Problem Solving Seminar Fall 2022. Problem Set 7: Calculus.

Classical results.

1. Every continuous mapping of a circle into a line carries some pair of diametrically opposite points to the same point.

2. **Leibniz formula.**
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots.$$

3. **Gaussian integral.**
$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

Problems.

1. **Putnam 1994. A1.** Suppose that a sequence $a_1, a_2, \dots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

2. **Putnam 2012. B1.** Let S be a class of functions from $[0, \infty)$ to $[0, \infty)$ that satisfies:

   (i) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x+1)$ are in S;

   (ii) If $f(x)$ and $g(x)$ are in S, the functions $f(x) + g(x)$ and $f(g(x))$ are in S;

   (iii) If $f(x)$ and $g(x)$ are in S and $f(x) \geq g(x)$ for all $x \geq 0$, then the function $f(x) - g(x)$ is in S.

   Prove that if $f(x)$ and $g(x)$ are in S, then the function $f(x)g(x)$ is also in S.

3. **Putnam 1991. B2.** Suppose f and g are non-constant, differentiable, real-valued functions on $\mathbb{R}$. Furthermore, suppose that for each pair of real numbers x and y,
$$f(x+y) = f(x)f(y) - g(x)g(y),$$
$$g(x+y) = f(x)g(y) + g(x)f(y).$$
If $f'(0) = 0$ prove that $(f(x))^2 + (g(x))^2 = 1$ for all real x.

4. **Putnam 2013. A3.** Suppose that the real numbers $a_0, a_1, \dots, a_n$ and x, with $0 < x < 1$, satisfy
$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \dots + \frac{a_n}{1-x^{n+1}} = 0.$$
Prove that there exists a real number y with $0 < y < 1$ such that
$$a_0 + a_1 y + \dots + a_n y^n = 0.$$

5. **Putnam 2008. A4.** Define $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) = \begin{cases} x & \text{if } x \leq e \\ x f(\ln x) & \text{if } x > e. \end{cases}$$
Does $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converge?

6. **Putnam 2021. B3.** Let $h(x, y)$ be a real-valued function that is twice continuously differentiable throughout $\mathbb{R}^2$, and define

$$\rho(x, y) = yh_x - xh_y.$$

Prove or disprove: For any positive constants d and r with $d > r$, there is a circle $\mathcal{S}$ of radius r whose center is a distance d away from the origin such that the integral of ρ over the interior of $\mathcal{S}$ is zero.