and $R_2 = P_2'[s_2, v_k] \cup P_1[v_k, t_1]$. For some $i \in \{1, 2\}$ $R_i \subseteq P_1 \cup P_2'$ is an $M$-alternating path with ends $s_2$ and $t_1$ such that $R_i$ and $P_1$ intersect transversally. Thus (2) holds.

It remains to consider the case when $s_2 = u'_0$ and $u_{k+1} = r$. If $k = 0$, then (3) holds, and so we may assume that $k \geq 1$. We claim that $E(P_1[v_{k+1}, v_k] \cap P_2) = \emptyset$. Suppose for a contradiction $P_2[x, y] \subseteq P_1[v_{k+1}, v_k]$ is a segment, and let $P_2[x, y]$ be chosen so that $P_2[y, t_2]$ is minimal. If $x \in V(P_1[v_k, y])$ define $Q_2 = P_2[s_2, v_k] \cup P_1[v_k, x] \cup P_2[x, t_2]$, and otherwise define $Q_2 = P_2[s_2, v_{k+1}] \cup P_1[v_{k+1}, x] \cup P_2[x, t_2]$. As $E(P_1[v_{k+1}, v_k] \cap P'_2) = \emptyset$ we see that $Q_2$ is an $M$-alternating path. We replace $P_2$ with $Q_2$ to contradict the minimality of $G$.

Now we claim $E(P_1[v_{k-1}, u_k] \cap P_2) = \emptyset$. Again suppose $P_2[x, y] \subseteq P_1[v_{k-1}, u_k]$ is a segment, and let $P_2[x, y]$ be chosen so that $P_2[y, t_2]$ is minimal. If $x \in V(P_1[v_{k-1}, y])$ define $Q_2 = P_2[s_2, v_{k-1}] \cup P_1[v_{k-1}, x] \cup P_2[x, t_2]$, and otherwise define $Q_2 = P_2[s_2, v_k] \cup P_1[v_k, x] \cup P_2[x, t_2]$. As $E(P_1[v_{k+1}, v_k] \cap P_2) = \emptyset$ we see that $Q_2$ is an $M$-alternating path. Again we replace $P_2$ with $Q_2$ to contradict the minimality of $G$.

Now let $Q_2 = P_2[s_2, v_{k-1}] \cup P_1[v_{k-1}, u_k] \cup P_2[u_k, t_2]$. As $E(P_1[v_{k-1}, u_k] \cap P_2) = \emptyset$ we see that $Q_2$ is an $M$-alternating path and replacing $P_2$ with $Q_2$ we once again contradict the minimality of $G$. □

We deduce several corollaries of (3.4). Let $\Omega$ be an octopus in a graph $G$, where $\Omega$ consists of two tentacles and a head $C$ with $V(C) = \{v\}$. Then the graph of $\Omega$ is a path. We say that $\Omega$ is a *path octopus* with head $v$. The head of a path octopus can be moved along $\Omega$ in the sense that if $v' \in V(\Omega)$ is at even distance from $v$ in $\Omega$, then there is another path octopus with the same graph and head $v'$. The next lemma will use this fact.

**(3.5)** *Let $G$ be a graph, let $\Omega$ be a path octopus in $G$ with head $v$ and ends $v_1$ and $v_2$, let $z$ be the neighbor of $v_1$ in $\Omega$, let $M$ be an $\Omega$-compatible matching, and let $P$ be an $M$-alternating path in $G \backslash v_1 \backslash v_2$ with ends $v$ and $w \notin V(\Omega)$. Then there exist a path octopus $\Omega'$ with head $z$ and ends $v_1$ and $v_2$, an $\Omega'$-compatible matching $M'$, and a path $P'$ with ends $z$ and $w$ such that $E(\Omega') \subseteq E(\Omega \cup P)$, $zv_1 \in E(\Omega')$, $v_1 \notin V(P')$, $M$ coincides with $M'$ on $G \setminus (V(P) \cup V(\Omega))$, $\Omega \cup P \setminus V(\Omega' \cup P')$ is $M'$-covered, and $P'$ intersects $\Omega' \backslash v_1$ transversally.*

*Proof.* Since $M$ is $\Omega$-compatible, $v$ is incident with no edge of $M$. Let $R = \Omega[z, v]$, let $M' = M \triangle E(R)$, and let $\Omega'$ be the octopus with graph $\Omega$ and head $z$. Then $M'$ is an $\Omega'$-compatible matching. By (3.2) there exists an $M'$-alternating path $P'$ with ends $z$ and $w$ such that $E(P') \subseteq E(P) \triangle E(R)$. By (3.3) we may assume, by replacing the tentacle $\Omega'[z, v_2]$ and path $P'$, that $P'$ intersects $\Omega' \setminus v_1 = \Omega'[z, v_2]$ transversally, as desired. □

Let $P_1, P_2, P_3$ be odd paths in a graph $H$. For $i = 1, 2, 3$ let $u_i$ and $v_i$ be the ends of $P_i$. If $u_1 = u_2 = u_3$ and otherwise $P_1, P_2, P_3$ are pairwise disjoint, then we say that the octopus with tentacles $P_1, P_2$ and $P_3$ and a head the graph with vertex-set $\{u_1\}$ is a *triad* in $H$. Assume now that $P_1, P_2, P_3$ are pairwise disjoint, and let $Q_1, Q_2, Q_3$ be three odd paths such that for $\{i, j, k\} = \{1, 2, 3\}$ the ends of $Q_k$ are $u_i$ and $u_j$. Assume further that $P_1, P_2, P_3, Q_1, Q_2, Q_3$ are pairwise disjoint, except for common ends in the set $\{u_1, u_2, u_3\}$. In those circumstances we say that an octopus with tentacles $P_1, P_2$ and $P_3$ and head $Q_1 \cup Q_2 \cup Q_3$ is a *tripod* in $H$.

**(3.6)** *Let $G$ be a graph. Let $T$ be a triad or tripod in $G$ with ends $v_1, v_2$ and $v_3$. Let $M$ be a $T$-compatible matching, and let $P$ be an $M$-alternating path in $G \setminus v_1 \setminus v_2$ with one end in the head of $T$ and another end $w \notin V(T)$. Assume that the edge of $P$ incident with $w$ does not belong to $M$. Then there exists a triad or tripod $T' \subseteq T \cup P$ with ends $v_1, v_2$ and $w$ and a $T'$-compatible matching $M'$ such that $M$ is identical to $M'$ on $G \setminus V(P \cup T)$ and $(T \cup P) \setminus V(T')$ is $M'$-covered.*

*Proof.* If $T$ is a triad then the result follows immediately from (3.5). If $T$ is a tripod, then for $i \in \{1, 2, 3\}$ let $P_i, Q_i, u_i, v_i$ be as in the definition of tripod. Extend $M$ to $Q_1, Q_2$ and $Q_3$ in such a way that $Q_1 \cup Q_2 \cup Q_3 \setminus u_1$ is $M$-covered. Let $T''$ be the path octopus with tentacles $P_1$ and $P_2 \cup Q_1 \cup Q_2$. Extend $P$ along $Q_1 \cup Q_2 \cup Q_3$ to a path $P''$ so that $P''$ is an $M$-alternating path with ends $w$ and $u_1$. It remains to apply (3.5) to $P''$ and $T''$ . □

Let $Q$ be an even path with ends $u_1$ and $u_3$, let $u_2 = u_1$ and $u_4 = u_3$, and for $i = 1, 2, 3, 4$ let $P_i$ be an odd path with ends $u_i$ and $v_i$, disjoint from $Q$ except for $u_i$, and such that the paths $P_i$ are pairwise disjoint, except that $P_1$ and $P_2$ share a common end

$u_1 = u_2$ and $P_3$ and $P_4$ share a common end $u_3 = u_4$. In those circumstances we say that the octopus with head $Q$ and tentacles $P_1, P_2, P_3, P_4$ is a *quadropod*.

Now let $P_1, P_2, P_3, Q_1, Q_2, Q_3$ be as in the definition of tripod, except that $Q_2$ and $Q_3$ are allowed to intersect beyond the vertex $u_1$. Suppose there exists a perfect matching $M$ of $Q_2 \cup Q_3 \setminus u_1 \setminus u_2 \setminus u_3$ such that $Q_2$ and $Q_3$ are $M$-alternating and intersect transversally. Then we say that the octopus $\Omega$ with tentacles $P_1, P_2$ and $P_3$ and a head $Q_1 \cup Q_2 \cup Q_3$ is a *quasi-tripod* in $H$. Clearly every tripod is a quasi-tripod. It follows from the definition of transversal intersection that $Q_2 \cap Q_3$ consists of one or two paths, one of which contains the vertex $u_1$. By shortening both $Q_2$ and $Q_3$ and extending $P_1$ we may assume that one of the components of $Q_2 \cap Q_3$ has vertex-set $\{u_1\}$. If that is the only component of $Q_2 \cap Q_3$, then $\Omega$ is a tripod; otherwise $\Omega$ looks as depicted in Figure 6.

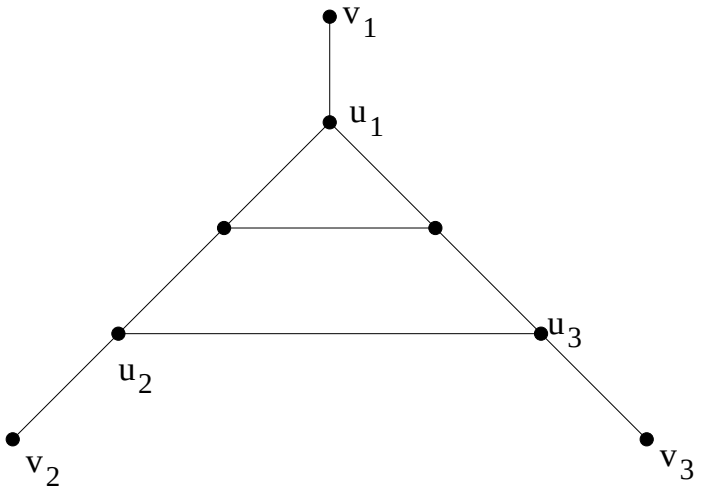



**Figure 6. A quasi-tripod.**

**(3.7)** *Let $G$ be a graph. Let $T$ be a triad or tripod in $G$ with ends $v_1, v_2$ and $v_3$. Let $M$ be a $T$-compatible matching, and let $P$ be an $M$-alternating path in $G \setminus \{v_1, v_2, v_3\}$ with one end in the head of $T$ and another end $w \notin V(T)$. Assume that the edge of $P$ incident with $w$ does not belong to $M$. Then there exists an octopus $T' \subseteq T \cup P$ and a $T'$-compatible matching $M'$, such that $M$ is identical to $M'$ on $G \setminus V(P \cup T)$, the graph $(T \cup P) \setminus V(T')$ is $M'$-covered and either $T'$ is a quasi-tripod with ends $v_i, v_j$ and $w$, for some distinct indices $i, j \in \{1, 2, 3\}$, or $T'$ is a quadropod with ends $v_1, v_2, v_3$ and $w$.*

*Proof.* We may assume that $G = T \cup P$ and that there do not exist a triad or tripod $T'$ with ends $v_1, v_2$ and $v_3$, a $T'$-compatible matching $M'$ and an $M'$-alternating path $P'$ in $G \setminus \{v_1, v_2, v_3\}$ with one end in the head of $T'$ and the other end $w$ such that $w \notin V(T')$, $(T \cup P) \setminus V(T' \cup P')$ is $M'$-covered and $P' \cup T'$ is a proper subgraph of $G$. We refer to this as the *minimality of* $G$.

Let the tentacles of $T$ be $P_1, P_2, P_3$, where $P_i$ has one end $v_i$, and let $u_i$ be the other end of $P_i$. If $T$ is a tripod, then let $Q_i$ be as in the definition of tripod, and otherwise let $Q_i$ be the null graph. We say that a vertex $v$ of $P_i$ is *inbound* if $P_i[v, u_i]$ is even and we say that $v$ is *outbound* otherwise.

Let $u_0 \in V(P \cap T)$ be chosen to minimize $P[w, u_0]$. If $T$ is a triad and $u_0$ is inbound, then $T \cup P[w, u_0]$ is a required quadropod. If $T$ is a tripod and $u_0 \in V(P_i)$ is inbound then by replacing $P_i[v_i, u_0]$ by $P[w, u_0]$ in $T$ we obtain a required quasi-tripod. If $T$ is a tripod and $u_0 \in V(Q_i)$, then we may assume from the symmetry that $Q_i[u_0, u_j]$ is even, in which case by replacing $P_j$ by $P[w, u_0]$ we obtain a required quasi-tripod.

Therefore for the rest of the proof we may assume that $u_0 \in V(P_1)$ and that $u_0$ is outbound. Let $r \in V(T) \cap V(P) - V(P_1)$ be chosen to minimize $P[w, r]$ and if no such $r$ exists let $r \neq w$ be the end of $P$. Apply (3.4) to $P_1$ and $P$ with $s_1 = v_1$, $t_1 = u_1$ and $s_2 = w$. Outcome (3.4)(1) does not hold by the minimality of $G$. If (3.4)(2) holds, then by considering the path guaranteed therein we obtain a desired quasi-tripod or quadropod. Thus we may assume (3.4)(3) holds, and hence $P_1$ intersects $P[w, r]$ transversally.

Let $v_0$ be such that $P[v_0, u_0]$ is a component of $P \cap P_1$, and let $u$ be such that $P[v_0, u]$ is a maximal path with no internal vertex or edge in $T$. If $u \in V(P_1)$, then by the definition of transversal intersection the vertices $v_1, v_0, u_0, u, u_1$ occur on $P_1$ in the order listed and $u$ is inbound. By considering $T \cup P[w, u]$ and deleting $P_3 \backslash u_3$ and the interior of $Q_3$ we obtain a required quasi-tripod. Thus we may assume that $u \notin V(P_1)$, and hence $u = r$. If $r$ is not outbound, then a similar argument gives a required quasi-tripod.

It follows that for the remainder of the proof we may assume that $r \in V(P_2)$, and that $r$ is outbound. Let $M_1$ be the unique perfect matching of $Q_1 \cup Q_2 \cup Q_3 \backslash u_1$, and let $M^+ = M \cup M_1$. We can extend $P$ along $Q_1 \cup Q_2 \cup Q_3$ to an $M^+$-alternating path $P^+$ so that $u_1$ is an end of $P^+$. Apply (3.2) to $P^+$ and $P_1[v_0, u_1]$ to produce an $M'$-alternating path

$P'$ with ends $w$ and $v_0$, where $M' = M^+ \triangle P_1[u_1, v_0]$. Let $T'$ be obtained from $T \cup P[v_0, r]$ by deleting the interiors of $P_2[r, u_2]$ and $Q_2$; then $T'$ is a triad with ends $v_1, v_2, v_3$. But now $T'$ and $P'$ contradict the minimality of $G$. □

## 4. EMBEDDINGS AND MAIN LEMMA

In this section we first formalize the notion of a matching minor by introducing the concept of an embedding, and show in (4.2) below that a graph $H$ has a matching minor isomorphic to a graph $G$ if and only if there is an embedding $H \hookrightarrow G$. Then we study the following question. Suppose that $\eta : H \hookrightarrow G$ is an embedding, $G$ is a brick, and $v_0 \in V(H)$ has degree two. Since bricks have no vertices of degree two, there is a subgraph of $G$ that "fixes" this violation of being a brick. What can we say about this subgraph? The answer leads to the notion of $v_0$-augmentation of $\eta$. We define this concept formally, and then prove two results about its existence. The second, (4.4), will be used when some graph obtained from $H$ by bisplitting a vertex is isomorphic to a matching minor of $G$; otherwise we will use (4.3), the first of these results. Finally, we classify all "minimal" $v_0$-augmentations into one of four types.

Let $T'$ be a tree, and let $T$ be obtained from $T'$ by subdividing every edge an odd number of times. Then $V(T') \subseteq V(T)$. The vertices of $T$ that belong to $V(T')$ will be called *old* and the vertices of $V(T) - V(T')$ will be called *new*. We say that $T$ is a *barycentric tree*. Please note that the partition into old and new vertices depends on $T'$ (there is an ambiguity concerning vertices of degree two). We shall assume that each barycentric tree has a fixed partition into new and old vertices. By a *branch* of a barycentric tree $T$ we mean a subpath of $T$ with ends old vertices and all internal vertices new.

We need to formalize the concept of matching minor. Let $H$ and $G$ be graphs. A *weak embedding of $H$ to $G$* is a mapping $\eta$ with domain $V(H) \cup E(H)$ such that for $v, v' \in V(H)$ and $e, e' \in E(H)$

(1) $\eta(v)$ is a barycentric subtree in $G$,

(2) if $v \neq v'$, then $\eta(v)$ and $\eta(v')$ are vertex-disjoint,

(3) $\eta(e)$ is an odd path with no internal vertex in any $\eta(v)$ or $\eta(e')$ for $e' \neq e$,

(4) if $e = u_1u_2$, then the ends of $\eta(e)$ can be denoted by $x_1, x_2$ in such a way that $x_i$ is an old vertex of $\eta(u_i)$, and

(5) $G \setminus \bigcup_{x \in V(H) \cup E(H)} V(\eta(x))$ has a perfect matching.

The next lemma will show that $H$ is isomorphic to a matching minor of $G$ if and only if there is a weak embedding of $H$ to $G$. Then we will show that such a weak embedding can be chosen with two additional properties. Thus we say that a weak embedding from $H$ to $G$ is an *embedding* if, in addition, it satisfies

(6) if $v$ has degree one then $\eta(v)$ has exactly one vertex,

(7) if $v \in V(H)$ has degree two and $e_1, e_2$ are its incident edges, then $\eta(v)$ is an even path with ends $x_1$, $x_2$, say, and $\eta(e_1)$, $\eta(e_2)$ both have length one, one has $x_1$ as its end and the other has $x_2$ as its end, and

(8) if $v$ has degree at least three and $x$ is an old vertex of $\eta(v)$ of degree $d$, then $x$ is an end of $\eta(e)$ for at least $3 - d$ distinct edges $e$.

For every subgraph $H'$ of $H$ define $\eta(H') = \bigcup_{x \in V(H) \cup E(H)} \eta(x)$. We denote the fact that $\eta$ is an embedding of $H$ into $G$ by writing $\eta : H \hookrightarrow G$.

Let $T \subseteq H$ be a barycentric tree, and let $(X, Y)$ be the unique partition of $V(T)$ into two independent sets with $X$ including all the old vertices. The vertices of $X$ will be called *protected* and the vertices of $Y$ will be called *exposed*.

**(4.1)** *Let $H$ and $G$ be graphs. There exists a weak embedding of $H$ to $G$ if and only if $H$ is isomorphic to a matching minor of $G$.*

*Proof.* If $\eta : H \hookrightarrow G$ then a graph isomorphic to $H$ can be obtained from the central subgraph $\eta(H)$ of $G$ by repeatedly bicontracting exposed vertices of $\eta(v)$ and internal vertices of $\eta(e)$ for $v \in V(H)$ and $e \in E(H)$. Thus $H$ is a matching minor of $G$.

To prove the converse we may assume that $H$ is a matching minor of $G$. Thus there exist graphs $H_1, H_2, \ldots, H_k$ such that $H_1 = H$, $H_k$ is a central subgraph of $G$, and for $i = 2, 3, \ldots, k$ the graph $H_{i-1}$ is obtained from $H_i$ by bicontracting a vertex. We define $\eta_k : H_k \hookrightarrow G$ by saying that if $v \in V(H_k)$, then $\eta_k(v)$ is the graph with vertex-set $\{v\}$, and if $e \in E(H_k)$, then $\eta_k(e)$ is the graph consisting of $e$ and its ends. It is clear that $\eta_k$ satisfies (1)-(5). We now construct a sequence of mappings satisfying (1)-(5).

Assuming that $\eta_i$ has been defined we define $\eta_{i-1}$ as follows. Let $v$ be the vertex of $H_i$ whose bicontraction produces $H_{i-1}$, let $x, y$ be the neighbors of $v$, and let $w$ be the the new vertex of $H_{i-1}$. For $z \in V(H_{i-1}) \cup E(H_{i-1}) - \{w\}$ let $\eta_{i-1}(z) = \eta_i(z)$, and let $\eta_{i-1}(w) = \eta_i(x) \cup \eta_i(y) \cup \eta_i(v) \cup \eta_i(xv) \cup \eta_i(yv)$. This completes the construction. It is clear that $\eta_1$ satisfies (1)-(5). □

We now show that if there is a weak embedding of $H$ to $G$, then there is an embedding of $H$ to $G$.

**(4.2)** *Let $H$ and $G$ be graphs. There exists an embedding of $H$ to $G$ if and only if $H$ is isomorphic to a matching minor of $G$.*

*Proof.* By (4.1) it suffices to show that if $\eta$ is a weak embedding of $H$ to $G$, then there exists an embedding of $H$ to $G$.

It is easy to modify $\eta$ so that it satisfies conditions (6) and (7). Thus we may choose a mapping $\eta$ with domain $V(H) \cup E(H)$ satisfying (1)-(7) such that the total number of old vertices in $\eta(v)$ over all vertices $v \in V(H)$ of degree at least three is minimum. We claim that $\eta$ satisfies (8) as well.

