CONNECTIVITY OF ADDABLE CLASSES OF FORESTS (DRAFT)

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ABSTRACT. We improve the bound from [1] on connectivity of addable graph classes.

Let $\mathcal{A}$ be a class of graphs and let $\mathcal{A}_n$ be the set of graphs in $\mathcal{A}$ on the vertex set $\{1,\dots,n\}$. Let $p(\mathcal{A}_n)$ denote the probability that an element of $\mathcal{A}_n$ chosen uniformly at random is connected. We say that $\mathcal{A}$ is *addable* if the graph $G+e$ belongs to $\mathcal{A}$ for every graph $G\in\mathcal{A}$ and every edge e joining to vertices in different components of G.

Theorem 1. *If $\mathcal{A}$ is an addable class of graphs then*

$$\lim_{n\to\infty}\inf p(\mathcal{A}_n)\geq e^{-2/3}.$$

As described in [2], the results of [1] can be deduced from the following rather technical theorem concerning the relationship between the classes of all trees and all two component forests on n vertices. We use the notation from [2]. Let $\mathcal{F}_1$ be the set of all trees on the vertex set $\{1,\dots,n\}$, and let $\mathcal{F}_2$ be the set of all two component forests on this vertex set. Let B_n be a bipartite graph with the bipartition $(\mathcal{F}_1,\mathcal{F}_2)$ and $T\in\mathcal{F}_1$ joined by an edge to $F\in\mathcal{F}_2$ if and only if $T\supset F$. For a vertex $G\in V(B_n)$ we denote by $\delta(G)$ the set of all edges of B_n incident to G.

Theorem 2. *There exists a weight function $w: E(B_n)\to\mathbb{R}_+$ such that*

(a) $w(\delta(T))=1$ *for every* $T\in\mathcal{F}_1$, *and*
(b) $w(\delta(F))\geq\frac{3}{2}-o_n(1)$ *for every* $F\in\mathcal{F}_2$.

Proof. We define w in several steps. We start by fixing $T\in\mathcal{F}_1$ and defining a discharging function $d_T: E(T)\times E(T)\to\mathbb{R}_+$. For $e\in E(T)$, let T_1 and T_2 be the two components of $T\setminus e$. If $|V(T_1)|=|V(T_2)|$, let $d_T(e,e)=1$ and $d_T(e,f)=0$ for every $f\in E(T)-\{e\}$. Otherwise, we assume without loss of generality that $|V(T_1)|<|V(T_2)|$ and set $d_T(e,f)=1/|V(T_1)|$ for every $f\in E(T_1)\cup\{e\}$, and $d_T(e,f)=0$ for every $f\in E(T_2)$. We have

$$\sum_{f\in E(T)} d_T(e,f)=1 \tag{1}$$

for every $e\in E(T)$.

We are now ready to define a weight function $w': E(B_n)\to\mathbb{R}_+$, which will be close to satisfying the theorem. We will later obtain w by smoothing w'. Fix $T\in\mathcal{F}_1$ and $F\in\mathcal{F}_2$ with $F\subset T$. Let $\{f\}=E(T)-E(F)$ and define $w'(T,F)=\frac{1}{n-1}\sum_{e\in E(T)} d_T(e,f)$. We have, using (1),

$$w'(\delta(T)) = \sum_{f\in E(T)} w'(T, T\setminus f) = \sum_{f\in E(T)} \left(\frac{1}{n-1}\sum_{e\in E(T)} d_T(e,f)\right)$$
$$= \frac{1}{n-1}\sum_{e\in E(T)}\left(\sum_{f\in E(T)} d_T(e,f)\right) = 1 \tag{2}$$

for every $T\in\mathcal{F}_1$.

We proceed to estimate $w'(\delta(F))$ for $F\in\mathcal{F}_2$, but need a small amount of preparation first. Let k be a positive integer and let T be a tree. We say that $e\in E(T)$ is *k-balanced* if both components of $T\setminus e$ have at least $(|V(T)|-k)/2$ vertices. If E is the set of k-balanced edges in a tree T then every component of $T\setminus E$ containing a leaf of T has at least $(|V(T)|-k)/2$ vertices. It follows that $|E|\le k+1$.

