Problem Solving Seminar Fall 2021. Problem Set 2: Number Theory.

Classical results.

1. **Euler.** For a positive integer n and any integer a relatively prime to n one has
$$a^{\phi(n)} \equiv 1 \pmod{n},$$
where $\phi(n)$ is the number of positive integers between 1 and n relatively prime to n.

2. **Polignac's formula.** If p is a prime number and n a positive integer, then the exponent of p in $n!$ is
$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \dots.$$

3. **Chinese Remainder theorem.** Let $m_1, m_2, \dots, m_k$ be pairwise positive integers greater than 1, such that $\gcd(m_i, m_j) = 1$ for $i \neq j$. Then for any integers $a_1, a_2, \dots, a_k$ the system of congruences
$$\begin{aligned} x &\equiv a_1 \pmod{m_1}, \\ x &\equiv a_2 \pmod{m_2}, \\ &\dots \\ x &\equiv a_k \pmod{m_k}. \end{aligned}$$
has solutions, and any two such solutions are congruent modulo $m = m_1 m_2 \dots m_k$.

Problems.

1. Prove that $n!$ is not divisible by 2^n for any positive integer n.

2. Prove that for every n, there exist n consecutive integers each of which is divisible by two different primes.

3. **Putnam 1956. A2.** Given any positive integer n, show that we can find a positive integer m such that mn uses all ten digits when written in the usual base 10.

4. **IMO 1970**. Prove that there are no positive integers n such that the set $\{n+1, n+2, \dots, n+6\}$ can be divided into two sets with the product of elements in one set equal to the product of elements in the other set.

5. **Putnam 2000. A2.** Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of the squares of two integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, $2 = 1^2 + 1^2$.]

6. **USA 1991.** Let n be an arbitrary positive integer. Show that the following sequence is eventually constant modulo n:
$$2, 2^2, 2^{2^2}, 2^{2^{2^2}}, 2^{2^{2^{2^2}}}, 2^{2^{2^{2^{2^2}}}}, \dots$$

7. **IMO 2002.** The positive divisors of an integer $n > 1$ are $1 = d_1 < d_2 < \dots < d_k = n$. Let $s = d_1 d_2 + d_2 d_3 + \dots + d_{k-1} d_k$. Prove that $s < n^2$ and find all n for which s divides n^2.

8. **Putnam 1996. A6.** The sequence a_n is defined by $a_1 = 1, a_2 = 2, a_3 = 24$, and, for $n \geq 4$,
$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1} a_{n-2}^2}{a_{n-2} a_{n-3}}$$
Show that, for all n, a_n is an integer multiple of n.