To prove that $\eta$ satisfies (8) let $v \in V(H)$ have degree at least three, let $x$ be an old vertex of $\eta(v)$, and let $d$ be the degree of $x$ in $\eta(v)$. If $d = 2$ and $x$ is not an end of $\eta(e)$ for any $e \in E(G)$, then we change the barycentric structure of $\eta(v)$ by declaring $x$ to be a new vertex. The new embedding thus obtained contradicts the minimality of $\eta$. If $d = 0$, then $x$ is the unique vertex of $\eta(v)$, and it is an end of $\eta(e)$ for all the (at least three) edges $e$ incident with $v$ by (4). Thus we may assume that $d = 1$. If $x$ is not an end of any $\eta(e)$, then we remove from $\eta(v)$ the vertex $x$ and all internal vertices of $Q$, where $Q$ is the unique subpath of $\eta(v)$ between $x$ and the nearest old vertex. Then set of vertices removed has a perfect matching, because $Q$ is even by the definition of barycentric subdivision, and hence the new embedding satisfies (5). Thus the new embedding contradicts the minimality of $\eta$. To complete the proof we may therefore suppose for a contradiction that $x$ is incident with $\eta(e)$ for exactly one $e \in E(H)$. By (4) one end of $e$ is $v$; let $u$ be the other end. If $u$ has degree at most two, then we define a new embedding by moving $x$ and the internal

vertices of $Q$ from $\eta(v)$ to $\eta(u)$, and changing $\eta(e)$ accordingly. If $u$ has degree at least three, then we move $x$ and all internal vertices of $Q$ from $\eta(v)$ to $\eta(e)$. In either case the new embedding contradicts the minimality of $\eta$. Thus $\eta$ satisfies (8), and hence it is an embedding $H \hookrightarrow G$, as desired. □

Let $T$ be an even subpath of a graph $H$, and let $T$ be regarded as a barycentric tree, with its ends designated as old and all internal vertices designated as new. Let us recall that the notions of protected and exposed were defined prior to (4.1). Let $P$ be a path with one end, say $v$, in the interior of $T$ and no other vertex in $T$. If $v$ is exposed, then let $Q$ be the null graph, and if $v$ is protected, then let $Q$ be a path with ends exposed vertices $q_1, q_2 \in V(T)$ and otherwise disjoint from $H \cup P$ such that $v$ lies on $T$ between $q_1$ and $q_2$. In those circumstances we say that *$Q$ is a cap for $P$ at $v$ with respect to $T$ and $H$*.

Let $\eta : H \hookrightarrow G$. For every edge $e = uv \in E(H)$ the path $\eta(e)$ is odd. Let $P_e$ denote its interior (that is, the path obtained by deleting the ends), and let $M_e$ be the unique perfect matching of $P_e$ (possibly $M_e = \emptyset$). We define $M(\eta)$ to be the union of $M_e$ over all $e \in E(H)$.

Now let $v_0 \in V(H)$ have degree two, and let $v_1, v_2$ be its neighbors. For $i = 1, 2$ let $E_i$ be the set of edges of $H$ incident with $v_i$, except for the edge $v_0v_i$, and let $E_1 \cap E_2 = \emptyset$. Let $M_1$ be a perfect matching of $G \backslash V(\eta(H))$, and let $M = M_1 \cup M(\eta)$. Let $P$ be an $M$-alternating path with one end $x \in V(\eta(v_0))$ and the other end $u$ in $\bigcup\{\eta(v) : v \in V(H) - \{v_0, v_1, v_2\}\}$ with the property that if $P$ intersects $\eta(e)$ for some $e \in E(H)$ not incident with $v_0$, $v_1$, or $v_2$, then $P$ and $\eta(e)$ intersect in a path and have a common end. Let $S$ denote the path $\eta(v_0) \cup \eta(v_0v_1) \cup \eta(v_0v_2)$; then $S$ is obtained from $\eta(v_0)$ by appending two edges, one at each end. Let $Q$ be an $M_1$-alternating cap for $P$ at $x$ with respect to $S$ and $\eta(H)$. We say that the pair $(P, Q)$ is a *$v_0$-augmentation of $\eta$*. It follows that $P$ and $Q$ have no internal vertices in $\bigcup_{v \in V(H)} \eta(v)$. We say that $x$ is the *origin* and $u$ is the *terminus* of $P$. See Figures 7-9 for examples.

Our first result about augmentations is the following.

**(4.3)** *Let $H$ be a graph on at least four vertices, let $v_0$ be a vertex of $H$ that has exactly two neighbors $v_1$ and $v_2$, and let $v_1$ and $v_2$ be not adjacent. Let $G$ be a brick and let*

*$\eta : H \hookrightarrow G$ be an embedding such that both $\eta(v_1)$ and $\eta(v_2)$ have exactly one vertex. Then there exists an embedding $\eta' : H \hookrightarrow G$ and a $v_0$-augmentation of $\eta'$.*

*Proof.* Define $E_1, E_2$ and $M$ as in the definition of $v_0$-augmentation. The path $\eta(v_0) \cup \eta(v_0v_1) \cup \eta(v_0v_2)$ is even and can therefore be regarded as path octopus, which we denote by $\Omega_1$. Let $\Omega_2$ be the octopus with the set of tentacles $\{\eta(e) : e \in E_1 \cup E_2\}$ and head $\eta(H \backslash v_0 \backslash v_1 \backslash v_2)$. The head of $\Omega_2$ is non-null, because $H$ has at least four vertices. We can convert $M$ to a matching $M^+$ so that $M^+$ is $\Omega_i$-compatible for $i = 1, 2$. We apply (2.3) to the frame $(\{\Omega_1, \Omega_2\}, V(\eta(v_1)) \cup V(\eta(v_2)))$ and denote the resulting path by $R$. Let $R$ have ends $r_1 \in V(\Omega_1)$ and $r_2 \in V(\Omega_2)$ and let $e \in E(R)$ be such that each of the components $R_i = R[s_i, r_i]$ of $R \backslash e$ intersects only one of the octopi $\Omega_1$ and $\Omega_2$.

By (3.5) we may assume, by changing $M^+$, $R_1$, and $\eta(v_0)$, that there exists an $M^+$-alternating path $P_1$ with ends $p_1 \in V(\eta(v_0))$ and $s_1$, and an $M^+$-alternating cap $Q_1$ for $P_1$ at $p_1$ with respect to $\Omega_1$ and $\eta(H)$ such that $P_1 \cup Q_1 \subseteq R_1$. We may also assume that $r_2$ is the only vertex of $R$ in the head of $\Omega_2$. If $r_2 \in \eta(v)$ for some $v \in V(H)$, then let $R'_2$ be the null graph, and if $r_2 \in \eta(e)$ for some $e \in E(H)$, then let $R'_2$ be an $M^+$-alternating subpath of $\eta(e)$ with one end $r_2$ and the other in $\eta(v)$ for some $v \in V(H)$. Then $(P_1 \cup R_2 \cup R'_2, Q_1)$ is a desired $v_0$-augmentation of $\eta$. □

In the next section we will need the following lemma.

**(4.4)** *Let $H$ be a graph, and let $v$ be a vertex of $H$ of degree at least four, let $G$ be a brick, and let $\eta : H \hookrightarrow G$ be such that $\eta(v)$ has at least two vertices. Then either*

*(1) there exists a graph $H_1$ obtained from $H$ by bisplitting $v$, an embedding $\eta_1 : H_1 \hookrightarrow G$ and a $v_0$-augmentation of $\eta_1$, where $v_0$ is the new inner vertex of $H_1$, or*

*(2) there exists an embedding $\eta_2 : H \hookrightarrow G$, a path $P$ with ends $p_1$ and $p_2$ in the interiors of different branches, say $B_1$ and $B_2$, of $\eta_2(v)$ and otherwise disjoint from $\eta_2(H)$ and for $i = 1, 2$ there exists a cap $Q_i$ for $P$ at $p_i$ with respect to $B_i$ and $\eta_2(H)$ such that $Q_1$ and $Q_2$ are disjoint.*

*Proof.* Denote the branches of $\eta(v)$ by $B_1, B_2, \ldots, B_n$. They can be considered as octopi, which we denote by $\Omega_1, \Omega_2, \ldots, \Omega_n$, respectively. Let $\Omega_0$ be the octopus with the set of

tentacles $\{\eta(e) : e \text{ is incident to } v\}$ and head $\eta(H\backslash v)$, let $X$ be the set of old vertices of $\eta(v)$, and let $\mathcal{F} = \{\Omega_0, \Omega_1, \Omega_2, \ldots, \Omega_n\}$. We can extend a perfect matching of $G \setminus \eta(H)$ to a matching $M$ so that $M$ is $\Omega$-compatible for every $\Omega \in \mathcal{F}$. Clearly $|X| = n+1$. Therefore $(\mathcal{F}, X)$ is a frame. We apply (2.3) to it and denote the resulting path by $R$. Furthermore, there is an edge $e \in E(R)$ such that each of the components $R_i = R[s_i, r_i]$ of $R\backslash e$ intersects only one of the octopi of $\mathcal{F}$.

If for some $i \in \{1, 2\}$ the path $R_i$ intersects $\Omega_j$ for $j \geq 1$ we may assume, by changing $M$, and $\Omega_j$, that there exists an $M$-alternating path $P_i$ with ends $p_i \in V(B_j)$ and $s_i$, and an $M$-alternating cap $Q_i$ for $P_i$ at $p_i$ with respect to $B_j$ and $\eta(H)$ such that $P_i \cup Q_i \subseteq R_i \cup B_j$. If this happens for both $R_1$ and $R_2$ define $P = P_1 \cup P_2 + e$ and outcome (2) holds.

Therefore we may assume that $R_2$ intersects $\Omega_0$ and $R_1$ intersects $\Omega_j$ for some $j \geq 1$, and furthermore that $r_2$ is the only vertex of $R$ in the head of $\Omega_0$. If $r_2 \in \eta(v)$ for some $v \in V(H)$, then let $R'_2$ be the null graph, and if $r_2 \in \eta(e)$ for some $e \in E(H)$, then let $R'_2$ be an $M$-alternating subpath of $\eta(e)$ with one end $r_2$ and the other in $\eta(v)$ for some $v \in V(H)$.

Let $T_1$ and $T_2$ be the two components of the graph obtained from $\eta(v)$ by removing the internal vertices of $B_j$. Let $H_1$ be obtained from $H$ by splitting $v$ into new outer vertices $v_1$ and $v_2$ and new inner vertex $v_0$ in such a way that $v_i$ is adjacent to a neighbor $u$ of $v$ in $H$ if $\eta(uv_i)$ has an end in $T_i$. Let $\eta_1(v_i) = T_i$, let $\eta_1(v_0)$ be $B_1$ minus its ends, let $\eta_1(v_1v_0)$ and $\eta_1(v_2v_0)$ be the two end-edges of $B_1$ and let $\eta_1(x) = \eta(x)$ for all other $x \in V(H_1) \cup E(H_1)$. Then $(P_1 \cup R'_2 \cup \{e\}, Q_1)$ is a $v_0$-augmentation of $\eta_1$ and outcome (1) holds. □

Let $H$ and $G$ be graphs, let $\eta : H \hookrightarrow G$, let $v_0$ be a vertex of $H$ of degree two, and let $(P, Q)$ be a $v_0$-augmentation of $\eta$. We say that $\eta$ is *minimal* if there exists no embedding $\eta' : H \hookrightarrow G$ and a $v_0$-augmentation $(P', Q')$ of $\eta'$ such that $\eta'(H) \cup P' \cup Q'$ is a proper subgraph of $\eta(H) \cup P \cup Q$. In applications we may assume that our $v_0$-augmentations are minimal. The next lemma will classify minimal augmentations into four types, which we now introduce.

Let $\eta : H \hookrightarrow G$, let $v_0 \in V(H)$ have degree two, let $v_1, v_2 \in V(H)$ be its neighbors,

and let $E_1, E_2$ be as in the definition of $v_0$-augmentation. Let $i \in \{1,2\}$ and $e \in E_i$. Let $x_e$ be the end of $\eta(e)$ that belongs to $V(\eta(v_i))$. We say that an internal vertex $x \in V(\eta(e))$ is an *inbound vertex* if it is at even distance from $x_e$ in $\eta(e)$, and otherwise we say that it is an *outbound vertex*.

Let $M$ be a matching containing $M(\eta)$, let $P$ be an $M$-alternating path with ends $x_0$ and $x_5$, and let the vertices $x_0, x_1, x_2, x_3, x_4, x_5$ appear on $P$ in the order listed. Assume that $P[x_1, x_2]$ and $P[x_3, x_4]$ are subpaths of $\eta(e)$, and that otherwise $P$ is disjoint from $\bigcup\{\eta(e) : e \in E_1 \cup E_2\}$. Assume also that $x_1$ is an inbound vertex of $\eta(e)$, that $x_2$ and $x_3$ are outbound, and that either $x_2 = x_3 = x_4$, or $x_1, x_2, x_4, x_3, x_e$ are pairwise distinct and occur on $\eta(e)$ in the order listed. In those circumstances we say that $P$ intersects $\eta(e)$ *regularly from $x_0$ to $x_5$*.

Let $(P, Q)$ be a $v_0$-augmentation of $\eta$ and let $P$ have ends $a$ and $b$ where $a \in V(\eta(v_0))$. We say that $(P, Q)$ is of type A if whenever $P$ intersects $\eta(e)$ for some $e \in E_1 \cup E_2$, then $P$ and $\eta(e)$ intersect in a path whose one end is a common end of $P$ and $\eta(e)$. Thus $P$ intersects at most one $\eta(e)$, because the common end must be $b$, and $b$ does not belong to $\eta(v_1) \cup \eta(v_2)$. See Figure 7.

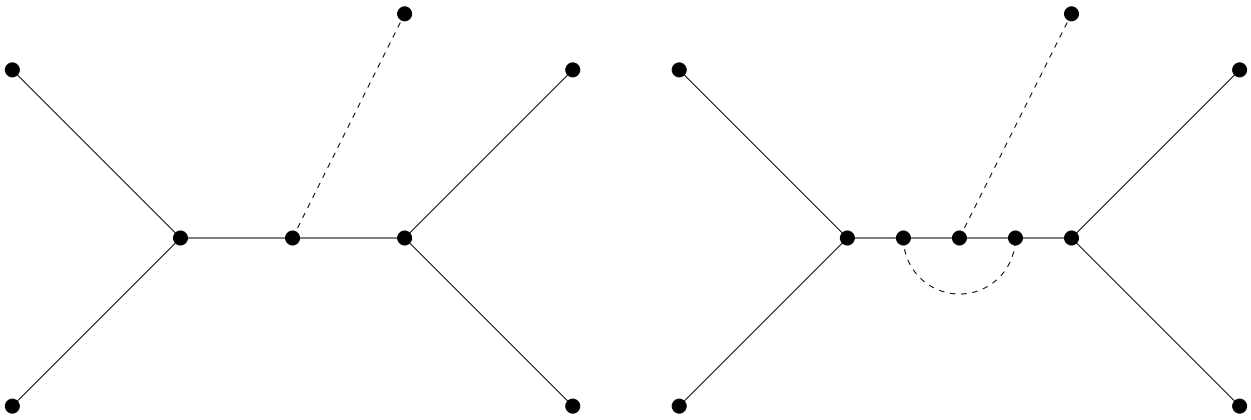


**Figure 7. Augmentations of type A.**

We say that $(P, Q)$ is of type B if there exist a vertex $x \in V(P)$, an index $i \in \{1,2\}$, and an edge $e \in E_i$ such that the vertex $v_i$ has degree at most three, the path $P[a, x]$ intersects $\eta(e)$ regularly from $a$ to $x$, and if $P[x, b] \backslash x$ intersects $\eta(e')$ for some $e' \in E(H)$,

then $P[x,b]\backslash x$ and $\eta(e')$ intersect in a path and have a common end. Moreover, if $e = e'$, then we require that $P[a,x] \cap \eta(e)$ be a path. We say that $(P,Q)$ *crosses* $\eta(e)$. See Figure 8.

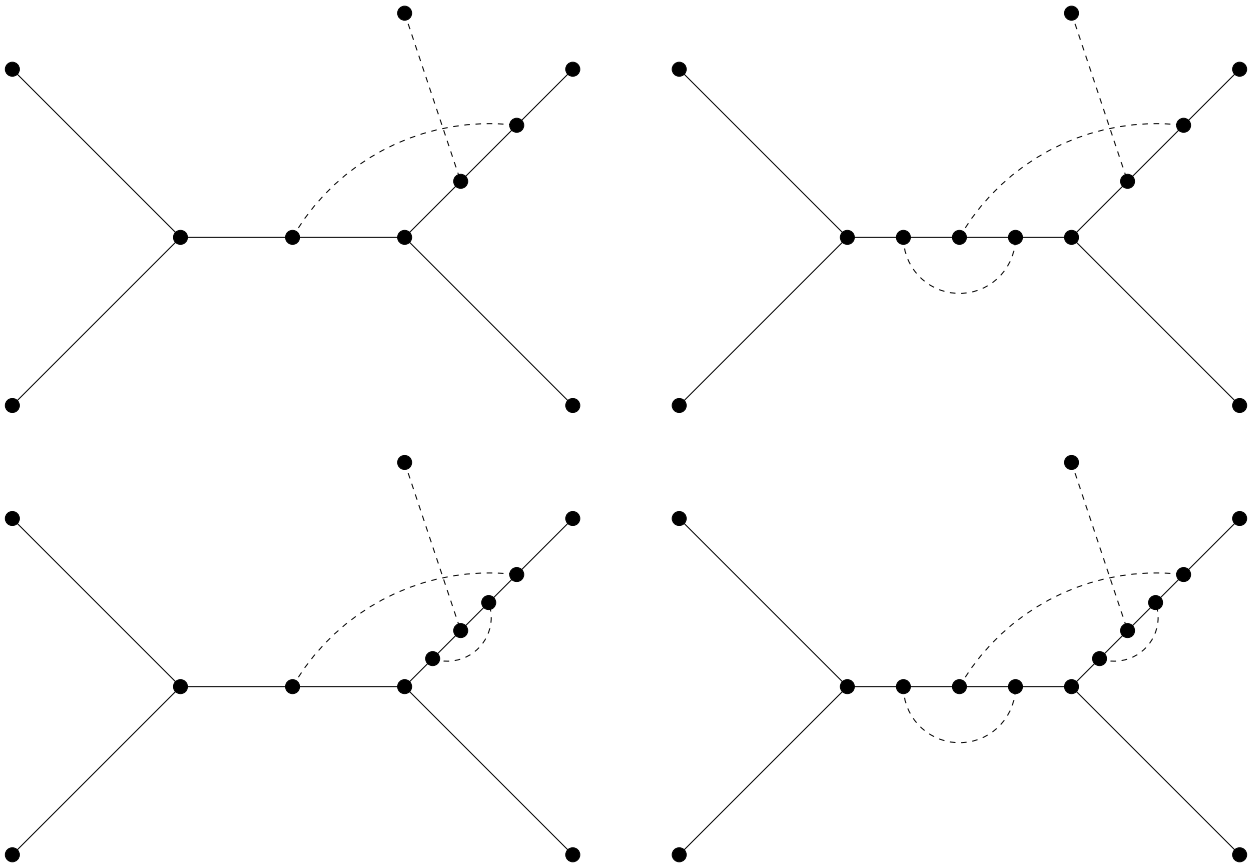


**Figure 8. Augmentations of type B.**

We say that $(P,Q)$ is of type C if there exist vertices $x_1, x_2 \in V(P)$ such that $a, x_1, x_2, b$ occur on $P$ in the order listed, and there exist distinct edges $e_1, e_2$, one in $E_1$ and the other in $E_2$, such that the end of $e_1$ in $\{v_1, v_2\}$ has degree at most three, $P[a,x_1]$ intersects $\eta(e_1)$ regularly from $a$ to $x_1$, $P[x_1,x_2]$ has no internal vertices in $\eta(H)$ and $x_2$ is an inbound vertex of $\eta(e_2)$. We say that $(P,Q)$ *crosses* $\eta(e_1)$. See Figure 9.

We say that $(P,Q)$ is of type D if for some $i \in \{1,2\}$ and some $e \in E_i$ the vertex $v_i$ has degree at least four and there exists an inbound vertex $x$ of $\eta(e)$ such that $x \in V(P)$ and $P[a,x]$ has no internal vertex in $\eta(H)$.

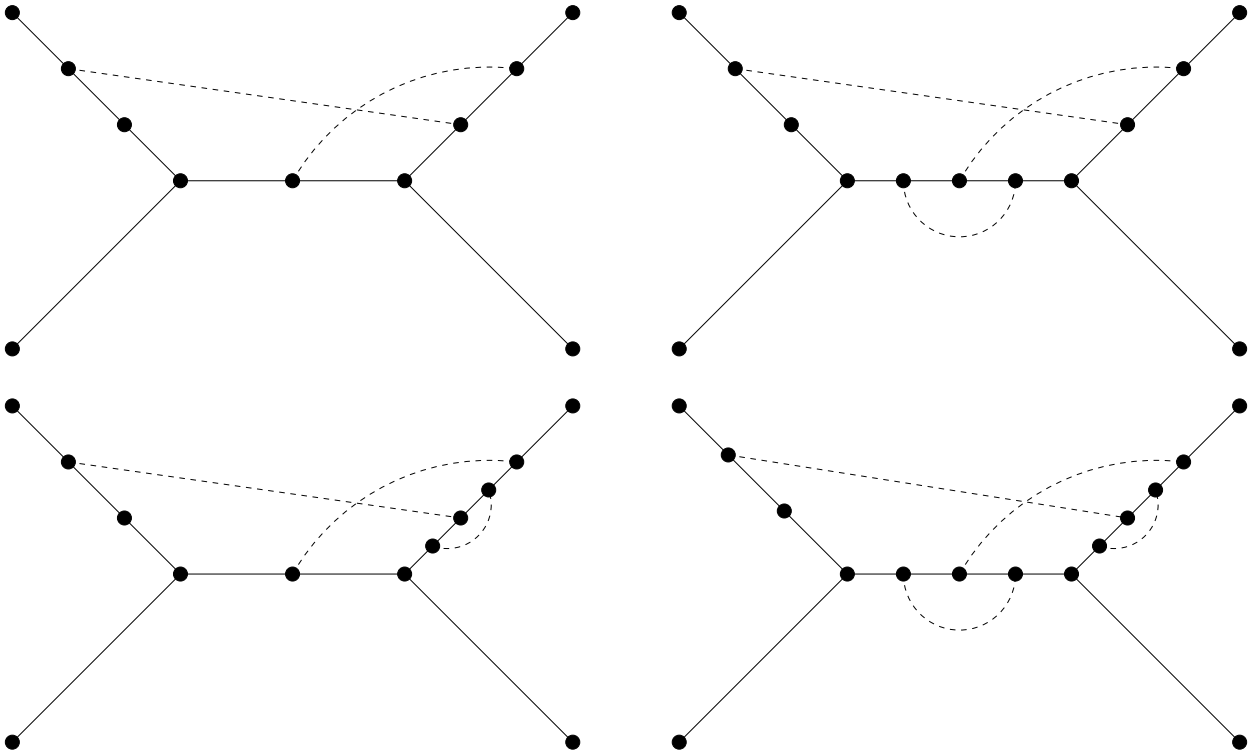


**Figure 9. Augmentations of type C.**

The following classification of minimal $v_0$-augmentations is the third main result of this section.

