Problem Solving Seminar Fall 2021. Problem Set 3: Inequalities.

Classical results.

1. **AM-GM.** For any non-negative real numbers $x_1, x_2, \ldots, x_n$,
$$\sqrt[n]{x_1 x_2 \ldots x_n} \le \frac{x_1 + x_2 + \ldots + x_n}{n}.$$

2. **Cauchy-Schwarz.** For any real $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$,
$$(x_1 y_1 + x_2 y_2 + \ldots + x_n y_n)^2 \le (x_1^2 + x_2^2 + \ldots x_n^2)(y_1^2 + y_2^2 + \ldots + y_n^2).$$

3. **Jensen.** For any convex function f and any real $x_1, x_2, \ldots, x_n$,
$$f\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right) \le \frac{f(x_1) + f(x_2) + \ldots + f(x_n)}{n}.$$

Problems.

1. Show that
$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$$

2. **Putnam 2003. A2.** Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that
$$(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \le [(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)]^{1/n}.$$

3. **Putnam 2004. B2.** Let m and n be positive integers. Show that
$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \frac{n!}{n^n}.$$

4. **USA 1997.** A set of $n > 3$ real numbers has sum at least n and the sum of the squares of the numbers is at least n^2. Show that the largest positive number is at least 2.

5. **IMO 1994.** Let m and n be positive integers. Let $a_1, a_2, \ldots, a_m$ be distinct elements of $\{1, 2, \ldots, n\}$ such that whenever $a_i + a_j \le n$ for some i, j (possibly the same) we have $a_i + a_j = a_k$ for some k. Prove that:
$$\frac{a_1 + a_2 + \ldots + a_m}{m} \ge \frac{n+1}{2}.$$

6. **Putnam 2003. A4.** Let a, b, c, A, B, C be real, a, A non-zero such that $|ax^2 + bx + c| \le |Ax^2 + Bx + C|$ for all real x. Show that $|b^2 - 4ac| \le |B^2 - 4AC|$.

7. **Putnam 2003. B6.** Show that
$$\int_0^1 \int_0^1 |f(x) + f(y)|\, dx\, dy \ge \int_0^1 |f(x)|\, dx$$
for any continuous real-valued function f on $[0, 1]$.