Problem Solving Seminar Fall 2022.
Problem Set 5: Combinatorics.

Classical results.

1. Show that the equation
$$x_1 + x_2 + \ldots + x_r = n$$
has exactly $\binom{n+r-1}{r-1}$ non-negative integer solutions.

2. **Erdős-Ko-Rado.** Let $\mathcal{F}$ be a family of k element subsets of an n element set, with $n \geq 2k$, such that every two sets in $\mathcal{F}$ have a non-empty intersection. Then
$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

3. **Turán.** Show that a graph with n vertices and more than $\frac{t-1}{t}\frac{n^2}{2}$ edges contains a complete subgraph on $t+1$ vertices.

Problems.

1. **Putnam 1954. A2.** Given any five points in the interior of a square with side length 1, show that two of the points are a distance apart less than $k = 1/\sqrt{2}$. Is this result true for a smaller k?

   Putnam 2003. A1. Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$ there are four ways: 4, $2+2$, $1+1+2$, $1+1+1+1$.

2. **Putnam 1964. B2.** Let S be a finite set, and suppose that a collection $\mathcal{F}$ of subsets of S has the property that any two members of $\mathcal{F}$ have at least one element in common, but $\mathcal{F}$ cannot be extended (while keeping this property). Prove that $\mathcal{F}$ contains exactly half of the subsets of S.

3. **Putnam 1993. A3.** Let $\mathcal{P}_n$ be the set of subsets of $\{1, 2, \ldots, n\}$. Let $c(n, m)$ be the number of functions $f : \mathcal{P}_n \to \{1, 2, \ldots, m\}$ such that $f(A \cap B) = \min\{f(A), f(B)\}$. Prove that
$$c(n, m) = \sum_{j=1}^{m} j^n.$$

4. **Putnam 1997. A5.** Let N_n denote the number of ordered n-tuples of positive integers $(a_1, a_2, \ldots, a_n)$ such that $1/a_1 + 1/a_2 + \ldots + 1/a_n = 1$. Determine whether N_{10} is even or odd.

5. **Putnam 2021. B5.** Say that an n-by-n matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ with integer entries is *very odd* if, for every nonempty subset S of $\{1, 2, \ldots, n\}$, the $|S|$-by-$|S|$ submatrix $(a_{ij})_{i,j \in S}$ has odd determinant. Prove that if A is very odd, then A^k is very odd for every $k \geq 1$.

6. **Putnam 2018. B6.** Let S be the set of sequences of length 2018 whose terms are in the set $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860. Prove that the cardinality of S is at most
$$2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018}.$$