Problem Solving Seminar Fall 2022.

Problem Set 2: Number Theory.

Classical results.

1. **Polignac's formula.** If p is a prime number and n a positive integer, then the exponent of p in $n!$ is
$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \dots.$$

2. **Wilson.**
$$(p-1)! \equiv -1 (\bmod p)$$
for any prime p.

3. **Chinese Remainder theorem.** Let $m_1, m_2, \dots, m_k$ be pairwise positive integers greater than 1, such that $\gcd(m_i, m_j) = 1$ for $i \neq j$. Then for any integers $a_1, a_2, \dots, a_k$ the system of congruences
$$\begin{aligned} x &\equiv a_1 \quad (\bmod m_1), \\ x &\equiv a_2 \quad (\bmod m_2), \\ &\dots \\ x &\equiv a_k \quad (\bmod m_k). \end{aligned}$$
has solutions, and any two such solutions are congruent modulo $m = m_1 m_2 \dots m_k$.

Problems.

1. Prove that $n!$ is not divisible by 2^n for any positive integer n.

2. The number 2^{29} has 9 distinct digits. Which digit is missing?

3. **Putnam 1956. A2.** Given any positive integer n, show that we can find a positive integer m such that mn uses all ten digits when written in the usual base 10.

4. **Putnam 2000. A2.** Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of the squares of two integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, $2 = 1^2 + 1^2$.]

5. **Putnam 2000. B2.** Prove that the expression
$$\frac{\gcd(m,n)}{n} \binom{n}{m}$$
is an integer for all pairs of integers $n \geq m \geq 1$.

6. **IMO 2002.** The positive divisors of an integer $n > 1$ are $1 = d_1 < d_2 < \dots < d_k = n$. Let $s = d_1 d_2 + d_2 d_3 + \dots + d_{k-1} d_k$. Prove that $s < n^2$ and find all n for which s divides n^2.

7. **Putnam 2021. B4.** Let $F_0, F_1, \dots$ be the sequence of Fibonacci numbers, with $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For $m > 2$, let R_m be the remainder when the product $\prod_{k=1}^{F_m - 1} k^k$ is divided by F_m. Prove that R_m is also a Fibonacci number.