**(4.5)** *Let $H$ and $G$ be graphs, and let $\eta : H \hookrightarrow G$. Let $v_0 \in V(H)$ have degree two, and let $v_1, v_2$ be its neighbors. Assume that $v_1$ is not adjacent to $v_2$. Then every minimal $v_0$-augmentation of $\eta$ is of type A, B, C, or D.*

*Proof.* Let $(P, Q)$ be a minimal $v_0$-augmentation of $\eta$, let $x_0$ be the end of $P$ in $\eta(v_0)$, and let $b$ be the other end of $P$. We wish to think of $P$ as being directed away from $x_0$; thus language such as "the first vertex of $P$ in a set $Z$" will mean the vertex of $V(P) \cap Z$ that is closest to $x_0$ on $P$. Let $E_1$ and $E_2$ be as in the definition of $v_0$-augmentation.

Let us assume for a moment that $P$ includes an internal vertex of some $\eta(e)$, where $e \in E(H)$ is not incident with $v_0$, $v_1$, or $v_2$. Let $z$ be the first such vertex on $P$. The vertex $z$ divides $\eta(e)$ into two subpaths, one even and one odd. Let $R$ be the even one. Then $(P[x_0, z] \cup R, Q)$ is a $v_0$-augmentation, and hence the minimality of $(P, Q)$ implies

that $R = P[z, b]$. If $e \in E_1 \cup E_2$ and $z$ is an outbound vertex, then the same conclusion holds. This will be later referred to as the *confluence property of $P$*.

If $P$ includes an internal vertex of $\eta(e_1)$ for no $e_1 \in E_1 \cup E_2$, then $(P, Q)$ is of type $A$. Thus we may assume that $P$ includes such a vertex, and let $x_1$ be the first such vertex on $P$. From the symmetry we may assume that $e_1 = v_1v_3 \in E_1$. If $x_1$ is an outbound vertex, then the confluence property of $P$ implies that $(P, Q)$ is of type A. Thus we may assume that $x_1$ is inbound. If $v_1$ has degree at least four, then $(P, Q)$ is of type $D$, and so we may assume that $v_1$ has degree at most three. It follows from axiom (7) in the definition of an embedding that $v_1$ has degree exactly three.

Let $x_2$ be the first vertex on $P$ that belongs to $\eta(z)$ for some $z \in V(H) \cup E(H)$ not equal, incident or adjacent to $v_0$ and not equal to $e_1$. Then $x_1$ lies on $P$ between $x_0$ and $x_2$. Let $P_1 = \eta(e_1)$. By (3.4) applied to $P_1$, $P_2 = P$, $r = x_2$, $s_2 = x_0$, $t_2 = b$ and the ends of $P_1$ numbered so that $s_1 \in V(\eta(v_0))$ and $t_1 \in V(\eta(v_3))$ we deduce that (1), (2), or (3) of (3.4) holds. But (1) does not hold by the minimality of $(P, Q)$, and if (2) holds, then $(R, Q)$ is a $v_0$-augmentation of type A or B. Thus we may assume that (3) of (3.4) holds. Since $x_1$ is an inbound vertex, this implies that either there exist vertices $y_1, y_2 \in V(P_1)$, such that $y_1$ and $y_2$ are outbound, $P[x_1, y_1] \subseteq P_1$, $x_1 \in P_1[y_1, y_2]$ and $P[y_1, y_2]$ has no internal vertices in $\eta(H)$, or $P[x_0, x_2] \setminus x_2$ intersects $\eta(e_1)$ regularly from $x_0$ to $x_2$. In the former case $(P[x_0, y_2] \cup P_1[y_2, t_1], Q)$ is a $v_0$-augmentation of $\eta$ of type B, and hence we may assume that the latter case holds. Thus $P[x_0, x_2] \setminus x_2$ intersects $\eta(e_1)$ regularly from $x_0$ to $x_2$, and if $x_2 = t_2$, then $P[x_0, x_2] \setminus x_2$ intersects $\eta(e_1)$ in a path.

If $x_2 \in \bigcup\{V(\eta(v)) : v \in V(H) - \{v_0, v_1, v_2\}\}$, then $(P, Q)$ is of type B. Therefore we may assume that $x_2 \in V(\eta(e_2))$ for some $e_2 \in E(H \backslash v_0) - \{e_1\}$. By the confluence property of $P$ we may assume that $e_2 \in E_1 \cup E_2$ and that $x_2$ is inbound, for otherwise $(P, Q)$ is of type B.

If $e_2 \in E_2$, then $(P, Q)$ is of type $C$, and the lemma holds. Thus we may assume that $e_2 \in E_1 - \{e_1\}$. Let $y$ be such that $\eta(e_2)[x_2, y]$ is a component of $\eta(e_2) \cap P$. For simplicity of notation assume that $Q$ is empty. The argument in the other case is similar. As $v_1$ has degree three, axiom (8) in the definition of an embedding implies that the tree $\eta(v_1)$ consists a single vertex, say $u_1$. Since $x_2$ is inbound it follows that $y$ lies between $u_1$ and

$x_2$ in $\eta(e_2)$. Let $C$ be the cycle $P[x_0, y] \cup \eta(e_2)[y, u_1] \cup S$, where $S = \eta(v_0)[x_0, u_1]$. The subgraph of $G$ with edge-set $E(P) \triangle E(C)$ includes a path with ends $x_0$ and $b$, say $P'$. Let $f$ be the edge of $P[x_0, x_1]$ incident to $x_1$. We define a new embedding $\eta' : H \hookrightarrow G$ by $\eta'(e_1) = \eta(e_1)[x_1, t_1]$, $\eta'(e_2) = P[x_1, x_2] \cup \eta(e_2)[x_2, z]$ (where $z \neq u_1$ is the other end of $\eta(e_2)$), $\eta'(v_1)$ is the graph with vertex-set $\{x_1\}$, we define $\eta'(v_0v_1)$ to be the path with edge-set $\{f\}$, we define $\eta'(v_0)$ to be the path obtained from $\eta(v_0)$ by replacing $\eta(v_0)[x_0, u_1]$ by $P[x_0, x_1] \backslash x_1$, and we define $\eta'(x) = \eta(x)$ for all other $x \in V(H) \cup E(H)$. It follows that $(P', Q)$ is a $v_0$-augmentation of $\eta'$, contrary to the minimality of $(P, Q)$, because $P' \cup Q \cup \eta'(H)$ does not include the edge of $\eta(e_2)[y, x_2]$ incident with $x_2$. □

## 5. DISPOSITION OF BISPLITS

The purpose of this section is to prove (1.11) under the additional hypothesis that a graph, say $H'$, obtained from $H$ by bisplitting some vertex is isomorphic to a matching minor of $G$. If that is the case we apply (4.4) and (4.5). We handle the four possible outcomes of (4.5) separately.

**(5.1)** *Let $H$ and $G$ be graphs, where $H$ has minimum degree at least three. Let $H'$ be obtained from $H$ by bisplitting a vertex $v$, and let $v_0$ be the new inner vertex. Let $\eta : H' \hookrightarrow G$, and assume that there exists a $v_0$-augmentation of $\eta$ of type A. Then a linear extension of $H$ is isomorphic to a matching minor of $G$.*

*Proof.* Let $v_1$ and $v_2$ be the new outer vertices of $H'$, let $(P, Q)$ be a $v_0$-augmentation of $\eta$ of type A, and let $a$ and $b$ be the ends of $P$, where $a \in V(\eta(v_0))$. Let $b \in \eta(u)$, where $u \in V(H) - \{v_0, v_1, v_2\}$. Let us assume first that $b$ is protected. If $Q$ is null, then $H' + (v_0, u)$ is isomorphic to a matching minor of $G$, and otherwise (by ignoring $Q$ and bicontracting its ends) we see that that $H + (v, u)$ is isomorphic to a matching minor of $G$ and is a linear extension of $H$ unless $vu \in E(H)$. If $vu \in E(H)$ we assume without loss of generality that $uv_1 \in E(H')$. Then $\eta(H' \setminus uv_1) \cup P \cup Q$ is isomorphic to a bisubdivision of a linear extension of $H$.

Now let us assume that $b$ is exposed. Let $T_1, T_2$ be the two components of $\eta(u)\backslash b$. For each neighbor $w$ of $u$ in $H$ the path $\eta(uw)$ has exactly one end in $\eta(u)$; that end is an old vertex by axiom (4) in the definition of an embedding, and hence belongs to either $T_1$ or $T_2$. For $i = 1, 2$ let $N_i$ be the set of all neighbors $w$ of $u$ such that the end of $\eta(uw)$ in $\eta(u)$ belongs to $T_i$. Let $H_1$ be obtained from $H$ by bisplitting $u$ so that one of the new outer vertices is adjacent to every vertex of $N_1$, and the other new outer vertex is adjacent to every vertex of $N_2$. (Here we use that $u$ has degree at least three.) Let $u_0$ be the new inner vertex of $H_1$. Let $H'_1$ be defined similarly, but starting from $H'$ rather than $H$, and let the new inner vertex be also $u_0$. If $Q$ is null, then $H'_1 + (v_0, u_0)$ is isomorphic to a matching minor of $G$; otherwise $H_1 + (v, u_0)$ is isomorphic to a matching minor of $G$, as desired. □

**(5.2)** *Let $H$ and $G$ be graphs, let $\eta : H \hookrightarrow G$ be an embedding, let $v_0$ be vertex of $H$ of degree two, and let $v_1$ be a neighbor of $v_0$ of degree three with neighbors $v_0, v'_1, v''_1$. Let $(P, Q)$ be a $v_0$-augmentation of $\eta$ of type B or C that crosses $\eta(v_1v'_1)$. Then there exists an embedding $\eta' : H \hookrightarrow G$ and a $v_0$-augmentation $(P', Q')$ of $\eta'$ of the same type as $(P, Q)$ that crosses $\eta'(v_1v''_1)$ such that $\eta'(H) \cup P' \cup Q' \subseteq \eta(H) \cup P \cup Q$ and $P$ and $P'$ have the same terminus.*

*Proof.* We first define $\eta'$. Let $x_0$ be the end of $P$ in $\eta(v_0)$, let $x_6$ be the other end of $P$, let $x_5 \in V(P)$, and let $x_0, x_1, \ldots, x_5$ be as in the definition of regular intersection, witnessing that $P[x_0, x_5]$ intersects $\eta(v_1v'_1)$ regularly from $x_0$ to $x_5$. We define $\eta'(v_1) = x_1$, we define $\eta'(v_1v'_1)$ to be the subpath of $\eta(v_1v'_1)$ with one end $x_1$ and the other end in $\eta(v'_1)$, we define $\eta'(v_1v''_1)$ to be the union of the complementary subpath of $\eta(v_1v'_1)$ and $\eta(v_1v''_1)$, we define $\eta'(v_0)$ to be a suitable subgraph of $\eta(v_0) \cup P \cup Q$, define $\eta'(v_0v_1)$ to be the edge of $P[x_0, x_1]$ incident with $x_1$, and we define $\eta'(x) = \eta(x)$ for all other $x \in V(H) \cup E(H)$. Then $\eta' : H \hookrightarrow G$.

It is now easy to find subpaths $Q'$ and $P''$ of $\eta(v_0) \cup \eta(v_0v_1) \cup \eta(v_1v'_1) \cup P \cup Q$ such that $(P'' \cup P[x_4, x_6], Q')$ is the desired $v_0$-augmentation of $\eta'$. □

**(5.3)** *Let $H$ and $G$ be graphs, where $H$ has minimum degree at least three. Let $H'$ be obtained from $H$ by bisplitting a vertex $v$, and let $v_0$ be the new inner vertex. Let $\eta : H' \hookrightarrow G$, and assume that there exists a $v_0$-augmentation of $\eta$ of type B. Then a linear extension of $H$ is isomorphic to a matching minor of $G$.*

*Proof.* Let $v_1$ and $v_2$ be the new outer vertices of $H'$. Let $(P, Q)$ be a $v_0$-augmentation of $\eta$ of type B, let $x_0, x_6$ be the ends of $P$, where $x_0 \in V(\eta(v_0))$ and $x_6 \in V(\eta(u))$, and let $P$ cross $e_1 = v_1v_1'$, where $v_1' \neq v_0$ is a neighbor of $v_1$ in $H'$. Let $x_5 \in V(P)$ be such that $P[x_0, x_5]$ intersects $\eta(e_1)$ regularly from $x_0$ to $x_5$, and let the vertices $x_0, x_1, x_2, x_3, x_4, x_5$ be as in the definition of regular intersection. Notice that $v_1$ has degree three; thus $\eta(v_1)$ consists of a unique vertex by condition (8) in the definition of embedding. Let $v_1''$ be the third neighbor of $v_1$. By (5.2) we may assume that $u \neq v_1'$.

Assume first that $x_2, x_3, x_4$ are pairwise distinct. The path $P[x_4, x_6]$ proves that a linear extension of $H$ is isomorphic to a matching minor of $G$, unless $x_6$ is a protected vertex of $\eta(u)$ and $u$ is adjacent to $v$ in $H$. Let $i \in \{1, 2\}$ be such that $u$ is adjacent to $v_i$ in $H'$. Consider the graph obtained from $\eta(H) \cup P[x_4, x_0]$ by deleting the interior of $\eta(v_iu)$; the path $P[x_2, x_3]$ proves that the linear extension $H'' + (v_0', v_1')$ of $H$ is isomorphic to a matching minor of $G$, where $H''$ is obtained from $H$ by bisplitting of the vertex $v$ so that one of the new outer vertices is adjacent to $v_1'$ and $u$, the other outer vertex is adjacent to all other neighbors of $v$ and $v_0'$ is the new inner vertex.

Thus we may assume that $x_2 = x_3 = x_4$. Again the path $P[x_4, x_6]$ proves that a linear extension of $H$ is isomorphic to a matching minor of $G$, unless $x_6$ is a protected vertex of $\eta(u)$ and $u$ is adjacent to $v_1'$ in $H$. Thus we may assume that $x_6$ is a protected vertex of $\eta(u)$ and $u$ is adjacent to $v_1'$ in $H$. If $v_1'$ has degree at least four, then let $H''$ be obtained from $H'$ by bisplitting $v_1'$ in such a way that one of the new vertices is adjacent to $v_1$ and $u$, and let $z$ be the new vertex. Then $H'' + (v_0, z)$ is a linear extension of $H$ and is clearly isomorphic to a matching minor of $G$. If $v_1'$ has degree three we replace $\eta(v_1'u)$ by $P[x_4, x_6]$ and notice that $(P, Q)$ can be easily converted to a $v_0$-augmentation $(P', Q')$ of type A of the embedding thus obtained. (Notice that the terminus of $P'$ does not belong to $\eta(v_2)$, because $H'$ is obtained from $H$ by bisplitting $v$.) Hence the theorem follows from (5.1). □

For the next lemma we need the following generalization of $v_0$-augmentations. Let $v_0 \in V(H)$ have degree two, and let $v_1, v_2$ be its neighbors. For $i = 1, 2$ let $E_i$ be the set of edges of $H$ incident with $v_i$, except for the edge $v_0v_i$, and let $E_1 \cap E_2 = \emptyset$. Let $R$ be the interior of $\eta(v_0)$, let $M_1$ be a perfect matching of $G \backslash V(\eta(H))$, let $x \in V(R)$, let $M_2$ be a perfect matching of $R \backslash x$, and let $M = M_1 \cup M_2 \cup M(\eta)$. Let $P$ be an $M$-alternating path with one end $x$ and the other end $u$ in $\bigcup\{\eta(v) : v \in V(H) - \{v_0, v_1, v_2\}\}$. We say that $P$ is a *weak $v_0$-augmentation of $\eta$*. It follows that $P$ has no internal vertex in $\bigcup_{v \in V(H) - \{v_0\}} \eta(v)$. This is indeed a generalization of $v_0$-augmentation. For let $(P, Q)$ be a $v_0$-augmentation of $\eta$. If $Q$ is null, then $P$ is a weak $v_0$-augmentation of $\eta$, and otherwise $Q \cup S \cup P$ is a weak $v_0$-augmentation of $\eta$, where $S$ is a subpath of $\eta(v_0)$ with one end the end of $P$ and the other end an end of $Q$.

**(5.4)** *Let $H, G$ be graphs, let $\eta : H \hookrightarrow G$ be an embedding, let $v_0$ be a vertex of $H$ of degree two belonging to no triangle of $H$, and let $R$ be a weak $v_0$-augmentation of $\eta$. Then there exist an embedding $\eta' : H \hookrightarrow G$ and a $v_0$-augmentation $(P, Q)$ of $\eta'$ such that $P \cup Q \cup \eta'(H) \subseteq R \cup \eta(H)$.*

*Proof.* We may assume that $R$ is minimal in the sense that there does not exist an embedding $\eta' : H \hookrightarrow G$ and a weak $v_0$-augmentation $R'$ of $\eta'$ such that $R' \cup \eta'(H)$ is a proper subgraph of $R \cup \eta(H)$. It follows that $R$ has the confluence property introduced in the proof of (4.5). Let $v_1, v_2$ be the neighbors of $v_0$, and let $E$ be the set of all edges of $H$ incident with a neighbor of $v_0$, but not with $v_0$ itself.

Let $a, b$ be the ends of $R$, where $a \in V(\eta(v_0))$ and let $z_1, z_2$ be the ends of $\eta(v_0)$. Assume first that $R$ has a vertex $x$ such that $R[a, x]$ includes an internal vertex of $\eta(e)$ for no edge $e \in E$, and $R[x, b]$ includes no vertex of $\eta(v_0)$. Let $\Omega$ be a path octopus with head $a$ and graph $\eta(v_0)$. We apply (3.5) to $\Omega$ and $R[a, x]$ to produce a path octopus $\Omega'$ with head $z$ and ends $z_1$ and $z_2$ and a path $R'$ with ends $z$ and $x$. Define $\eta'$ so that $\eta'(v_0)$ is the graph of $\Omega'$ and otherwise $\eta'$ coincides with $\eta$. Let $P$ be a maximal subpath of $R' \cup R[x, b]$ with no internal vertex in $\eta'(v_0)$ containing $b$ and let $Q$ be a maximal non-empty subpath of $R' \backslash V(P)$ with no internal vertex in $\eta'(v_0)$ if such a path exists, and otherwise let $Q$ be the null graph. It is easy to check that $(P, Q)$ is a $v_0$-augmentation of $\eta'$.

Thus we may assume that the assumption of the previous paragraph does not hold. Thus there exists an edge $e \in E$ such that when following $R$ starting from $a$ at some point we encounter an internal vertex of $\eta(e)$, and later an internal vertex of $\eta(v_0)$, say $t$. Let $T$ be the component of $R \cap \eta(v_0)$ containing $t$. Let the ends of $e$ be $v_1$ and $v_1'$, where $v_1$ is adjacent to $v_0$, and let the ends of $\eta(e)$ be $u_1$ and $u_1'$, where $u_1$ belongs to $\eta(v_1)$ and $u_1'$ belongs to $\eta(v_1')$. Let $S$ be the component of $R[a,t] \cap \eta(e)$ that is closest to $u_1'$ on $\eta(e)$. Let $t_1, t_2$ be the ends of $T$, where $a, t_1, t_2, b$ occur on $R$ in the order listed, and let $s_1, s_2$ be the ends of $S$ chosen similarly. If $t_2$ lies at an even distance from $a$ on $\eta(v_0)$, then $R[t_2, b]$ is a weak $v_0$-augmentation of $\eta$, contrary to the minimality of $R$. Thus $t_1$ lies at an even distance from $a$ on $\eta(v_0)$. It follows from the confluence property that $s_1$ is an inbound vertex of $\eta(e)$ (that is, its distance from $u_1$ on $\eta(e)$ is even). Thus $s_2$ is an outbound vertex, and hence $R[t_1, s_2] \cup \eta(e)[s_2, u_1']$ is a weak $v_0$-augmentation of $\eta$, contrary to the minimality of $R$. □

Let $H$ and $G$ be graphs, let $H'$ be obtained from $H$ by bisplitting a vertex $v$, and let $v_0$ be the new inner vertex. Let $\eta : H' \hookrightarrow G$, and let $(P, Q)$ be a $v_0$-augmentation of $\eta$. We say that $(P, Q)$ is *strongly minimal* if there exists no graph $H''$ obtained from $H$ by bisplitting $v$, an embedding $\eta'' : H'' \hookrightarrow G$ and (letting $v_0''$ denote the new inner vertex of $H_0''$) a $v_0''$-augmentation $(P'', Q'')$ of $\eta''$ such that $\eta''(H'') \cup P'' \cup Q''$ is a proper subgraph of $\eta(H') \cup P \cup Q$.

