Problem Solving Seminar Fall 2021. Problem Set 1: Induction.

Classical results.

1. Finitely many lines divide the plane into regions. Show that these regions can be colored by two colors in such a way that neighboring regions have different colors.
2. **Fermat's little theorem.** Let p be a prime number, and n a positive integer. Show that $n^p - n$ is divisible by p.
3. An *Hadamard matrix* is an $n \times n$ square matrix, all of whose entries are $+-1$ or 1, such that every pair of distinct rows is orthogonal. In other words, if the rows are considered to be vectors of length n, then the dot product between any two distinct row-vectors is zero. Show that there exist infinitely many Hadamard matrices.
4. Prove that any positive integer can be represented as $\pm 1^2 \pm 2^2 \pm \ldots \pm n^2$ for some positive integer n and some choice of the signs.

Problems.

1. **Putnam 2001. A2.** You have coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.
2. **GA 32.** Show that if $a_1, a_2, \ldots, a_n$ are non-negative real numbers, then
$$(1+a_1)(1+a_2)\ldots(1+a_n) \geq (1+\sqrt[n]{a_1 a_2 \ldots a_n})^n.$$
3. **IMO 2001. B1.** 100 cards are numbered 1 to 100 (each card different) and placed in 3 boxes (at least one card in each box). How many ways can this be done so that if two boxes are selected and a card is taken from each, then the knowledge of their sum alone is always sufficient to identify the third box?
4. **Putnam 2004. A3.** Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that
$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$
for all $n \geq 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)
5. **Putnam 2006. B3.** Let S be a finite set of points in the plane. A linear partition of S is an unordered pair $\{A, B\}$ of subsets of S such that $A \cup B = S$, $A \cap B = \emptyset$, and A and B lie on opposite sides of some straight line disjoint from S (A or B may be empty). Let L_S be the number of linear partitions of S. For each positive integer n, find the maximum of L_S over all sets S of n points.
6. **Putnam 1996. A4.** Let S be the set of ordered triples (a, b, c) of distinct elements of a finite set A. Suppose that

   (a) $(a, b, c) \in S$ if and only if $(b, c, a) \in S$;
   (b) $(a, b, c) \in S$ if and only if $(c, b, a) \notin S$;
   (c) (a, b, c) and (c, d, a) are both in S if and only if (b, c, d) and (d, a, b) are both in S.

   Prove that there exists a one-to-one function g from A to $\mathbb{R}$ such that $g(a) < g(b) < g(c)$ implies $(a, b, c) \in S$.
7. **Putnam 2000. B5.** Let S_0 be a finite set of positive integers. We define finite sets $S_1, S_2, \ldots$ of positive integers as follows: the integer a is in S_{n+1} if and only if exactly one of $a-1$ or a is in S_n. Show that there exist infinitely many integers N for which $S_N = S_0 \cup \{N + a : a \in S_0\}$.