Let $F\in\mathcal{F}_2$ be a tree with components T_1 and T_2 such that $k=|V(T_1)|\le|V(T_2)|$. We have

$$w'(\delta(F)) = \sum_{u\in V(T_1)}\sum_{v\in V(T_2)} w'(F+uv, F) = \frac{1}{n-1}\sum_{u\in V(T_1)}\sum_{v\in V(T_2)}\left(\frac{1}{n-1}\sum_{e\in E(F+uv)} d_{F+uv}(e,uv)\right)$$
$$= \frac{1}{n-1}\sum_{e\in E(T_2)\cup\{uv\}}\left(\sum_{u\in V(T_1)}\sum_{v\in V(T_2)} d_{F+uv}(e,uv)\right). \tag{3}$$

Consider now $e\in E(T_2)$, suppose that e is not k-balanced and let T_e be the smaller component of $T_2\setminus e$. Then $|V(T_e)|<\frac{(n-k)-k}{2}$. It follows that for every $u\in V(T_1)$ and $v\in V(T_e)$, the tree $(T_1\cup T_e)+\{uv\}$ is the smaller component of $(F+uv)\setminus e$. Thus $d_{F+uv}(e,uv)=1/(|V(T_e)|+k)$ for every such pair u,v. It follows that

$$\sum_{u\in V(T_1)}\sum_{v\in V(T_2)} d_{F+uv}(e,uv) \ge \frac{k|V(T_e)|}{|V(T_e)|+k} \ge \frac{k}{k+1}. \tag{4}$$

for every not k-balanced $e\in E(T_2)$. Further, we have

$$\sum_{u\in V(T_1)}\sum_{v\in V(T_2)} d_{F+uv}(uv,uv) \ge \frac{k(n-k)}{k} = n-k. \tag{5}$$

Substituting the estimates in (4) and (5) into (3) and using the lower bound on the number of edges which are not k-balanced, we obtain

$$w'(\delta(F)) \ge \frac{1}{n-1}\left(\frac{k}{k+1}((n-k-1)-(k+1))+(n-k)\right) \ge 1+\frac{k}{k+1}-\frac{3k+1}{n-1}.$$

Note that $w'(\delta(F))\ge 3/2-o(1)$, when $k=o(n)$.

We now define $w'': E(B_n)\to\mathbb{R}_+$ be identically $1/(n-1)$. Then $w''(\delta(T))=1$ for every $T\in\mathcal{F}_1$. For $F\in\mathcal{F}_2$ with components T_1 and T_2 such that $k=|V(T_1)|\le|V(T_2)|$ we have $w''(\delta(F))=k(n-k)/(n-1)$. We will use w'' to correct w' when $k=\Omega(n)$.

Let $\alpha := 1/\sqrt{n}$ and let $w := (1-\alpha)w' + \alpha w''$. We have $w(\delta(T)) = 1$ for every $T \in \mathcal{F}_1$, as desired. For $F \in \mathcal{F}_2$ and k defined as above we use (5) to deduce

$$w(\delta(F)) \geq 1 + \frac{k}{k+1} - \frac{3k+1}{n-1} + \alpha\left(\frac{k(n-k)}{n-1} - 2\right).$$

If $k = 1$ then clearly $w(\delta(F)) \geq 3/2 + o(1)$. If $1 < k \leq n/18$ then

$$w(\delta(F)) \geq 1 + \frac{2}{3} - \frac{3n/18+1}{n-1} + \frac{1}{\sqrt{n}}\left(\frac{2(n-2)}{n-1} - 2\right) \geq 3/2 + o(1).$$

Finally, if $k \geq n/18$ then

$$w(\delta(F)) \geq \frac{1}{\sqrt{n}}\left(\frac{(n/18)(n-n/18)}{n-1} - 2\right) = \Omega(\sqrt{n}).$$

It follows that $w(\delta(F)) \geq 3/2 + o(1)$ for every F, as desired. $\square$

References

1. Paul Balister, Béla Bollobás, and Stefanie Gerke, *Connectivity of addable graph classes*, J. Combin. Theory Ser. B **98** (2008), no. 3, 577–584. MR 2401130 (2009b:05234)
2. Colin McDiarmid, *Connectivity for random graphs from abridge-addable class*, manuscript.