**(5.5)** *Let $H$ and $G$ be graphs. Let $H'$ be obtained from $H$ by bisplitting a vertex $v$, let $v_0$ be the new inner vertex, and let $\eta : H' \hookrightarrow G$. Then no $v_0$-augmentation of $\eta$ of type C is strongly minimal.*

*Proof.* Let $v_1, v_2$ be the new outer vertices of $H'$, let $(P, Q)$ be a $v_0$-augmentation of $\eta$ of type C, let $a, b$ be the ends of $P$ with $a \in V(\eta(v_0))$, and let $x_1, x_2, e_1, e_2$ be as in the definition of augmentation of type C. The vertex $v_1$ has degree three; let $e_1' \notin \{e_1, v_1v_0\}$ be the third incident edge. Let $H''$ be obtained from $H$ by bisplitting $v$ into new outer vertices $v_1'', v_2''$ and new inner vertex $v_0''$, where $v_1''$ is incident with $e_1$ and $e_2$, and $v_2''$ is incident with all the remaining edges of $H$ incident with $v$. The embedding $\eta$ can be modified to

produce an embedding $\eta'' : H'' \hookrightarrow G$ with $\eta''(H) \subseteq P \cup \eta(H)$ by defining $\eta''(v_2'') = \eta(v_2)$, by defining $\eta''(v_1'')$ to be the graph with vertex set $\{x_2\}$, by letting $\eta''(e_2)$ be a subpath of $\eta(e_2)$ with end $x_2$, by letting $\eta''(e_1)$ be the union of a subpath of $P[x_2, a]$ with a suitable subpath of $\eta(e_1)$, and by letting $\eta''(e_1') = \eta(v_0) \cup \eta(v_1v_0) \cup \eta(v_2v_0) \cup \eta(e_1')$. Now $P[x_2, b] \backslash x_2$ is a weak $v_0''$-augmentation of $\eta''$. By (5.4) there exists an embedding $\xi : H'' \hookrightarrow G$ and a $v_0''$-augmentation $(P'', Q'')$ of $\eta''$ such that

$$P'' \cup Q'' \cup \xi(H'') \subseteq P[x_2, b] \cup \eta''(H'') \subseteq P \cup \eta(H),$$

but $P'' \cup Q'' \cup \xi(H'')$ does not use the edge of $P$ incident with $a$, contrary to the weak minimality of $(P, Q)$. □

**(5.6)** *Let $H$ and $G$ be graphs, let $H'$ be obtained from $H$ by bisplitting a vertex $v$, let $v_0$ be the new inner vertex, and let $\eta : H' \hookrightarrow G$. Then no $v_0$-augmentation of $\eta$ of type D is strongly minimal.*

*Proof.* Let $v_1, v_2$ be the new outer vertices of $H'$, let $(P, Q)$ be a $v_0$-augmentation of $\eta$ of type D, let $a, b$ be the ends of $P$ with $a \in V(\eta(v_0))$, and let $i, e, x$ be as in the definition of augmentation of type D. We may assume that $i = 1$. Let $H''$ be obtained from $H$ by bisplitting $v$ into new outer vertices $v_1'', v_2''$ and new inner vertex $v_0''$, where $v_1''$ is incident with all the edges of $H$ incident with $v_1$ in $H'$ except $e$ (note that $v_0v_1 \notin E(H)$), and $v_2''$ is incident with all the remaining edges of $H$ incident with $v$. The embedding $\eta$ can be modified to produce an embedding $\eta'' : H'' \hookrightarrow G$ with $\eta''(H) \subseteq P \cup \eta(H)$ by defining $\eta''(v_1'') = \eta(v_1')$ and letting $\eta''(v_2'')$ be a suitable subgraph of $\eta(v_2) \cup \eta(v_0) \cup \eta(v_0v_2) \cup P[a, x] \cup Q$. Now $P[x, b] \backslash x$ includes a weak $v_0''$-augmentation of $\eta''$. By (5.4) there exists an embedding $\xi : H'' \hookrightarrow G$ and a $v_0''$-augmentation $(P'', Q'')$ of $\eta''$ such that

$$P'' \cup Q'' \cup \xi(H'') \subseteq P[x, b] \cup \eta''(H'') \subseteq P \cup \eta(H),$$

but $P'' \cup Q'' \cup \xi(H'')$ does not use one of the edges of $\eta(v_0)$ incident with $a$, contrary to the weak minimality of $(P, Q)$. □

We summarize (5.1), (5.3), (5.5), and (5.6) into the following.

**(5.7)** *Let $H$ and $G$ be graphs, where $H$ has minimum degree at least three, let $H'$ be obtained from $H$ by bisplitting a vertex $v$, let $v_0$ be the new inner vertex, let $\eta : H' \hookrightarrow G$ be an embedding and assume that there exists a $v_0$-augmentation of $\eta$. Then a linear extension of $H$ is isomorphic to a matching minor of $G$.*

*Proof.* We may assume that the $v_0$-augmentation is strongly minimal. By (4.5) it is of type A, B, C, or D. By (5.5) and (5.6) it is of type A or B, and so the result holds by (5.1) and (5.3). □

We say that an embedding $\eta : H \hookrightarrow G$ is a *homeomorphic embedding* if $\eta(v)$ has exactly one vertex for every $v \in V(H)$ of degree at least three. The next lemma motivates this definition.

**(5.8)** *Let $H$ and $G$ be graphs. Then there exists an embedding $\eta : H \hookrightarrow G$ which is not a homeomorphic embedding if and only if a graph obtained from $H$ by bisplitting a vertex is isomorphic to a matching minor of $G$.*

*Proof.* Suppose that $\eta : H \hookrightarrow G$ and that for some vertex $v \in V(H)$ of degree at least three its image $\eta(v)$ has more than one vertex. Then there exists a branch $B$ of $\eta(v)$ with length greater than zero. The argument from the last paragraph of the proof of (4.4) applied to $\eta(v)$ and $B$, provides us with an embedding into $G$ of a graph $H_1$ obtained from $H$ by bisplitting $v$ and therefore by (4.2) the graph $H_1$ is isomorphic to a matching minor of $G$.

On the other hand let a graph $H'$, obtained from $H$ by bisplitting some vertex $v$ into new outer vertices $v_1$ and $v_2$ and new inner vertex $v_0$, be isomorphic to a matching minor of $G$. Then by (4.2) there exists an embedding $\eta' : H' \hookrightarrow G$. Let $J$ be the subgraph of $H$ induced by $\{v_0, v_1, v_2\}$. Define an embedding $\eta : H \hookrightarrow G$ by saying that $\eta(v) = \eta'(J)$, $\eta(vu) = \eta'(v_i u)$ for $i \in \{1, 2\}$ and all neighbors $u \neq v_0$ of $v_i$, and otherwise $\eta$ coincides with $\eta'$. Clearly $\eta(v)$ has more than one vertex and therefore $\eta$ is not a homeomorphic embedding. □

The following theorem and its corollary are the main results of this section.

**(5.9)** *Let $G$ be a brick, let $H$ be a graph of minimum degree at least three, and let $\eta : H \hookrightarrow G$. If $\eta$ is not a homeomorphic embedding, then a linear extension of $H$ is isomorphic to a matching minor of $G$.*

*Proof.* Let $v$ be a vertex of $H$ of degree at least three such that $\eta(v)$ has at least two vertices. By axiom (8) in the definition of an embedding the vertex $v$ has degree at least four. We apply (4.4) to $H$, $G$, $\eta$ and $v$. If outcome (4.4)(1) holds then (5.9) holds by (5.7).

Therefore we may assume that (2) of (4.4) holds, and let $\eta_2$, $P$, $p_1$, $p_2$, $B_1$, $B_2$, $Q_1$ and $Q_2$ be as in (4.4). Let $G'$ be the graph obtained from $\eta_2(H) \cup P \cup Q_1 \cup Q_2$ by bicontracting all exposed vertices, except those in $B_1 \cup B_2$. Note that $G'$ is a matching minor of $G$ and therefore it suffices to prove that a linear extension of $H$ is isomorphic to a matching minor of $G'$. If both $Q_1$ and $Q_2$ are null, then the graph $G'$ is isomorphic to a bisubdivision of a graph obtained from $H$ by two bisplits and adding an edge joining the two new inner vertices. Thus a linear extension of $H$ is isomorphic to a matching minor of $G$.

Therefore we may assume that $Q_2$ is not null. Let $u$ be the common end of $B_1$ and $B_2$ in $G'$ and let $u_1$ and $u_2$ be the other ends of $B_1$ and $B_2$ correspondingly. If $Q_1$ is not null, denote its ends by $q$ and $q'$ so that $q \in B_1[p_1, u_1]$ and let $q = q' = p_1$ otherwise. If $u$ has degree at least four in $G'$ then the graph $G''$ obtained from $G'$ by deleting the interiors of $B_1[u, q']$, $B_1[p_1, q]$ and $Q_2$ can be bicontracted to a graph obtained from $H$ by two bisplits and $Q_2$ can be bicontracted to an edge joining the two new inner vertices. Thus again a linear extension of $H$ is isomorphic to a matching minor of $G$.

Therefore we may assume that $u$ has degree three in $G'$. Hence there exists a unique vertex $w \in V(H)$ such that $u \in \eta_2(vw)$. Now $G''$ can be bicontracted to a graph obtained from $H$ by bisplitting $v$ and $Q_2$ can be bicontracted to an edge joining the new inner vertex to $w$. We deduce that a linear extension of $H$ is isomorphic to a matching minor of $G$, as desired. □

The next result follows immediately from (5.8) and (5.9).

**(5.10)** *Let $G$ be a brick, let $H$ be a graph of minimum degree at least three, and assume that a graph obtained from $H$ by bisplitting a vertex is isomorphic to a matching minor*

*of $G$. Then a linear extension of $H$ is isomorphic to a matching minor of $G$.* □

## 6. THE HIERARCHY OF EXTENSIONS

For the sake of exposition let us define a *split extension* of a graph $H$ to be any graph obtained from $H$ by bisplitting a vertex. We have seen in the previous section that if a split extension of $H$ is isomorphic to a matching minor of $G$, then the conclusion of Theorem (1.11) holds. The purpose of this short section is to define other types of extensions and to give an ordering on these extensions, and to reformulate (4.5). The ordering reflects the order in which these extensions will be dealt with. We will be proving theorems of the form "if such an such extension is isomorphic to a matching minor of $G$, then an extension that is higher on our list of priorities is also isomorpchic to a matching minor of $G$". Of course, the highest priority extensions are linear extensions.

Let us begin the definitions. The lowest on our list will be the following. Let $H$ be a graph, let $v \in V(H)$ be a vertex of degree at least three, and let $v_1, v_2$ be two distinct neighbors of $v$ in $H$. Let $H'$ be obtained from $H$ by bisubdividing the edge $vv_1$, and let $x, y$ be the new vertices numbered so that $x$ is adjacent to $v$. We say that the graph $H + (y, v_2v)$ is a *vertex-parallel extension of $H$*. We say that $H + (y, v_2)$ is an *edge-parallel extension of $H$*.

Let $v$ be a vertex of degree 3 in a graph $H$ and let $v_1$, $v_2$ and $v_3$ be its neighbors. We say that $K$ is obtained from $H$ by *replacing $v$ by a triangle* if $K$ is obtained from $H$ by deleting the vertex $v$ and adding the vertices $u_1, u_2, u_3$ and edges $u_1u_2, u_2u_3, u_3u_1, u_1v_1, u_2v_2$ and $u_3v_3$.

Let $H$ be a graph, let $v$ be a vertex of $H$ of degree at least three in a graph $H$, and let $v_1$ and $v_2$ be two neighbors of $v$. Let $K$ be obtained from $H$ by bisubdividing the edges $v, v_1$ and $v, v_2$ and let $x_1, y_1, x_2, y_2$ be the new vertices numbered so that $v_1y_1x_1vx_2y_2$ is a path in $K$. Let $K' = K + (x_1, y_2) + (x_2y_1)$, and let $J = K'$, or let $J$ be obtained from $K'$ by replacing one or both of the vertices $x_1$, $x_2$ by triangles. We say that $J$ is a *cross extension* of $H$, and that $v$ is its *hub*. See Figure 10.

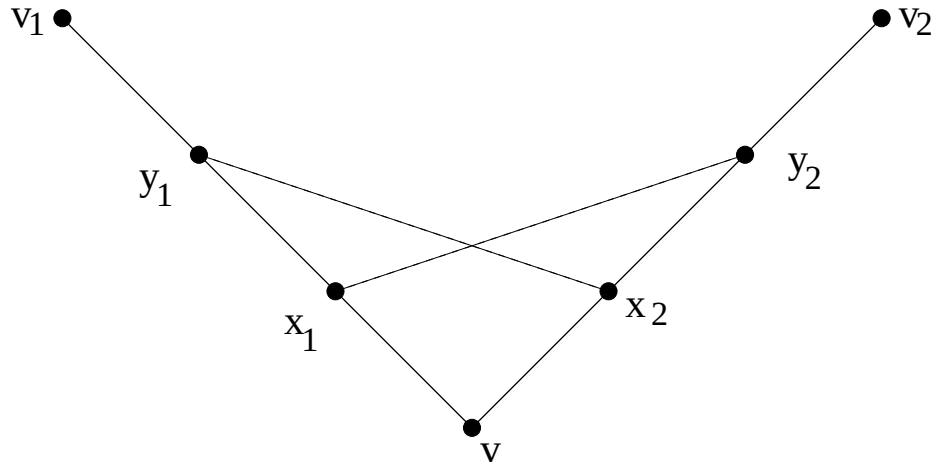



**Figure 10. A cross extension.**

Let $u$ be a vertex of $H$ of degree three and let $u_1$,$u_2$ and $u_3$ be its neighbors. Let $H_0$ be obtained from $H$ by bisubdividing each of the edges $uu_1$, $uu_2$ and $uu_3$. Let the new vertices be $y_1, y_2, y_3$ and $z_1, z_2, z_3$ in such a way that $u_1y_1z_3u$, $u_2y_2z_1u$ and $u_3y_3z_2u$ are paths. Let $H_1 := H_0 + (y_1, z_2) + (y_2, z_3) + (y_3, z_1)$, let $H_2$ be obtained from $H_1$ by replacing $z_1$ by a triangle, let $H_3$ be obtained from $H_2$ by replacing $z_2$ by a triangle, and let $H_4$ be obtained from $H_3$ by replacing $z_3$ by a triangle. Then each of the graphs $H_1, H_2, H_3, H_4$ is called a *cube extension of* $H$. See Figure 11.

Let $H$ be a graph, let $uv \in E(H)$, and let $H'$ be obtained from $H$ by bisubdividing $uv$, where the new vertices $x, y$ are such that $x$ is adjacent to $u$ and $y$. Let $x' \in V(H) - \{u\}$ and $y' \in V(H) - \{v\}$ be not necessarily distinct vertices such that not both belong to $\{u, v\}$. In those circumstances we say that $H' + (x, x') + (y, y')$ is a *quadratic extension of* $H$. Now let $ab \in E(H) - \{uv\}$ be such that $a \neq v$ and $u \neq b$, let $H''$ be obtained from $H'$ by bisubdividing $ab$, and let $x', y'$ be the new vertices. Then the graph $H'' + (x, x') + (y, y')$ is called a *quartic extension of* $H$.

We are now ready to define the promised linear order on extensions. We define that linear extensions are better than quartic extensions, quartic extensions are better than quadratic extensions, which in turn are better than cross extensions, which are better than cube extensions, which are better than edge-parallel extensions, and those are better than vertex-parallel extensions. The linear order is depicted on Figure 12.

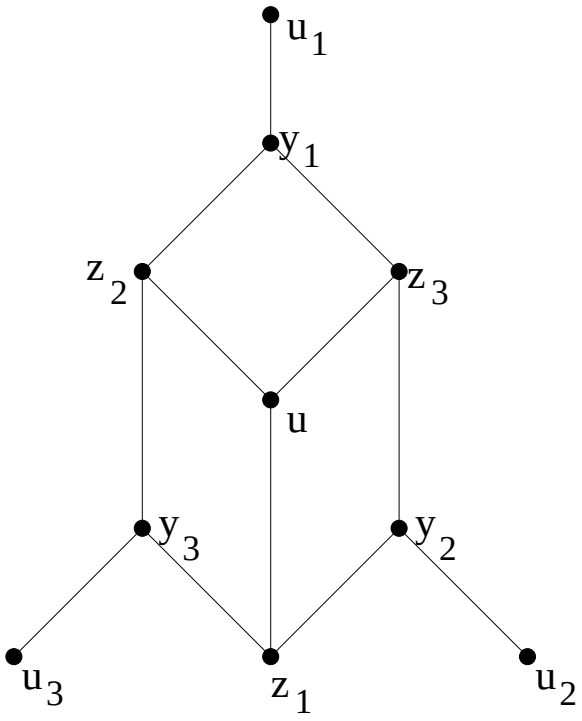



**Figure 11. A cube extension.**

For later convenience we reformulate (4.5) in a form more suitable for applications. To do so we will need a definition, but before we can state it, we need to introduce a convention. Let $G$ be a graph, let $w \in V(G)$, and let $uv$ be an edge of $G$ not incident with $w$. Then the graph $G' = G + (w, uv)$ has two new vertices, and it will be convenient to have a default notation for them. We shall use $\tau_1$ and $\tau_2$ to denote the new vertices, so that $\tau_1$ is adjacent to $u$, $w$ and $\tau_2$ in $G'$. We shall extend this convention naturally to more complicated scenarios, as exemplified by the following illustration. For instance, if $ab \in E(G) - \{uv\}$, then $G'' = G + (w, uv) + (\tau_2, ab)$ means the graph $G' + (\tau_2, ab)$, and its new vertices are denoted by $\tau_3$ and $\tau_4$ so that $\tau_3$ is adjacent to $a$, $\tau_2$ and $\tau_4$ in $G''$. In general, the new vertices will be numbered $\tau_1, \tau_2, \tau_3, \ldots$ in the order they arise as the input graph is read from left to right. Sometimes we will use $\rho_1, \rho_2, \ldots$ rather than $\tau_1, \tau_2, \ldots$ in order to avoid confusion.

Now we are ready for the definition. Let $J, G$ be graphs, let $v_0$ be a vertex of $J$ of degree two, and let $v_1, v_2$ be the neighbors of $v_0$. We wish to reformulate the outcomes of (4.5). Let $v \in V(J) - \{v_0, v_1, v_2\}$, let $i \in \{1, 2\}$, and for $j = 1, 2$ let $v'_j$ be a neighbor of $v_j$

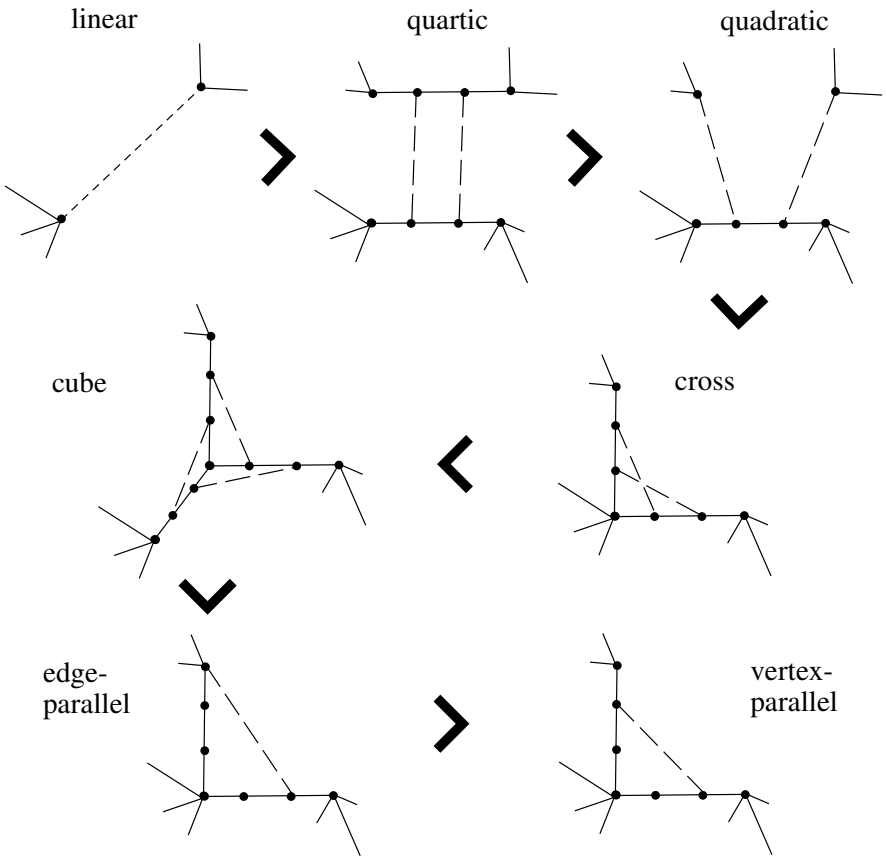



**Figure 12. The linear order on extensions.**

other than $v_0$. We define the following graphs:

- $A_1(v) = J + (v_0, v)$,
- $A_2(v) = J + (v_0, v_1v_0) + (\tau_2, v)$,
- $B_1(v_i'v_i, v) = J + (v_0, v_i'v_i) + (\tau_2, v)$,
- $B_2(v_i'v_i, v) = J + (v_0, v_i'v_i) + (\tau_2, v_i\tau_2) + (\tau_4, v)$,
- $B_3(v_i'v_i, v) = J + (v_0, v_iv_0) + (\tau_2, v_i'v_i) + (\tau_4, v)$,
- $B_4(v_i'v_i, v) = J + (v_0, v_iv_0) + (\tau_2, v_i'v_i) + (\tau_4, v_i\tau_4) + (\tau_6, v)$,
- $C_1(v_i'v_i, v_{3-i}'v_{3-i}) = J + (v_0, v_i'v_i) + (\tau_2, v_{3-i}'v_{3-i})$,
- $C_2(v_i'v_i, v_{3-i}'v_{3-i}) = J + (v_0, v_i'v_i) + (\tau_2, v_i\tau_2) + (\tau_4, v_{3-i}'v_{3-i})$,
- $C_3(v_i'v_i, v_{3-i}'v_{3-i}) = J + (v_0, v_iv_0) + (\tau_2, v_i'v_i) + (\tau_4, v_{3-i}'v_{3-i})$,
- $C_4(v_i'v_i, v_{3-i}'v_{3-i}) = J + (v_0, v_iv_0) + (\tau_2, v_i'v_i) + (\tau_4, v_i\tau_4) + (\tau_6, v_{3-i}'v_{3-i})$.

Sometimes we will omit the arguments when they will be clear from the context and write,

e.g., $B_3$ instead of $B_3(v_i'v_i, v)$. The graphs $A_1, \ldots, C_4$ correspond to augmentations of types $A, B$ and $C$, shown in Figures 7-9. The following lemma gives the promised reformulation of the outcomes of (4.5).

**(6.1)** *Let $J$ be a graph, let $G$ be a brick, let $v_0$ be a vertex of $J$ of degree two, let $v_1, v_2$ be the neighbors of $v_0$, for $i = 1, 2$ let $v_i' \neq v_0$ be a neighbor of $v_i$, assume that $v_1$ is not adjacent to $v_2$, assume that every vertex $v \in V(J) - \{v_0\}$ has a neighbor in $V(J) - \{v_1, v_2\}$, and assume that there exists an embedding $J \hookrightarrow G$. Then one of the following holds:*

(A) *there exists a vertex $v \in V(J) - \{v_0, v_1, v_2\}$ such that $A_1(v) \hookrightarrow G$ or $A_2(v) \hookrightarrow G$,*

(B) *there exist a vertex $v \in V(J) - \{v_0, v_1, v_2\}$ and indices $i \in \{1, 2\}$ and $j \in \{1, 2, 3, 4\}$ such that $v_i$ has degree three and $B_j(v_i'v_i, v) \hookrightarrow G$,*

(C) *there exist indices $i \in \{1, 2\}$ and $j \in \{1, 2, 3, 4\}$ such that $v_1, v_2$ have degree three and $C_j(v_i'v_i, v_{3-i}'v_{3-i}) \hookrightarrow G$,*

(D) *some split extension of $J$ is isomorphic to a matching minor of $G$.*

*Proof.* Let $\eta : J \hookrightarrow G$. We may assume that $\eta$ is a homeomorphic embedding, for otherwise (D) holds by (5.8). By changing $\eta$ we may assume that $\eta(v_1)$ and $\eta(v_2)$ each have exactly one vertex, even if $v_1$ or $v_2$ has degree less than three. By (4.4) there exists an embedding $\eta' : J \hookrightarrow G$ and a $v_0$-augmentation $(P, Q)$ of $\eta'$. We may assume that $(P, Q)$ is minimal, and hence by (4.5) it is of type A, B, C or D. Similarly as above, we may assume that $\eta'$ is a homeomorphic embedding. Let $P$ have origin $a \in V(\eta'(v_0))$ and terminus $b \in V(\eta'(u))$. We say that $(P, Q)$ is *good* if either $u$ has degree not equal to two, or $u$ has degree two and $b$ is at even distance from either end of $\eta'(u)$ (recall that $\eta'(u)$ is an even path when $u$ has degree two, and otherwise $\eta'(u)$ has exactly one vertex).

Suppose $(P, Q)$ is not good. Then $u$ has degree two and $b$ is at odd distance from the ends of $\eta(u)$. Let $u'$ be a neighbor of $u$ in $V(J) - \{v_1, v_2\}$ and let $b'$ and $b''$ be the ends of $\eta(u)$, such that $b' \in V(\eta(uu'))$. Let $G'$ be obtained from $\eta(H) \cup P \cup Q$ by contracting the even path $\eta(u)[b, b'] \cup \eta(uu')$. Define $\eta' : J \hookrightarrow G'$ as follows. Let $\eta'(u) = \eta(u)[b'', b] \backslash b$, $\eta'(uu')$ is a length one subpath of $\eta(u)[b, b'']$ with one end at $b$ and $\eta'$ is otherwise equal to $\eta$. Note that $(P, Q)$ is a good augmentation of $\eta'$. Note also that $G'$ is a matching minor of $G$.

Therefore we may assume that $(P, Q)$ is a good augmentation of $\eta$ of type A, B, C or D. Now if $(P, Q)$ is of type A, then outcome (A) holds, and similarly for type D, and, by (5.2), for type B. Thus we may assume that $(P, Q)$ is of type C. From the symmetry we may assume that $P$ crosses an edge incident with $v_1$, and by (5.2) we may assume that it crosses the edge $v_1v_1'$. In particular, $v_1$ has degree at most three. But it has degree exactly three by axiom (7) in the definition of an embedding, because $\eta(v_1v_1')$ has at least one internal vertex. The existence of $(P, Q)$ implies, by the same argument as above, that there is an integer $j \in \{1, 2, 3, 4\}$ such that $C_j(v_1'v_1, v_2''v_2) \hookrightarrow G$ for some neighbor $v_2''$ of $v_2$ other than $v_0$. Let $L = C_1(v_1'v_1, v_2''v_2)\backslash v_0v_2\backslash\tau_1\tau_2$ if $j = 1$, and let it be defined analogously for $j \geq 2$. If $v_2$ has degree at least four, then $L$ is isomorphic to a bisubdivision of a split extension of $H$, and hence the lemma holds. Thus we may assume that $v_2$ has degree at most three, but it has degree exactly three by the same reason as $v_1$. If $v_2' = v_2''$, then (C) holds, and so we may assume not. If $j = 1$, then by considering $L$ and the edges $\tau_1\tau_2$ and $v_0v_2$ we deduce that $C_1(v_1'v_1, v_2'v_2) \hookrightarrow G$. An analogous argument works for $j = 4$, while for $j \in \{2, 3\}$ the analogous argument proves that $C_{5-j}(v_1'v_1, v_2'v_2) \hookrightarrow G$. Thus (C) holds, as desired. □

## 7. USING 3-CONNECTIVITY

A graph $G$ is *matching covered* if it is connected and every edge of $G$ belongs to a perfect matching of $G$. Thus every brick is matching covered.

**(7.1)** *Let $H$ and $G$ be graphs such that $H$ has minimum degree at least three, $G$ is matching covered, and $H$ is isomorphic to a matching minor of $G$. If $H$ is not isomorphic to $G$, then either a linear or split extension of $H$ is isomorphic to a matching minor of $G$, or there exists a homeomorphic embedding $\eta' : H \hookrightarrow G$ such that $\eta'(e)$ has at least three edges for some $e \in E(H)$.*

*Proof.* By (4.2) there exists an embedding $\eta : H \hookrightarrow G$. We may assume that $\eta$ is a homeomorphic embedding, for otherwise the lemma holds by (5.8). We may also assume that $\eta(e)$ has exactly one edge for each $e \in E(H)$. Thus $\eta(H)$ is isomorphic to $H$. But

$G$ is not isomorphic to $H$, and hence there exists an edge $e$ of $G$ with exactly one end in $\eta(H)$. Let $M_1$ be a perfect matching of $G$ containing $e$, and let $M_2$ be a perfect matching of $G\backslash V(\eta(H))$. (This exists, because $\eta(H)$ is a central subgraph of $G$.) The component of $M_1 \triangle M_2$ containing $e$ is a path with both ends in $\eta(H)$; let $u, v \in V(H)$ be such that $P$ has one end in $\eta(v)$ and the other end in $\eta(u)$. If $u$ and $v$ are not adjacent in $H$, then by letting $\eta(uv) = P$ the embedding $\eta$ can be extended to an embedding $H + uv \hookrightarrow G$, and hence a linear extension of $H$ is isomorphic to a matching minor of $G$. On the other hand, if $u$ and $v$ are adjacent in $H$, then $P$ has at least three edges, because in that case the unique edge of $G$ between $\eta(u)$ and $\eta(v)$ belongs to $\eta(uv)$. Thus we obtain the desired homeomorphic embedding by replacing $\eta(uv)$ by $P$. □

Let $G$ be a graph, let $A, B \subseteq V(G)$, let $M$ be a perfect matching of $G\backslash(A \cup B)$, and let $k \geq 0$ be an integer. We say that the sequence of paths $(P, Q_1, Q_2, \ldots, Q_k)$ is an *$(A, B)$-hook with respect to $M$* if the following conditions hold:

(1) $P$ has ends $p_0 \in A - B$ and $p_{k+1} \in B - A$, and is otherwise disjoint from $A \cup B$,

(2) for $i = 1, 2, \ldots, k$, $Q_i$ is an even path with ends $p_i \in V(P) - \{p_0, p_{k+1}\}$ and $q_i \in A \cap B$, and is otherwise disjoint from $A \cup B \cup V(P)$,

(3) $V(Q_i) \cap V(Q_j) \subseteq \{q_i, q_j\}$ for all distinct indices $i, j \in \{1, 2, \ldots, k\}$,

(4) the graph $P \cup Q_1 \cup Q_2 \cup \ldots \cup Q_k \backslash (A \cup B)$ is $M$-covered, and

(5) the vertices $p_0, p_1, p_2, \ldots, p_k, p_{k+1}$ are distinct and occur on $P$ in the order listed.

See Figure 13.

**(7.2)** *Let $G$ be a matching covered graph, let $A, B \subseteq V(G)$, and let $M$ be a perfect matching of $G\backslash(A \cup B)$. If $A - B$ and $B - A$ are both nonempty and belong to the same component of $G\backslash(A \cap B)$, then there exists an $(A, B)$-hook with respect to $M$.*

*Proof.* Suppose for a contradiction that the graph $G$ does not satisfy the lemma, and choose $(A, B)$ violating the lemma with $A \cup B$ maximum. Let $e$ be an edge of $G$ with one end in $A - B$ and the other end in $V(G) - A$. Let $M'$ be a perfect matching of $G$ containing $e$, and let $P_0$ be the component of $M \triangle M'$ containing $e$. Then $P_0$ is a path with one end

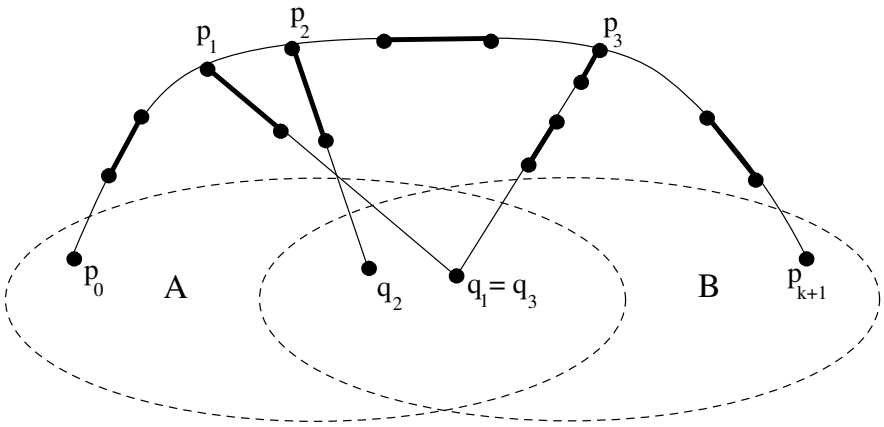



**Figure 13. An $(A, B)$-hook.**

in $A - B$, the other end in $A \cup B$, and otherwise disjoint from $A \cup B$. If the other end of $P_0$ is in $B - A$, then the sequence with sole term $P_0$ is a required $(A, B)$-hook, and so we may assume that the other end of $P_0$ is in $A$. Let $A' := A \cup V(P_0)$. Then $A' \cap B = A \cap B$. By the maximality of $A \cup B$ there exists an $(A', B)$-hook $h = (P, Q_1, Q_2, \ldots, Q_k)$.

Let $p_0 \in A'$ be an end of $P$. If $p_0 \in A$, then $h$ is an $(A, B)$-hook, and the lemma holds. Thus we may assume that $p_0$ is an internal vertex of $P_0$. Let $P_1$ and $P_2$ be the two subpaths of $P_0$ with common end $p_0$ and union $P_0$. Exactly one of them, say $P_1$, has the property that $P_1 \cup P \cup Q_1 \cup Q_2 \cup \ldots \cup Q_k \backslash (A \cup B)$ is $M$-covered. If the other end of $P_1$ is in $A - B$, then $(P \cup P_1, Q_1, Q_2, \ldots, Q_k)$ is a desired $(A, B)$-hook. Thus we may assume that $P_1$ has one end in $A \cap B$, in which case $(P \cup P_2, P_1, Q_1, Q_2, \ldots, Q_k)$ is a desired $(A, B)$-hook. □

**(7.3)** *Let $H$ and $G$ be graphs, where $H$ has minimum degree at least three and is isomorphic to a matching minor of $G$ and $G$ is a brick. If $H$ is not isomorphic to $G$, then a vertex-parallel, edge-parallel or a linear extension of $H$ is isomorphic to a matching minor of $G$.*

*Proof.* By (4.2) and (5.9) we may assume that there exists a homeomorphic embedding $\eta : H \hookrightarrow G$. By (7.1) we may assume that there exists an edge $uv \in E(H)$ such that $\eta(uv)$ has at least three edges. Let $A = V(\eta(uv))$ and let $B$ consist of $V(\eta(H))$, except the

internal vertices of $\eta(uv)$. Then $A - B$ and $B - A$ are nonempty and $|A \cap B| = 2$. Thus $A-B$ and $B-A$ belong to the same component of $G\backslash(A\cap B)$, because $G$ is 3-connected. We have $A\cup B = V(\eta(H))$, and hence $G\backslash(A\cup B)$ has a perfect matching, say $M$, because $\eta(H)$ is a central subgraph of $G$. By (7.2) there exists an $(A, B)$-hook $h = (P, Q_1, Q_2, \ldots, Q_k)$ with respect to $M$. We may choose $\eta$, $uv$ and $h$ so that $k$ is minimum. If $k = 0$, then by considering the path $P$ we conclude that a required extension is isomorphic to a matching minor of $G$.

Thus we may assume that $k > 0$. Let the notation be as in the definition of $(A, B)$-hook. Thus $p_0$ is an internal vertex of $\eta(uv)$, and from the symmetry we may assume it is located at even distance from $\eta(v)$ on $\eta(uv)$. We have $q_i \in \{\eta(u), \eta(v)\}$ for all $i = 1, 2, \ldots, k$. We properly two-color the graph $\eta(uv) \cup P$ using the colors black and white so that $\eta(v)$ is black and $\eta(u)$ is white. For convenience let $q_0 := p_0$. We will show that $q_0, q_1, q_2, ..., q_k$ all have the same color. Indeed, suppose for some $i \in \{0, 1, \ldots, k-1\}$ the vertices $q_i$ and $q_{i+1}$ have different color. We replace $\eta(uv)[q_i, q_{i+1}]$ by $Q_i \cup P[p_i, p_{i+1}] \cup Q_{i+1}$ to obtain an embedding $\eta'' : H \hookrightarrow G$. Then the sequence $h' = (P[p_{i+1}, p_{k+1}], Q_{i+2}, Q_{i+3}, \ldots, Q_k)$ is an $(A', B')$-hook, where $A'$ and $B'$ are defined in the same way as $A$ and $B$ but relative to $\eta''$. But $h'$ contradicts the minimality of $k$. This proves our claim that $q_0, q_1, q_2, ..., q_k$ all have the same color; in particular, $q_1 = q_2 = \cdots = q_k = \eta(v)$.

The graph $\eta(H) \cup Q_k \cup P[p_k, p_{k+1}]$ has a matching minor isomorphic to a desired extension of $H$, unless $p_{k+1}$ belongs to $\eta(vw)$ for some neighbor $w$ of $v$ other than $u$. By using the argument of the previous paragraph we deduce that $p_{k+1}$ is an internal vertex of $\eta(vw)$ located at even distance from $\eta(v)$ on $\eta(vw)$ for some neighbor $w \neq u$ of $v$. Let $J$ be obtained from $H$ as follows. First we bisubdivide the edges $uv$ and $vw$; let the new vertices be $p'_0$,$r_0$ and $p'_{k+1}$,$r_{k+1}$ correspondingly, where $p'_0$ is adjacent to $u$ and $p'_{k+1}$ is adjacent to $w$. Denote resulting graph by $H'$. Then we add new vertices $p'_1, p'_2, \ldots, p'_k$ and $r_1, r_2, \ldots, r_k$ in such a way that $p'_0p'_1 \ldots p'_kp'_{k+1}$ is a path, and $p'_ir_iv$ is a path for all $i = 1, 2, \ldots, k$, and there are no other edges incident with the new vertices. This completes the definition of $J$. See Figure 14. Now $\eta$ can be converted to an embedding $\eta' : J \hookrightarrow G$ in a natural way; thus, for instance, $\eta'(p'_i)$ is the graph with vertex-set $\{p_i\}$.

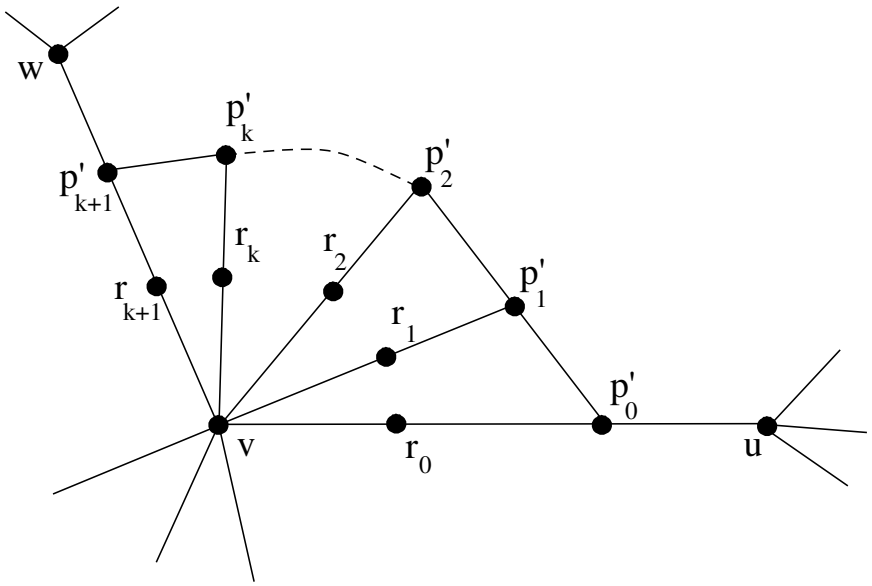



**Figure 14. An extension $J$ of $H$.**

We apply (6.1) to the graphs $J$ and $G$ and the vertex $r_0$; let $J'$ be the resulting graph, and let $\eta'' : J' \hookrightarrow G$. Suppose outcome (D) of (6.1) holds. Then either a split extension of $H$ is isomorphic to a matching minor of $G$, in which case the desired result follows from (5.9), or $J'$ is obtained from $J$ by splitting $v$. Let $v_1$ and $v_2$ be the new outer vertices and $v_0$ the new inner vertex. As we may assume that no split extension of $H$ is isomorphic to a matching minor of $G$, we have that $|N_{J'}(v_i) \cap N_{H'}(v)| \geq 2$ for at most one $i \in \{1, 2\}$, where $N_{J'}(v_i)$ and $N_{H'}(v)$ denote the neighborhoods of $v_i$ and $v$ in $J'$ and $H'$ correspondingly. Without loss of generality let $|N_{J'}(v_1) \cap N_{H'}(v)| \leq 1$. Assume first $N(v_1) \cap N_{H'}(v) = \emptyset$, then we can choose $1 \leq i < i' \leq k$ such that $r_i, r_{i'} \in N(v_1)$ and $r_j \notin N(v_1)$ for every $j$ such that $1 \leq j < i$ or $i' < j \leq k$. The image of the hook $h' = (p'_0 p'_1 \dots p'_i r_i v_1 r_{i'} p'_{i'} p'_{i'+1} \dots p'_{k+1}, p'_1 r_1 v_2, \dots, p'_{i-1} r_{i-1} v_2, v_1 v_0 v_2, p'_{i'+1} r_{i'+1} v_2, \dots)$ under $\eta''$ contradicts the minimality of $k$. Assume now $|N_{J'}(v_1) \cap N_{H'}(v)| = 1$. From the symmetry between $r_0$ and $r_{k+1}$ we may assume $r_0 \in N(v_2)$. Let $i$ be minimal such that $r_i \in N(v_1)$ then $i \leq k$ and the image of the hook $h' = (p'_0 p'_1 \dots p'_i r_i v_1, p'_1 r_1 v_2, \dots, p'_{i-1} r_{i-1} v_2)$ under $\eta''$ contradicts the minimality of $k$. We assume now that one of the outcomes (A),(B) or (C) of (6.1) holds.

Throughout the rest of the proof let $z \in V(J) - \{v, p_0', r_0\}$. Outcome (C) cannot hold, because $v$ has degree at least four in $J$. Assume next that either $J' = A_1(z)$, in which case we put $\tau_1 = \tau_2 = v_0$, or that $J' = A_2(z) = J + (r_0, p_0'r_0) + (\tau_2, z)$, in which case $\tau_1$ and $\tau_2$ have their usual meaning. If $z \in (V(H) - \{u\}) \cup \{r_{k+1}, p_{k+1}'\}$, then $J + (\tau_2, z)$ is isomorphic to a bisubdivision of a suitable extension of $H$. If $z = u$ we replace $\eta(uv)$ by $\eta''(u\tau_2 r_0 \tau_1 p_0' p_1' r_1 v)$ and the hook $h' = (P[p_1, p_{k+1}], Q_2, Q_3, \ldots, Q_k)$ contradicts the minimality of $k$. If $z = r_i$ for some $1 \leq i \leq k$ then the hook $h' = (\eta''(\tau_2 r_i p_i') \cup P[p_i, p_{k+1}], Q_{i+1}, Q_{i+2}, \ldots, Q_k)$ contradicts the minimality of $k$. Finally if $z = p_i'$ for some $1 \leq i \leq k$ we replace $\eta(uv)$ by $\eta''(up_0'\tau_1 r_0 \tau_2 p_i' r_i v)$ and the hook $h' = (P[p_i, p_{k+1}], Q_{i+1}, Q_{i+2}, \ldots, Q_k)$ contradicts the minimality of $k$. This completes the case $J' = A_i$.

Since $v$ has degree at least four in $J$ we assume that $J' = B_i(p_1'p_0', z)$ for some $i \in \{1, 2, 3, 4\}$. Note that $J'$ contains $J'' = A_j(z) \backslash p_1'p_0'$ as a matching minor for some $j \in \{1, 2\}$, and unless $z = u$ the argument from the previous paragraph provides us with a suitable extension or a contradiction. If $z = u$ we replace $\eta(uv)$ by $\eta'(u\tau\tau'\tau'' p_1' r_1 v)$, where $\tau = \tau' = \tau'' = \tau_{2i}$ if $i \in \{1, 2\}$ and $\tau = \tau_{2i-2}$, $\tau' = \tau_{2i-3}$, $\tau'' = \tau_{2i-4}$ if $i \in \{3, 4\}$. The hook $h' = (P[p_1, p_{k+1}], Q_2, Q_3, \ldots, Q_k)$ now contradicts the minimality of $k$. □

## 8. VERTEX-PARALLEL AND EDGE-PARALLEL EXTENSIONS

The purpose of this section is to replace vertex-parallel and edge-parallel extensions in the statement of (7.3) by extensions that are closer to linear extensions. Our first goal is to prove that if a brick $G$ has a matching minor isomorphic to a vertex-parallel extension of a 2-connected graph $H$, then it also has a matching minor isomorphic to a better extension of $H$. We will proceed in two steps; in the intermediate step we will produce a better extension or one that is "almost better", the following. Let $H$ be a graph, let $u$ be a vertex of degree at least three, let $u_1$ and $u_2$ be distinct neighbors of $u$, and let $H' = H + (u_1, uu_1) + (\tau_2, u_2u)$. We say that $H'$ is a *semi-edge-parallel* extension of $H$.

**(8.1)** *Let $H$ be a graph of minimum degree at least three, and let $G$ be a brick. If a vertex-parallel extension of $H$ is isomorphic to a matching minor of $G$, then an edge-parallel, a semi-edge-parallel, a linear, a cross, or a split extension of $H$ is isomorphic to a matching minor of $G$.*

*Proof.* Let $u_0$ be the vertex of $H$ with neighbors $u_1$ and $u_2$ such that the graph $H_2$ defined below is isomorphic to a matching minor of $G$. Let $H_1$ be obtained from $H$ by bisubdividing the edges $u_0u_1$ and $u_0u_2$ exactly once, and let $x_1, y_1, x_2, y_2$ be the new vertices, numbered so that $u_2y_2x_2u_0x_1y_1u_1$ is a path. The graph $H_2$ is defined as $H_1 + (y_1, y_2)$. By (6.1) applied to $J = H_2$ and the vertex $x_1$ there exists a graph $J' \hookrightarrow G$ satisfying (A), (B), (C) or (D) of that lemma. If $J'$ is a split extension of $J$, then the graph obtained from $J'\backslash y_1y_2$ by bicontracting $y_1$ and $y_2$ is a split extension of $H$. Thus if (D) holds, then the theorem holds, and so we may assume that (A), (B) or (C) holds. Throughout this proof let $v \in V(J) - \{u_0, x_1, y_1\}$. The symbols $\tau_1, \tau_2, \ldots$ will refer to the new vertices of $J'$ according to the convention introduced prior to (6.1).

Assume first that $J' = A_1 = J + (x_1, v)$. If $v = u_1$, then $J'$ is isomorphic to a semi-edge-parallel extension of $H$. If $v = x_2$, then $H + (u_1, u_0u_2) \hookrightarrow G$; if $v = y_2$, then $H + (u_1, u_2u_0) \hookrightarrow G$; and in all other cases $H + (v, u_0u_1) \hookrightarrow G$. In the last case, if $v$ is not adjacent to $u_1$, then $H + (v, u_1)$ is a linear extension of $H$, and otherwise $H + (v, u_0u_1)$ is an edge-parallel extension of $H$. The same argument will be used later. We will also use later the fact that the inclusions above did not use the edge $y_1y_2$. This completes the case $J' = A_1$.

Now we assume that $J' = A_2 = J + (x_1, x_1u_0) + (\tau_2, v)$. If $v = x_2$, then $H + (u_2, u_1u_0) \hookrightarrow G$; if $v = y_2$, then by deleting the edge $y_1y_2$ and bicontracting $y_1$ we see that a semi-edge-parallel extension of $H$ is isomorphic to a minor of $G$; if $v = u_1$, then the graph $A_1\backslash x_1\backslash y_1$ is isomorphic to a bisubdivision of $H$, and by considering the path $y_2y_1x_1\tau_1$ we deduce that $H + (u_1, u_2u_0) \hookrightarrow G$; and in all other cases $H + (v, u_1u_0) \hookrightarrow G$. This completes the case $J' = A_2$.

Let $j \in \{1, 2, 3, 4\}$ and let $J' = B_j(y_2y_1, v)$. We have $A_1(v)\backslash y_1y_2 \hookrightarrow B_j(y_2y_1, v)$ for $j = 1, 2$ and $A_2(v)\backslash y_1y_2 \hookrightarrow B_j(y_2y_1, v)$ for $j = 3, 4$ (if $j = 1$ we delete the edges $y_2\tau_1$ and

$y_1\tau_2$ and analogously for $j \geq 2$). Since the arguments of the previous two paragraphs did not use the edge $y_1y_2$, except for the cases of $A_1(u_1)$ and $A_2(u_1)$, we may assume that $J' = B_j(y_2y_1, u_1)$, for some $j \in \{1, 2, 3, 4\}$. But $H + (u_1, u_2u_0) \hookrightarrow B_j(y_2y_1, u_1)$ (consider the path $u_1\tau_2\tau_1y_2$ when $j = 1$). This completes the cases $J' = B_j(y_2y_1, v)$.

Our next step is to handle the cases $J' = B_j(x_2u_0, v)$ and $J' = C_j(x_2u_0, y_2y_1)$. If $j \leq 2$, then $H + (u_1, u_2u_0) \hookrightarrow G$, and if $j \geq 3$, then $H_1 + (x_1, x_1u_0) + (\rho_2, x_2u_0) \hookrightarrow G$ and $H_1 + (x_1, x_1u_0) + (\rho_2, x_2u_0)$ after bicontraction of $y_1$ and $y_2$ becomes isomorphic to a semi-edge parallel extension of $H$. (We are using "$\rho$" instead of "$\tau$", because the "$\tau$" notation is reserved for vertices of $J'$.)

Thus the only remaining cases are $J' = C_j(y_2y_1, x_2u_0)$. If $j = 1$, then by considering the path $x_1\tau_1\tau_2\tau_3$ we deduce that $H+(u_1, u_2u_0) \hookrightarrow G$; for $j = 2$ the argument is analogous. For $j = 3$ notice that $C_3(y_2y_1, x_2u_0)\backslash\tau_1y_1\backslash y_2\tau_3\backslash x_1\tau_2\backslash\tau_4\tau_5$ is isomorphic to a bisubdivision of $H$. By considering the edge $\tau_4\tau_5$ we see that $H+(u_1, u_2u_0) \hookrightarrow G$. Finally, $C_4(y_2y_1, x_2u_0)$ has a matching minor isomorphic to a semi-edge-parallel extension of $H$. To see that, consider the edge $x_1\tau_1$ and path $\tau_2\tau_3\tau_4\tau_5\tau_6\tau_7$. (The last argument applies to $j = 3$ as well, but for the sake of the next proof we wish to avoid semi-parallel extensions as much as possible.) □

**(8.2)** *Let $H$ be a 2-connected graph of minimum degree at least three, and let $G$ be a brick. If a semi-edge-parallel extension of $H$ is isomorphic to a matching minor of $G$, then an edge-parallel, a linear, a cross, a cube or a split extension of $H$ is isomorphic to a matching minor of $G$, unless $H$ is isomorphic to $K_4$ and $G$ has a matching minor isomorphic to the Petersen graph.*

*Proof.* By hypothesis there exists a vertex $u_0$ of $H$ with distinct neighbors $u_1$ and $u_2$ such that the graph $H_3$ is isomorphic to a matching minor of $G$, where $H_1, H_2, x_1, y_1, x_2, y_2$ are defined as in the proof of (8.1), and $H_3 = H_2 + (x_2, u_2)$. We may assume that $u_0$ has degree exactly three, for otherwise $H_3\backslash u_2y_2\backslash x_2u_0$ is isomorphic to a bisubdivision of a split extension of $H$, and hence a split extension of $H$ is isomorphic to a matching minor of $G$. Let $u_3$ be the third neighbor of $u_0$. Since $H_3 \hookrightarrow G$, either a split extension of $H$

is isomorphic to a matching minor of $G$, or one of the graphs $H_3$, $H_4 = H_2 + (x_2, y_2u_2)$, $H_5 = H_2 + (x_2, u_2'u_2)$, where $u_2' \neq u_0$ is a neighbor of $u_2$, has a homeomorphic embedding into $G$. Graphs $H_3, H_4$ and $H_5$ are shown on Figure 15. Let $J$ denote that graph, and let it be chosen so that $J \neq H_3$, if possible. This choice implies that if a split extension of $J$ is isomorphic to a matching minor of $G$, then so is a split extension of $H$. Let $x_2', y_2'$ be the new vertices of $H_4$ and $H_5$. We apply (6.1) to $J$ and the vertex $x_1$, and so we may assume that (A), (B), or (C) holds, for otherwise the theorem holds. Let $J'$ be the graph satisfying (A), (B) or (C). The symbols $\tau_1, \tau_2, \ldots$ will again refer to the new vertices of $J'$.

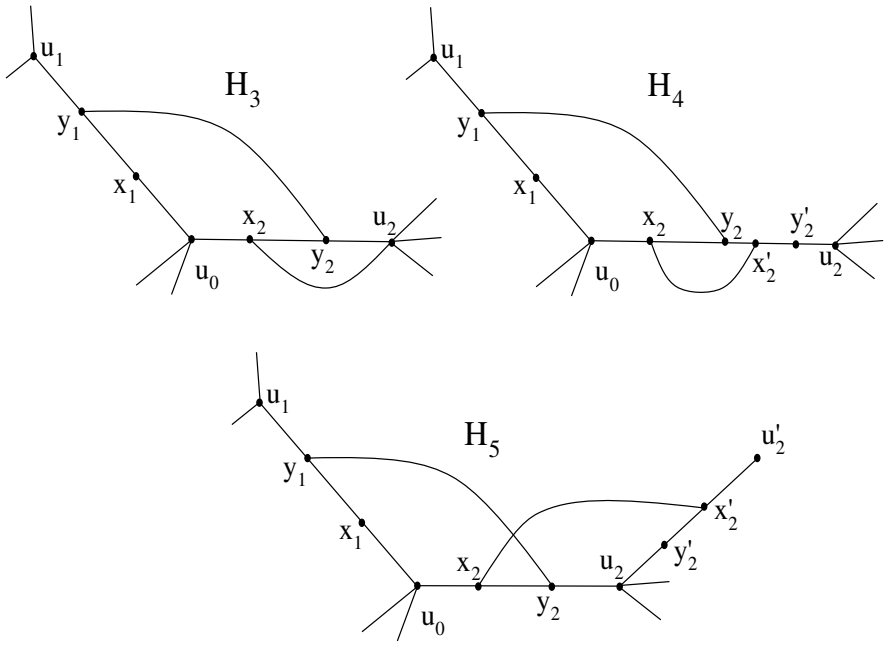



**Figure 15. The graphs $H_3, H_4$ and $H_5$ used in (8.2).**

Let us assume first that either $J = H_3$, or that $y_2'$ has degree two in $J'$. Then by deleting the edge $x_2u_2$ (and bicontracting $y_2'$ if $J \neq H_3$) we may use the proof of (8.1). By that argument the theorem holds, unless $J' = A_1(u_1)$, $J' = A_2(y_2)$, $J' = B_j(y_2y_1, y_2)$, $J' = B_j(x_2u_0, v)$, $J' = C_j(x_2u_0, y_2y_1)$ or $J' = C_4(y_2y_1, x_2u_0)$ for some $j \in \{3, 4\}$ and $v \in V(J) - \{x_1, y_1, u_0\}$.

If $J' = A_1(u_1)$, then $J' \backslash u_1y_1 \backslash x_1u_0 \backslash x_2u_2$ is isomorphic to a bisubdivision of $H$, and by

considering the edge $u_2x_2$ we deduce that $H + (u_2, u_0u_3) \hookrightarrow G$. If $J' = A_2(y_2)$ we delete the edge $y_1y_2$, bicontract the vertex $y_1$ and apply the previous argument.

Next, let $J' = B_3(y_2y_1, y_2)$. The graph obtained from $J'$ by deleting the edges $y_1\tau_4$ and $\tau_3y_2$ and bicontracting the vertices $y_1$ and $\tau_4$ is isomorphic to $A_2(y_2)$. Thus $H + (u_2, u_0u_3) \hookrightarrow G$. Similarly if $J' = B_4(y_2y_1, y_2)$ we delete the edges $y_1\tau_5$, $\tau_4\tau_6$ and $\tau_3y_2$ and bicontract the vertices $y_1$, $\tau_4$ and $\tau_6$ to demonstrate that $H + (u_2, u_0u_3) \hookrightarrow G$.

Our next step is to handle the cases $J' = B_j(x_2u_0, v)$ and $J' = C_j(x_2u_0, y_2y_1)$. Assume first that $j = 3$. If $v \notin \{u_2, x_2, y_2\}$, then by considering the edge $\tau_4v$ we deduce that $H + (v, u_0u_2) \hookrightarrow B_3(x_2u_0, v) \hookrightarrow G$, and similarly $H + (u_2, u_1u_0) \hookrightarrow C_3(x_2u_0, y_2y_1) \hookrightarrow G$. For the cases $v \in \{u_2, x_2, y_2\}$ let $L_3 = B_3(x_2u_0, v)\backslash y_1y_2\backslash x_1\tau_2\backslash\tau_1u_0\backslash\tau_4v\backslash x_2u_2$. By considering the edge $\tau_4v$ we deduce that $H + (u_2, u_0u_3) \hookrightarrow G$ if $v \in \{u_2, x_2\}$ and $H + (u_3, u_2u_0) \hookrightarrow G$ if $v = y_2$. Now assume $j = 4$. If $v \notin \{u_2, x_2, y_2\}$, then by considering the edge $\tau_6v$ we deduce that $H + (v, u_2u_0) \hookrightarrow B_4(x_2u_0, v) \hookrightarrow G$. If $v = u_2$ then let $L_4 = B_4(x_2u_0, u_2)\backslash x_2\backslash y_2\backslash x_1\tau_1\backslash\tau_4\tau_6\backslash\tau_5u_0$. By considering the edge $x_1\tau_1$ we deduce that $H + (u_3, u_0u_1) \hookrightarrow G$. If $v = x_2$ we get the same result by the graph obtained from $L_4$ by adding the path $x_2y_2u_2$, and if $v = y_2$ we add the path $y_2x_2u_2$ instead. The graph $C_4(x_2u_0, y_2y_1)$ has a matching minor isomorphic to a cross extension of $H$ (delete the edges $\tau_7y_2$ and $x_2u_2$; the cross extension has two vertices replaced by triangles). This concludes the cases $J' = B_j(x_2u_0, v)$ and $J' = C_j(x_2u_0, y_2y_1)$.

The graph $C_4(y_2y_1, x_2u_0)$ also has a matching minor isomorphic to a cross extension of $H$. To see that, delete the edges $u_2y_2$ and $x_2\tau_7$; the cross extension has two vertices replaced by triangles.

We may therefore assume that $J = H_4$ or $J = H_5$, and that $y_2'$ has degree three in $J'$. Thus $J' = A_j(y_2')$ or $J' = B_j(y_2y_1, y_2')$ or $J' = B_j(x_2u_0, y_2')$ for some $j$. Assume first that $J' = A_j(y_2')$. If $J = H_4$, then $J'$ is isomorphic to a cross extension of $H$ (with one or two vertices replaced by triangles depending on the value of $j$), and so we may assume that $J = H_5$. If $j = 2$, then by considering the edge $\tau_2y_2'$ we deduce that $H + (u_0, u_2u_2') \hookrightarrow G$, and so we may assume that $j = 1$. We may assume that $u_2' = u_1$, for otherwise by considering the edge $x_1y_2'$ we deduce that $H + (u_1, u_2u_2') \hookrightarrow G$. Now there is symmetry among $u_0, u_1, u_2$, and since we could assume $u_0$ had degree three, we may also assume $u_1$

and $u_2$ have degree three in $H$. The graph $K := J'\backslash u_0x_1\backslash x_2y_2\backslash u_2y_2'$ is isomorphic to a bisubdivision of $H$. If $u_2$ is not adjacent to $u_3$, then let $u_2''$ be the third neighbor of $u_2$; by considering $K$ and the edge $x_2y_2$ we see that $H + (u_3, u_2u_2'') \hookrightarrow G$, as desired. Thus we may assume that $u_2$ is adjacent to $u_3$, and by symmetry we may also assume that $u_1$ is adjacent to $u_3$. But $H$ is 2-connected, and hence $u_3$ is not a cutvertex; thus $H$ is isomorphic to $K_4$. It follows that $J'$ is isomorphic to the Petersen graph, as desired. This completes the case $J' = A_j(y_2')$.

Now let $J' = B_j(y_2y_1, y_2')$ or $J' = B_j(x_2u_0, y_2')$. If $J = H_4$, then $J'$ is isomorphic to a cube extension of $H$, and so we may assume that $J = H_5$. If $J' = B_j(y_2y_1, y_2')$ and $j = 1$, then by considering the path $y_2\tau_1\tau_2y_2'$ we deduce that $H + (u_2', u_2u_0) \hookrightarrow G$. The argument for $j > 1$ is analogous. Thus we may assume that $J' = B_j(x_2u_0, y_2')$. If $j = 1$, then by considering the path $\tau_2y_2'$ we deduce that $H + (u_2', u_0u_2) \hookrightarrow G$. The argument is analogous for $j > 1$ with the proviso that when $j$ is even the conclusion is $H + (u_2', u_2u_0) \hookrightarrow G$. □

We now turn our attention to edge-parallel extensions. Let us recall that $G/v$ denotes the graph obtained from the graph $G$ by bicontracting the vertex $v$.

**(8.3)** *Let $H$ be a graph of minimum degree at least three, and let $G$ be a brick. If an edge-parallel extension of $H$ is isomorphic to a matching minor of $G$, then a cross, cube, linear, quadratic, quartic or split extension of $H$ is isomorphic to a matching minor of $G$.*

*Proof.* By hypothesis there exists a vertex $u_0 \in V(H)$ of degree at least three with neighbors $u_1$ and $u_2$ such that the graph $H_2 := H + (u_2, u_1u_0)$ is isomorphic to a matching minor of $G$. Let $y_1, x_1$ be the new vertices of $H_2$; thus $u_0x_1y_1u_1$ is a path of $H_2$. Let $H_1 := H_2\backslash u_2y_1$. Since $H_2 \hookrightarrow G$, either a split extension of $H$ is isomorphic to a matching minor of $G$, or one of the graphs $H_2$, $H_3 = H_1 + (y_1, u_0u_2)$, $H_4 = H_1 + (y_1, u_2'u_2)$, where $u_2' \neq u_0$ is a neighbor of $u_2$, has a homeomorphic embedding into $G$. Graphs $H_3, H_4$ and $H_5$ are shown on Figure 15. Let $J$ denote that graph, and let it be chosen so that $J \neq H_2$, if possible. This choice implies that if a split extension of $J$ is isomorphic to a matching minor of $G$, then so is a split extension of $H$. Let $x_2, y_2$ be the new vertices of $H_3$ and $H_4$. If $J = H_2$ let $x_2 := u_2$ and let $y_2$ be undefined. We apply (6.1) to $J$ and the vertex $x_1$,

and so we may assume that (A), (B), or (C) holds, for otherwise the theorem holds. Let $J'$ be the graph satisfying (A), (B) or (C). Throughout this proof let $v \in V(J) - \{x_1, y_1, u_0\}$ and once again the symbols $\tau_1, \tau_2, \ldots$ will again refer to the new vertices of $J'$.

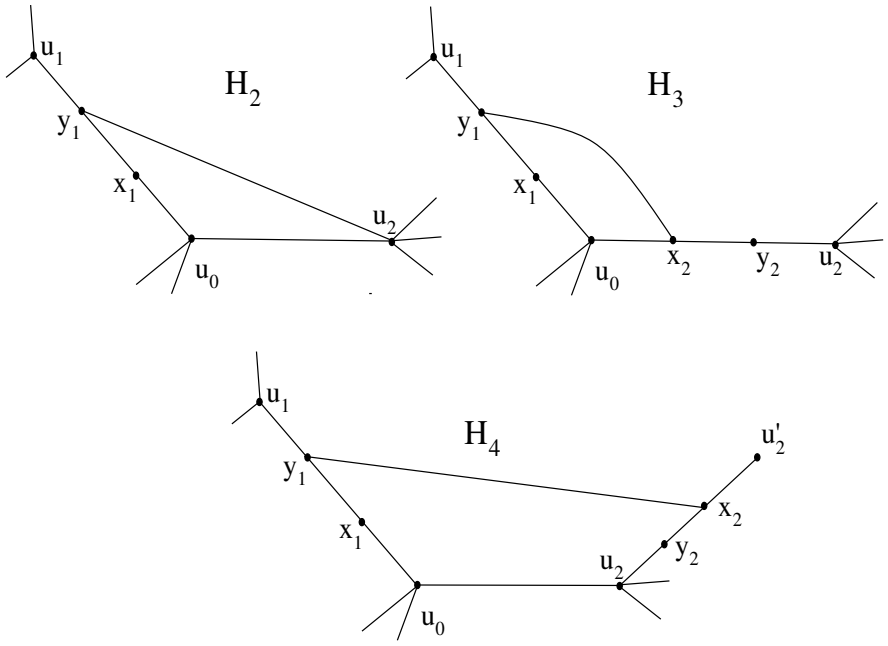



**Figure 16. The graphs $H_2, H_3$ and $H_4$ used in (8.3).**

We first notice that if $u_0$ has degree at least four, then $H_2 \backslash u_0u_2$ is isomorphic to a split extension of $H$, and so we may and will assume that $u_0$ has degree three. Let $u_3$ be the third neighbor of $u_0$. We now show that we may assume that if $J = H_4$, then $u_2$ has degree three. Indeed, if $J = H_4$ and $u_2$ has degree at least four then $H_4 \backslash u_0u_2/x_1$ is isomorphic to a split extension of $H$. So in the case $J = H_4$ let $u_2''$ be the third neighbor of $u_2$. Let $L$ be obtained from $J'$ by deleting $u_0u_2$ and all the "new" edges. Thus, for instance, if $J' = A_2(v)$, then $L = J' \backslash u_0u_2 \backslash x_1\tau_1 \backslash \tau_2 v$. Then $L/u_0/y_2$ is isomorphic to $H$.

Assume first that $J' = A_1(v) = J + (x_1, v)$. If $v = y_2$, then $J \in \{H_3, H_4\}$, and $J'$ is a cross extension of $H$ if $J = H_3$, and a quartic or cross extension of $H$ if $J = H_4$. Thus we may assume that $v \neq y_2$, and hence we may assume (by bicontracting $y_2$) that $J = H_2$. It follows that $J'$ is a quadratic extension of $H$, as desired. This completes the case $J' = A_1$.

Next we assume that $J' = A_2(v) = J + (x_1, u_0x_1) + (\tau_2, v)$. Assume first that $v = y_2$. If $J = H_3$, then $J'$ is a cross extension of $H$, and so we may assume that $J = H_4$. But then $J'\backslash x_1\tau_1/x_1/\tau_1$ is isomorphic to a quadratic extension of $H$. Thus we may assume that $v \neq y_2$, and hence, by bicontracting $y_2$, we may assume that $J = H_2$. If $v \neq u_1$, then $J'\backslash y_1u_2/y_1$ is a quadratic extension of $H$, and so we may assume that $v = u_1$. But then by considering the graph $L/u_0$ and edges $x_1\tau_1$ and $\tau_2u_1$ we deduce that a quadratic extension of $H$ is isomorphic to a matching minor of $G$. This completes the case $J' = A_2$.

Next we handle the cases $J' = B_j(x_2y_1, v)$. We start by assuming that $v = y_2$. If $J = H_3$, then $J'$ is isomorphic to a cube extension of $H$, and so we may assume that $J = H_4$. Recall the definiton of $L$ and that $u_2$ has degree three. If $j = 1$, then by considering $L$ and the edges $x_1\tau_1$ and $\tau_2y_2$ we deduce that a quadratic extension of $H$, namely $H + (u_2'', u_0u_2) + (\rho_2, u_3)$, is isomorphic to a matching minor of $G$. If $j = 2$, then by considering the edges $\tau_2\tau_3$ and $\tau_4y_2$ we deduce that the quadratic extension $H + (u_2'', u_2u_0) + (\rho_2, u_2)$ is isomorphic to a matching minor of $G$. An analogous argument applies when $j = 4$. If $j = 3$ then by deleting the edge $x_1\tau_1$ and bicontracting $x_1$ and $\tau_1$ we deduce that $H + (u_2'', u_0u_2) + (\rho_2, u_0) \hookrightarrow G$, as desired. Thus we may assume that $v \neq y_2$, and hence, by bicontracting $y_2$, we may assume that $J = H_2$. If $j = 1$, then by considering $L$ and the edges $x_1\tau_1$ and $\tau_2v$ we deduce that the quadratic extension $H+(u_3, u_2u_0)+(\rho_2, v)$ is isomorphic to a matching minor of $G$. Let $j = 2$. If $v \neq u_2$, then by considering $L$ and the edges $\tau_2\tau_3$ and $\tau_4v$ we deduce that the quadratic extension $H + (v, u_2u_0) + (\rho_2, u_2)$ is isomorphic to a matching minor of $G$. If $v = u_2$ then by considering the graph obtained from $L$ by replacing the edge $x_1y_1$ by $\tau_1x_1$ and considering the edges $\tau_2\tau_3$ and $\tau_4u_2$ we deduce that the quadratic extention $H + (u_2, u_1u_0) + (\rho_2, u_1)$ is isomorphic to a matching minor of $G$. Thus we may assume $j \in \{3, 4\}$. Let us assume that $v = u_1$. Then we may assume that $u_1$ is adjacent to $u_2$, for otherwise $H + (u_1, u_2) \hookrightarrow G$ (consider the path $u_1\tau_4\tau_3u_2$ when $j = 3$ and the analogous path for $j = 4$). If $j = 3$, then by replacing the edge $u_1u_2$ by the path $u_1\tau_4\tau_3u_2$ we obtain a graph isomorphic to a bisubdivision of $H$, and by considering the edges $y_1\tau_4$ and $\tau_2\tau_3$ we deduce that a quadratic extension of $H$, namely $H + (u_0, u_2u_1) + (\rho_2, u_0)$, is isomorphic to a matching minor of $G$. If $j = 4$ then by replacing the edge $u_1u_2$ by the path $u_1\tau_6\tau_5\tau_4\tau_3u_2$, by considering the edges $\tau_4\tau_6$ and

$y_1\tau_5$ and by bicontracting $x_1$ and $\tau_3$ we deduce that a quadratic extension of $H$, namely $H + (u_0, u_2u_1) + (\rho_2, u_2)$, is isomorphic to a matching minor of $G$. Thus we may assume that $v \neq u_1$. If $j = 3$, then by considering the edge $x_1\tau_1$ and path $\tau_2\tau_3\tau_4 v$ we see that the quadratic extension $H + (v, u_1u_0) + (\rho_2, u_1)$ is isomorphic to a matching minor of $G$; an analogous argument gives the same conclusion when $j = 4$.

The cases $J' = B_j(u_2u_0, v)$ can be reduced to the cases just handled by noticing that $J\backslash u_0u_2$ is isomorphic to a bisubdivision of $H$, and hence $J$ is isomorphic to the edge-parallel extension $H + (u_2, u_3u_0)$. Similarly the cases $J' = C_j(u_2y_1, u_2u_0)$ can be reduced to $J' = C_j(u_2u_0, u_2y_1)$, and so it remains to handle the cases $J' = C_j(u_2u_0, u_2y_1)$. But in all four of those cases a cross extension of $H$ is isomorphic to a matching minor of $G$. □

The results of this section allow us to strengthen (7.3) as follows.

**(8.4)** *Let $H$ and $G$ be graphs, where $H$ is 2-connected, has minimum degree at least three and is isomorphic to a matching minor of $G$, and $G$ is a brick. Assume that if $H$ is isomorphic to $K_4$, then $G$ has no matching minor isomorphic to the Petersen graph. If $H$ is not isomorphic to $G$, then a cross, cube, linear, quadratic or quartic extension of $H$ is isomorphic to a matching minor of $G$.*

*Proof.* By (7.3) we may assume that a vertex-parallel or an edge-parallel extension of $H$ is isomorphic to a matching minor of $G$. Thus the result follows from (8.1), (8.2) and (8.3). □

## 9. CUBE AND CROSS EXTENSIONS

In this section we strengthen (8.4) by eliminating cube and cross extensions from the conclusion.

**(9.1)** *Let $H$ be a graph, let $u$ be a vertex of $H$ of degree three, and let $u_1$ and $u_2$ be two neighbors of $u$. Let $H_1$ be obtained from $H$ by bisubdividing the edges $uu_1$ and $uu_2$ once, and let $x_1, y_1, x_2, y_2$ be the new vertices so that $u_1y_1x_1ux_2y_2u_2$ is a path. Let $H_2 := H_1 + (x_2, y_2x_2) + (\tau_2, x_1)$, let $H_3 := H_1 + (x_2, y_2x_2) + (\tau_2, x_1y_1) + (\tau_4, x_1)$, and let $H_4$*

*be obtained from $H_2$ or $H_3$ by replacing exactly one of the vertices $x_2, \tau_1, \tau_2$ by a triangle. Then each of $H_2, H_3, H_4$ has a matching minor isomorphic to a quadratic extension of $H$.*

*Proof.* Throughout this proof let $\tau_1, \tau_2$ denote the new vertices of $H_2$, and let $\tau_1, \tau_2, \tau_3, \tau_4$ denote the new vertices of $H_3$ with the usual numbering convention. We can naturally embed $H$ into $H_2$. By bicontracting $y_1$ and $y_2$ and considering edges $x_2\tau_1$ and $x_1\tau_2$, we see that $H_2$ is isomorphic to a bisubdivision of a quadratic extension of $H$. The graph $H_3\backslash\tau_1x_2\backslash x_1u\backslash\tau_3\tau_4$ is isomorphic to a bisubdivision of $H$ and by bicontracting $y_1, \tau_3$ and $\tau_4$ and considering edges $\tau_1x_2$ and $x_1u$ we deduce that $H_3$ has a matching minor isomorphic to a quadratic extension of $H$. This completes the proof for $H_2$ and $H_3$.

Suppose $H_4$ is obtained from $H_2$ by replacing $\tau_2$ with a triangle, then $H_4\backslash x_2\tau_1/x_2/\tau_1/y_1$ is isomorphic to a quadratic extension $H + (u_1, uu_2) + (\rho_2, u)$ of $H$. Similarly if $H_4$ is obtained from $H_2$ by replacing $x_2$ or $\tau_1$ with a triangle then $H_4\backslash x_1u/x_1/u/y_1$ is isomorphic to a quadratic extension of $H$.

It remains to consider the case when $H_4$ be obtained from $H_3$ by replacing exactly one of the vertices $x_2, \tau_1, \tau_2$ by a triangle. We need to make the following easy observation. If a graph $G_1$ is obtained from a graph $G$ by replacing a vertex $t \in V(G)$ of degree three with a triangle $T$ and $G_2$ is obtained from $G_1$ by replacing one of the vertices of $T$ by a triangle, then $G$ is isomorphic to a matching minor of $G_2$. Let $H_2' = H_1 + (x_1, y_1x_1) + (\rho_2, x_2)$. Clearly a graph obtained from $H_3$ by contracting a triangle with vertex set $\{x_2, \tau_1, \tau_2\}$ is isomorphic to $H_2'$. Therefore, by the observation above, $H_4$ contains $H_2'$ as a matching minor and $H_2'/y_1/y_2$ is isomorphic to a quadratic extension of $H$. □

**(9.2)** *Let $H$ be a graph of minimum degree at least three, and let $G$ be a brick. If a cube extension of $H$ is isomorphic to a matching minor of $G$, then a linear, cross or quadratic extension of $H$ is isomorphic to a matching minor of $G$.*

*Proof.* Let $u$ be a vertex of $H$ of degree three and let $u_1$,$u_2$ and $u_3$ be its neighbors. Let $H_0$ be obtained from $H$ by bisubdividing each of the edges $uu_1$, $uu_2$ and $uu_3$. Let the new vertices be $y_1, y_2, y_3$ and $z_1, z_2, z_3$ in such a way that $u_1y_1z_3u$, $u_2y_2z_1u$ and $u_3y_3z_2u$ are paths. Let $H_1 := H_0 + (y_1, z_2) + (y_2, z_3) + (y_3, z_1)$, and let $J$ be obtained from $H_1$ by

replacing a subset of $\{z_1, z_2, z_3\}$ by triangles. If $z_i$ is replaced by a triangle, then let the triangle be $Z_i$; otherwise, let $Z_i$ denote the graph with vertex-set $\{z_i\}$. By hypothesis the vertex $u$ and graph $J$ may be selected so that $J$ is isomorphic to a matching minor of $G$. Let $\eta : J \hookrightarrow G$. We may assume that $\eta$ is a homeomorphic embedding, for otherwise a split extension of $H$ is isomorphic to a matching minor of $G$ and the result holds by (5.9).

When $v \in V(J)$ we will abuse notation and use $\eta(v)$ to denote the unique vertex of the graph $\eta(v)$. With that in mind let $J' = \eta(J)$, let $u'_i = \eta(u_i)$, $u' = \eta(u)$ and $z'_i = \eta(z_i)$. For $i = 1, 2, 3$ let $P_i$ denote the path $\eta(u_i y_i)$. We may assume that $J$ and $\eta$ are chosen so that $|V(P_1)| + |V(P_2)| + |V(P_3)|$ is minimum.

Let $\Omega_1$ be the octopus with head $\eta(Z_1)$ and tentacles the paths of $\eta(J)$ joining $u'$, $y'_2$ and $y'_3$ to $Z_1$, and let $\Omega_2$ and $\Omega_3$ be defined analogously. Let $\Omega_4$ be the octopus with head $\eta(J \backslash V(Z_1) \backslash V(Z_2) \backslash V(Z_3) \backslash \{y_1, y_2, y_3, u\})$ and tentacles $P_1, P_2, P_3$, let $\mathcal{F} = \{\Omega_1, \Omega_2, \Omega_3, \Omega_4\}$, and let $Y' = \{y'_1, y'_2, y'_3, u'\}$. Then $(\mathcal{F}, Y')$ is a frame in $G$. Let $M$ be a perfect matching of $G \backslash V(\eta(J))$; then $M$ has a unique extension to a matching $M'$ that is $\Omega_i$-compatible for all $i = 1, 2, 3, 4$. By (2.3) there exist distinct integers $i, j \in \{1, 2, 3, 4\}$ and an $M'$-alternating path $S$ joining vertices $v_i$ and $v_j$, where $v_i$ belongs to the head of $\Omega_i$ and $v_j$ belongs to the head of $\Omega_j$, such that for some edge $e \in E(S) \backslash M'$ the two components of $S \setminus e$ may be denoted by $S_i$ and $S_j$ so that $V(S_i) \cap V(\mathcal{F}) \subseteq V(\Omega_i)$ and $V(S_j) \cap V(\mathcal{F}) \subseteq V(\Omega_j)$.

Assume first that $j = 4$. Then from the symmetry we may assume that $i = 2$. In this case it will be convenient to allow $v_4$ to be an internal vertex of a tentacle of $\Omega_4$. By doing so we may assume (by replacing $S$ by its subpath) that $v_4$ is the only vertex of $S \cap \Omega_4$. If for some $l \in \{1, 2, 3\}$ we have $v_4 \in V(P_l)$ and $P_l[u'_l, v_4]$ is even, then let $v = u_l$; if $v_4 \in V(P_l)$ and $P_l[u'_l, v_4]$ is odd, then $v$ is undefined. If $v_4$ belongs to $V(\eta(z))$ for some $z \in V(J)$, then let $v = z$. Finally, if $v \in V(\eta(zz'))$ for some edge $zz' \in E(H \setminus u)$, then $v_4$ is at even distance on $\eta(zz')$ from exactly one of $\eta(z)$, $\eta(z')$, say from $\eta(z)$. In that case we put $v = z$. Notice that if $v$ is defined, then $v \in V(H) - \{u\}$. From the symmetry we may assume $v \neq u_1$ and $v_4 \notin V(P_1)$. By (3.6) the graph $\Omega_2 \cup S_2 + e$ includes a triad or tripod $T$ with ends $y'_1, u', v_4$.

We claim that if $v_4$ belongs to $P_3$, then the path $P_i[v_4, u'_3]$ is even. Indeed, otherwise

by making use of $T$, $\Omega_1$ and $\Omega_3$ we obtain contradiction to the minimality of $|V(P_1)| + |V(P_2)| + |V(P_3)|$. This proves that if $v$ is undefined then $v_4 \in V(P_2)$. In that case by deleting the path of $\eta(J)$ joining $y_2'$ and $Z_1$ and by considering the path of $\eta(J)$ joining $y_1'$ and $Z_3$ and using $T$ we deduce that a cross extension of $H$ is isomorphic to a matching minor of $G$. If $v$ is defined, then one of the following graphs is isomorphic to a matching minor of $G$:

- $H + (v, uu_1) + (\tau_2, uu_2)$, if $T$ is a triad and $Z_3 = \{z_3\}$,
- $H + (v, uu_1) + (\tau_2, u_2u)$, if $T$ is a triad and $Z_3$ is a triangle,
- $H + (v, u_1u) + (\tau_2, \tau_1u_1)$, if $T$ is a tripod.

But each of the above graphs has a matching minor isomorphic to a quadratic extension of $H$. This completes the case $j = 4$.

Thus we may assume that $i = 1$ and $j = 2$. By (3.6) $\Omega_1 \cup S_1 + e$ includes a triad or tripod $T_1$ with ends $y_3', u', s_2$ and $\Omega_2 \cup S_2 + e$ includes a triad or tripod $T_2$ with ends $y_1', u', s_1$, where $s_1 \in V(S_1), s_2 \in V(S_2)$ are the ends of $e$. If either $T_1$ or $T_2$ is a tripod then the required result follows from (9.1) by deleting the path of $\eta(J)$ joining $y_1'$ and $Z_3$ and making use of $T_1$ and $T_2$. If both $T_1$ and $T_2$ are triads then one of the following graphs is isomorphic to a matching minor of $G$:

- $H + (uu_3, uu_1) + (\tau_4, uu_2)$, if $Z_3$ is not a triangle,
- $H + (uu_3, uu_1) + (\tau_4, u_2u)$, if $Z_3$ is a triangle.

Both of these graphs have matching minors isomorphic to quadratic extensions of $H$. □

**(9.3)** *Let $H$ be a graph, let $J$ be a cross extension of $H$ and let $v$ be the hub of $J$. If the degree of $v$ in $H$ is at least four then a split extension of $H$ is isomorphic to a matching minor of $J$.*

*Proof.* Let $x_1, y_1, x_2, y_2$ and $K'$ be as in the definition of cross extension. If $J = K'$ then $J\backslash vx_1\backslash x_2y_1/x_1$ is isomorphic to a split extension of $H$. If $J \neq K'$ the argument is analogous. □

**(9.4)** *Let $H$ be a graph of minimum degree at least three, and let $G$ be a brick. If a cross extension of $H$ is isomorphic to a matching minor of $G$, then a linear or quadratic extension of $H$ is isomorphic to a matching minor of $G$.*

*Proof.* Let $u$ be a vertex of $H$ of degree three and let $u_1$,$u_2$ and $u_3$ be its neighbors. Let $H_1$ be a cross extension of $H$ obtained by deleting the vertex $u$ and adding the vertices $x_1, x_2, y_1, y_2, y_3$ and edges $y_j u_j$ and $y_j x_i$ for all $i = 1, 2$ and $j = 1, 2, 3$. Let $H_2$ be obtained from $H_1$ by replacing $x_1$ by the triangle $X_1$, and let $H_3$ be obtained from $H_2$ by replacing $x_2$ by the triangle $X_2$. Let the vertices of $X_1$ be $a_1, a_2, a_3$ such that $a_i$ is adjacent to $y_i$, and let the vertices of $X_2$ be $b_1, b_2, b_3$ such that $b_i$ is adjacent to $y_i$. By hypothesis, (9.3) and (5.10) we may assume that there exist a vertex $u$ of $H$ of degree three, a graph $J \in \{H_1, H_2, H_3\}$, and an embedding $\eta : J \hookrightarrow G$. If $J \neq H_3$ we define $X_2$ to be the subgraph of $J$ with vertex-set $\{x_2\}$ and let $b_1 = b_2 = b_3 = x_2$, and if $J = H_1$ we define $X_1$ to be the subgraph of $J$ with vertex-set $\{x_1\}$ and let $a_1 = a_2 = a_3 = x_1$. By (5.9) we may assume that $\eta$ is a homeomorphic embedding. Let $J' = \eta(J)$, let $u'_i = \eta(u_i)$, and $y'_i = \eta(y_i)$. Let $P_i$ denote the path $\eta(u_i y_i)$. We may assume that $J$ and $\eta$ are chosen so that $|V(P_1)| + |V(P_2)| + |V(P_3)|$ is minimum.

Let $\Omega_1$ be the octopus with head $\eta(X_1)$ and tentacles $\eta(a_j y_j)$, where $j = 1, 2, 3$, and let $\Omega_2$ be defined analogously. Let $\Omega_3$ be the octopus with head $\eta(J \backslash V(X_1) \backslash V(X_2) \backslash \{y_1, y_2, y_3\})$ and tentacles $P_1, P_2, P_3$, let $\mathcal{F} = \{\Omega_1, \Omega_2, \Omega_3\}$, and let $Y' = \{y'_1, y'_2, y'_3\}$. Then $(\mathcal{F}, Y')$ is a frame in $G$. Let $M$ be a perfect matching of $G \backslash V(\eta(J))$; then $M$ has a unique extension to a matching $M'$ that is $\Omega_i$-compatible for all $i = 1, 2, 3$. By (2.3) there exist distinct integers $i, j \in \{1, 2, 3\}$ and an $M'$-alternating path $S$ joining vertices $v_i$ and $v_j$, where $v_i$ belongs to the head of $\Omega_i$ and $v_j$ belongs to the head of $\Omega_j$, and an edge $e \in E(S) \backslash M'$ such that the components of $S \backslash e$ may be denoted by $S_i$ and $S_j$ so that $V(S_i) \cap V(\mathcal{F}) \subseteq V(\Omega_i)$ and $V(S_j) \cap V(\mathcal{F}) \subseteq V(\Omega_j)$.

Assume first that $j = 3$. In this case it will be convenient to allow $v_3$ to be an internal vertex of a tentacle of $\Omega_3$. By doing so we may assume (by replacing $S$ by its subpath) that $v_3$ is the only vertex of $S \cap \Omega_3$. If $v_3 \in V(P_i)$, then let $v = u_i$. If $v_3$ belongs to $V(\eta(z))$ for some $z \in V(J)$, then let $v := z$. Finally, if $v \in V(\eta(zz'))$ for some edge

$zz' \in E(J)$, then $v_3$ is at even distance on $\eta(zz')$ from exactly one of $\eta(z)$, $\eta(z')$, say from $\eta(z)$. In that case we put $v := z$. We may assume that $v \in V(H) - \{u, u_1, u_2\}$, and that if $v_3 \in V(P_1 \cup P_2 \cup P_3)$ then $v_3 \in V(P_3)$. By (3.6) we may assume that $S \cup \Omega_i$ includes a triad or tripod $T$ with ends $y_1', y_2', v_3$. We claim that if $v_3$ belongs to $P_3$, then the path $P_3[v_3, u_3']$ is even. Indeed, otherwise by making use of $T$ and $\Omega_{3-i}$ we obtain contradiction to the minimality of $|V(P_1)| + |V(P_2)| + |V(P_3)|$. We deduce that of the following graphs is isomorphic to a matching minor of $G$:

- $H_1 \backslash x_1 y_3 + (x_1, v)$,
- $H_2 \backslash x_2 y_3 + (x_2, v)$,
- $H_2 \backslash a_3 y_3 + (a_3, v)$,
- $H_3 \backslash a_3 y_3 + (a_3, v)$.

But each of the above graphs has a matching minor isomorphic to a quadratic extension of $H$ (in the first case we bicontract $y_3$ and consider the edges $x_1 v$ and $y_1 x_2$). In the second case delete $a_2 a_3$, bicontract its ends and consider the edges $y_1 a_1$ and $x_2 v$; in the third case delete $y_1 x_2$, bicontract its ends, and consider the edges $a_1 a_2$ and $a_3 v$; and in the fourth case consider the same two edges, delete $y_1 b_1$ and $b_2 b_3$ and bicontract their ends. This completes the case $j = 3$.

Thus we may assume that $i = 1$ and $j = 2$. Let $s_1 \in V(S_1)$ and $s_2 \in V(S_2)$ be the ends of $e$. We apply (3.7) to $S_2 \cup \Omega_2$ to conclude that $\Omega_2 \cup S_2 + e$ has a central subgraph $T_2$ such that $T_2$ is either a quadropod with ends $y_1', y_2', y_3', s_1$, or a quasi-tripod, in which case we may assume by symmetry that its ends are $y_1', y_2', s_1$. By (3.6) the graph $\Omega_1 \cup S_1 + e$ has a central subgraph $T_1$ that is a triad or tripod with ends $y_1', y_3', s_2$. If $T_2$ is a quasi-tripod then the theorem holds by (9.1). If $T_2$ is a quadropod with ends $y_1', y_2', y_3', s_1$, then one of the following graphs is isomorphic to a matching minor of $G$:

- $H_1 \backslash x_1 y_2 + (x_1, x_2)$,
- $H_2 \backslash a_2 y_2 + (a_2, x_2)$.

Both of these graphs have a matching minor isomorphic to a suitable extension of $H$. In the first case we get a quadratic extension by bicontracting $y_2$ and considering the edges $x_2 y_1$ and $x_2 x_1$. In the second case we get a quadratic extension by deleting $x_2 y_1$, bicontracting $y_1$ and $y_2$ and considering the edges $x_2 a_2$ and $a_1 a_3$ . □

Using (9.2) and (9.4) we can upgrade (8.4) to the following statement.

**(9.5)** *Let $H$ and $G$ be graphs, where $H$ is 2-connected and has minimum degree at least three, $G$ is a brick and $H$ is isomorphic to a matching minor of $G$. Assume that if $H$ is isomorphic to $K_4$, then $G$ has no matching minor isomorphic to the Petersen graph. If $H$ is not isomorphic to $G$, then a linear, quadratic or quartic extension of $H$ is isomorphic to a matching minor of $G$.*

*Proof.* This follows immediately from (8.4), (9.2) and (9.4). □

## 10. EXCEPTIONAL FAMILIES

We now handle quadratic extensions. The next lemma will show that a quadratic extension gives rise to a linear extension, unless it is of one of the following two types. Let $H, u, v, x, y, x', y', H'$ be as in the definition of quadratic extension; that is, $H$ is a graph, $uv \in E(H)$, $H'$ is obtained from $H$ by bisubdividing $uv$, where the new vertices $x, y$ are such that $x$ is adjacent to $u$ and $y$. Further, $x' \in V(H) - \{u\}$ and $y' \in V(H) - \{v\}$ do not both belong to $\{u, v\}$. Let $H_1 = H' + (x, x') + (y, y')$ be a quadratic extention of $H$. If $y' = u$, $x'$ is adjacent to $v$, and $v$ has degree three, then we say that $H_1$ is an *alpha extension of $H$*. If $x', y' \in V(H) - \{u, v\}$, $x'$ is adjacent to $v$, $y'$ is adjacent to $u$ and both $u$ and $v$ have degree three, then we say that $H_1$ is a *prism extension of $H$*. An alpha and a prism extension are shown in Figure 17.

**(10.1)** *Let $H$ be a graph of minimum degree at least three, and let $K$ be a quadratic extension of $H$. Then $K$ has a matching minor isomorphic to a linear, alpha or prism extension of $H$. Furthermore, if $H, u, v, x, y, x', y', H'$ are as in the definition of quadratic extension and $x', y' \in V(H) - \{u, v\}$, then $K$ has a matching minor isomorphic to a linear or prism extension of $H$.*

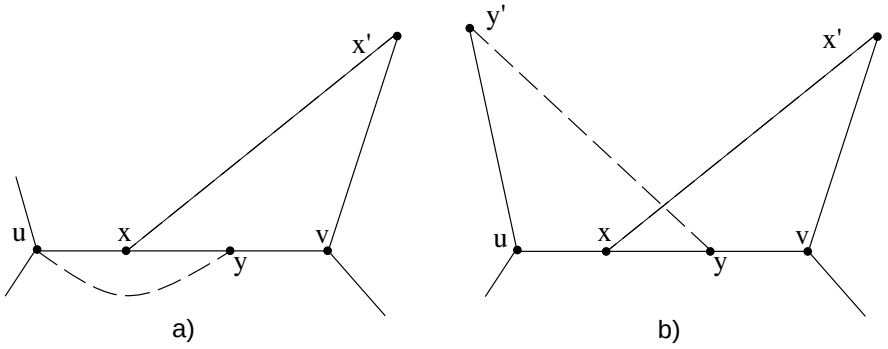



**Figure 17. (a) An alpha extension; (b) a prism extension.**

*Proof.* Let $H, u, v, x, y, x', y', H'$ be as in the definition of quadratic extention, and let $K = H' + (x, x') + (y, y')$ be a quadratic extension of $H$. By symmetry we may assume that $y' \neq u$. If $y'$ is not adjacent to $u$, then $H + (u, y') \hookrightarrow K$, as desired. Thus we may assume that $y'$ is adjacent to $u$. If $u$ has degree at least four, then $K \backslash uy'$ is isomorphic to a linear extension of $H$, as desired. Thus we may assume that $u$ has degree three. If $x' \neq v$, then by symmetry $K$ is a prism extension of $H$, and if $x' = v$, then $K$ is an alpha extension of $H$, as desired. □

**(10.2)** *Let $K$ be an alpha extension of a graph $H$ of minimum degree at least three. Then $K$ has a matching minor isomorphic to a linear or prism extension of $H$.*

*Proof.* Let $H, u, v, x, y, x', y', H'$ be as in the definition of quadratic extention, and let $K = H' + (x, x') + (y, y')$ be an alpha extension of $H$, where $y' = u$. Thus $v$ has degree three and is adjacent to $x'$. There exists a homeomorphic embedding $\eta : H \hookrightarrow K$ with $\eta(v) = x$ and $\eta(z) = z$ for $z \in V(H) - \{v\}$, and by considering $\eta(H)$ and the edges $vx'$ and $uy$ we deduce that $K$ is isomorphic to a quadratic extension of $H$ that satisfies the second statement of (10.1). Thus the lemma holds by that statement. □

Let $H$ be a graph. By a *fan* in $H$ we mean a sequence of vertices $(x, y, u_1, u_2, \ldots, u_k)$ such that these vertices are pairwise distinct, except that possibly $x = y$, and further $k \geq 2$, $u_1, u_2, \ldots, u_k$ all have degree three and form a path in $H$ in the order listed, and

for $i = 1, 2, \ldots, k$ the vertex $u_i$ is adjacent to $x$ if $i$ is even, and otherwise it is adjacent to $y$.

**(10.3)** *Let $K$ be a prism extension of a 3-connected graph $H$. If $K$ is not a prismoid, a wheel or a biwheel, then $K$ has a matching minor isomorphic to a linear extension of $H$.*

*Proof.* By hypothesis there exists a fan $(x, y, u_1, u_2)$ in $H$ such that $K = H + (x, u_1u_2) + (y, \tau_2)$. Let $t_1, t_2$ denote the new vertices $\tau_1, \tau_2$ of $K$, respectively. Let us choose a maximum integer $k$ such that $H$ has a fan $(x, y, u_1, u_2, \ldots, u_k)$ such that $H + (x, u_1u_2) + (y, \tau_2) \hookrightarrow K$. Let $u_0$ be the neighbor of $u_1$ other than $u_2$ and $y$. Now $u_0 \neq u_k$, for otherwise $H$ is a wheel or a biwheel (depending on whether $x$ and $y$ are distinct or not). Assume first that $u_0 \neq x$. There exists an embedding $\eta : H \hookrightarrow K$ such that $\eta(u_1) = t_2$. By considering the edges $u_1y$ and $xt_1$ we deduce that $H + (y, u_0u_1) + (x, \tau_2) \hookrightarrow K$, and by using the proof of (10.1) we deduce that either a linear extension of $H$ is isomorphic to a matching minor of $K$, or that $x$ is adjacent to $u_0$ and that $u_0$ has degree three. But then the fan $(y, x, u_0, u_1, \ldots, u_k)$ contradicts the maximality of $k$. Thus we may assume that $u_0 = x$, and by symmetry we may assume that $u_k$ is adjacent to both $x$ and $y$. It follows from the 3-connectivity of $H$ that $K$ is a prismoid, as desired. □

We now turn to quartic extensions. Again, we will show that a quartic extension gives rise to a linear extension, unless it is of two special types, the following ones. Let $H$ be a graph, and let $u, v, H', x, y, a, b$ be as in the definition of a quartic extension. That is, $uv \in E(H)$, $H'$ is obtained from $H$ by bisubdividing $uv$, where the new vertices are $x, y$ numbered so that $x$ is adjacent to $u$ and $y$, and let $K = H + (x, ab) + (\tau_2, y)$ be a quartic extension of $H$. If $b = v$ and the vertices $u$ and $a$ are adjacent and both have degree three, then we say that $K$ is a *staircase extension of $H$*. If $a, b, u, v$ are pairwise distinct, all have degree three, $a$ is adjacent to $u$ and $b$ is adjacent to $v$, then we say that $K$ is a *ladder extension of $H$*. We also say that the extension is *based at $u, v, b, a$* (in that order). A staircase and a ladder extension are shown on Figure 17.

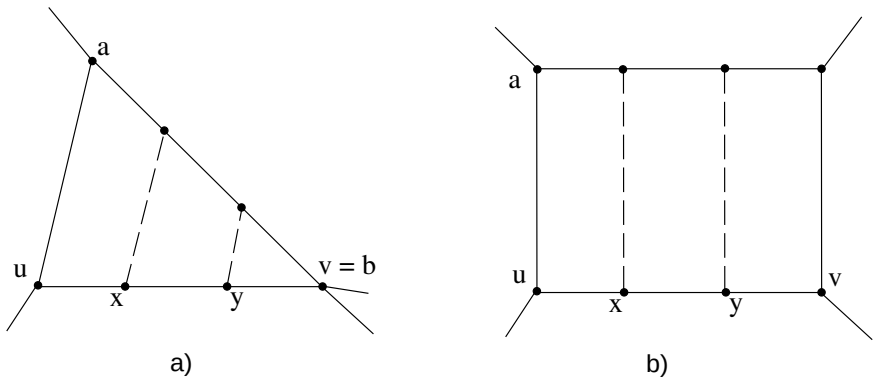



**Figure 18. (a) A staircase extension; (b) a ladder extension.**

**(10.4)** *Let $H$ be a graph of minimum degree at least three, and let $K$ be a quartic extension of $H$. Then $K$ has a matching minor isomorphic to a linear, staircase or ladder extension of $H$.*

*Proof.* If $a$ and $u$ are not equal or adjacent, then $H + au \hookrightarrow K$ (delete $x\tau_1$ and bicontract its ends), and hence the theorem holds. Assume now that $a$ and $u$ are adjacent. If both $u$ and $a$ have degree at least four, then $K\backslash au$ is a linear extension of $H$. If exactly one of $a$, $u$ has degree three, say $a$ does, then the graph obtained from $K\backslash au$ by bicontracting $a$ is isomorphic to a linear extension of $H$. Thus if $a \neq u$, and either they are not adjacent or one of them has degree at least four, then a linear extension of $H$ is isomorphic to a matching minor of $K$. By symmetry the same conclusion holds about the vertices $v$ and $b$, and the lemma follows. □

**(10.5)** *Let $K$ be a staircase extension of a 3-connected graph $H$. If $H$ has at least five vertices, then a linear or ladder extension of $H$ is isomorphic to a matching minor of $K$.*

*Proof.* Let $K = H' + x_1x_2 + y_1y_2$, where $H'$ is obtained from $H$ by bisubdividing the edges $vv_1$ and $vv_2$ so that $v_1y_1x_1vx_2y_2v_2$ is a path of $H'$, and assume that $v_1$, $v_2$ have degree three and are adjacent to each other. Let $v_1', v_2'$ be the third neighbors of $v_1$ and $v_2$, respectively. If $v_1'$ and $v_2'$ are not equal or adjacent, then $H + v_1'v_2' \hookrightarrow K$ (bicontract $v_1$ and $v_2$ in $K\backslash v_1v_2$), and so the lemma holds. If $v_1'$ and $v_2'$ are adjacent, then $K$ can be

regarded as a ladder extension of $H$, and if $v_1' = v_2'$, then the 3-connectivity of $H$ implies that it is isomorphic to $K_4$, contrary to hypothesis. □

A *fence* in a graph $H$ is a sequence $(u_1, v_1, u_2, v_2, \ldots, u_k, v_k)$ of distinct vertices of $H$ such that $k \geq 2$, each of theses vertices has degree three, $u_1u_2 \ldots u_k$ and $v_1v_2 \ldots v_k$ are paths and $u_i$ is adjacent to $v_i$ for all $i = 1, 2, \ldots, k$.

**(10.6)** *Let $K$ be a ladder extension of a 3-connected graph $H$ on an even number of vertices. If $K$ is not a ladder or a staircase, then $K$ has a matching minor isomorphic to a linear extension of $H$.*

*Proof.* By hypothesis there exists a fence $(u_1, v_1, u_2, v_2, \ldots, u_k, v_k)$ in $H$ such that $K = H' + x_1y_1 + x_2y_2$, where $H'$ is obtained from $H$ by bisubdividing $u_1u_2$ and $v_1v_2$ and $x_1, x_2, y_1, y_2$ are the new vertices numbered so that $u_1x_1x_2u_2v_2y_2y_1v_1$ is a cycle in $H'$. We may assume that the fence is chosen with $k$ maximum. Let $u_0, v_0$ be the third neighbors of $u_1, v_1$, respectively. Assume first that $u_0 \neq v_0$. Since the quartic extension of $H$ based at $u_0, u_1, v_1, v_0$ is isomorphic to $K$, the argument in the proof of (10.4) shows that either a linear extension of $H$ is isomorphic to a matching minor of $K$, or that $u_0$ and $v_0$ are adjacent and both have degree three. We may assume the latter, for otherwise the lemma holds. By the maximality of $k$ the sequence $(u_0, v_0, u_1, v_1, \ldots, u_k, v_k)$ is not a fence in $H$, and hence we may assume that $u_0 = u_k$ or $u_0 = v_k$. But $H$ is 3-connected, and so in the former case $K$ is a planar ladder, and in the latter case it is a Möbius ladder. Thus we may assume that $u_0 = v_0$. The ladder extension of $H$ based at $u_{k-1}u_kv_kv_{k-1}$ is clearly isomorphic to $K$, and hence the above argument shows that we may assume that the third neighbors of $u_k$ and $v_k$ are equal. Since $H$ is 3-connected and has an even number of vertices, it is a staircase. □

The following result summarizes the previous lemmas.

**(10.7)** *Let $K$ be a quadratic or quartic extension of a 3-connected graph $H$ on an even number of vertices, and assume that $K$ is not a prismoid, wheel, biwheel, ladder or staircase. Then a linear extension of $H$ is isomorphic to a matching minor of $K$.*

*Proof.* If $H$ is isomorphic to $K_4$, then $K$ is not a staircase extension of $H$, because $K$ is not a staircase. Thus the lemma follows from the results of this section. □

We are now ready to prove Theorem (1.11). Let $H$ and $G$ be as stated therein, and assume that they are not isomorphic. Assume first that either $H$ is not isomorphic to $K_4$, or $G$ has no matching minor isomorphic to the Petersen graph. By (9.5) we may assume that a quadratic or quartic extension $K$ of $H$ is isomorphic to a matching minor of $G$. It follows from the hypothesis of (1.11) that $K$ is not a prismoid, wheel, biwheel, ladder or staircase. Thus $K$ has a matching minor isomorphic to a linear extension of $H$ by (10.7), and hence so does $G$, as desired. Thus we may assume that $H$ is isomorphic to $K_4$ and $G$ has a matching minor isomorphic to the Petersen graph. But $G$ is not isomorphic to the Petersen graph by hypothesis. Since we have already shown that (1.11) holds when $H$ is the Petersen graph, we may now apply it to deduce that G has a matching minor isomorphic to a linear extension of the Petersen graph. The Petersen graph has, up to isomorphism, a unique linear extension, and this linear extension has a matching minor isomorphic to the staircase on eight vertices. But the latter graph has a matching minor isomorphic to $K_4$, the staircase on four vertices, contrary to hypothesis. □

## 11. A GENERALIZATION

In this section we state a generalization of (1.11), and point out how it follows from the theory that we developed. Let $G$ be a graph with a perfect matching. Let us recall that a barrier in $G$ is a set $X \subseteq V(G)$ such that $G\backslash X$ has at least $|X|$ odd components, and that bricks are 3-connected graphs with perfect matchings and no barriers of size at least two. Braces almost have no barriers, either, for if $X$ is a barrier in a brace and $X$ has at least two elements, then $X$ is one of the two color classes of $G$. We use this fact to weaken the condition on bricks. Let $s \geq 0$ be an integer. We say that a set $X \subseteq V(G)$ is an $s$-barrier in $G$ if $G\backslash X$ has $|X|-1$ odd components such that the union of the remaining components of $G\backslash X$ has at least $s$ vertices. We say that a graph is an *$s$-brick* if it is 3-connected and has no $s$-barrier of size at least two. Thus bricks are 1-bricks and braces are 2-bricks. Our

proof of (1.11) actually proves the following more general theorem. A *pinched staircase* is a graph obtained from a staircase by contracting the edge $v_1v_2$, where the vertices $v_1$ and $v_2$ are as in the definition of a staircase.

**(11.1)** *Let $s \geq 0$ be an integer, $G$ be an $s$-brick other than the Petersen graph, and let $H$ be a 3-connected matching minor of $G$ on at least $s+1$ vertices. Assume that if $H$ is a planar ladder, then there is no strictly larger planar ladder $L$ with $H \hookrightarrow L \hookrightarrow G$, and similarly for Möbius ladders, wheels, lower biwheels, upper biwheels, staircases, pinched staircases, lower prismoids and upper prismoids. If $H$ is not isomorphic to $G$, then some matching minor of $G$ is isomorphic to a linear extension of $H$.*

*Proof.* The proof follows the proof of (1.11), with the following minor modifications. In (2.2) the set $R_k$ is not required to be odd, but instead must have at least $s$ vertices. The proof goes through with the obvious changes. Then the definition of octopus needs to be changed to permit heads with even number of vertices, and in the definition of frame we need to add a condition guaranteeing that the heads of $\Omega_1, \Omega_2, \ldots, \Omega_{k-1}$ are odd and that the head of $\Omega_k$ has at least $s$ vertices. The assumption that $H$ has at least $s+1$ vertices will guarantee that this additional condition is satisfied whenever (2.3) is applied. Finally, in (10.6) the assumption that $H$ has an even number of vertices can be replaced by assuming that $K$ is not a pinched staircase. □

Clearly (11.1) implies (1.11) on taking $s = 1$. Let us now turn to braces. Let $L$ be a linear extension of a brace $H$. Then $L$ need not be a brace, but if $L$ is bipartite, then it is a brace. Furthermore, if $L$ is isomorphic to a matching minor of a bipartite graph, then $L$ itself is bipartite. Thus (11.1) implies (1.9) by taking $s = 2$. The third application of (11.1) is to factor-critical graphs. A graph $G$ is *factor-critical* if $G\backslash v$ has a perfect matching for every vertex $v \in V(G)$. It is easy to see that every 1-brick on an odd number of vertices is factor-critical. Thus the following immediate corollary of (11.1) gives a generation theorem for a subclass of factor-critical graphs.

**(11.2)** *Let $G$ be a 1-brick on an odd number of vertices, and let $H$ be a 3-connected matching minor of $G$. Assume that if $H$ is a wheel, then there is no strictly larger wheel $W$ with $H \hookrightarrow W \hookrightarrow G$, and similarly for pinched staircases, lower prismoids and upper prismoids. If $H$ is not isomorphic to $G$, then some matching minor of $G$ is isomorphic to a linear extension of $H$.*

Unfortunately, a linear extension of a 1-brick need not be a 1-brick.

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