Problem Solving Seminar Fall 2021. Problem Set 6: Combinatorics.

Classical results.

1. Show that the equation
$$x_1 + x_2 + \ldots + x_r = n$$
has exactly $\binom{n+r-1}{r-1}$ non-negative integer solutions.

2. Consider a convex polygon with n vertices so that no 3 diagonals go through the same point.
   (a) How many intersection points do the diagonals have?
   (b) Into how many regions do the diagonals divide the interior of the polygon?

3. **Erdős-Ko-Rado.** Let $\mathcal{F}$ be a family of k element subsets of an n element set, with $n \geq 2k$, such that every two sets in $\mathcal{F}$ have a non-empty intersection. Then
$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

Problems.

1. **Putnam 1964. B2.** Let S be a finite set, and suppose that a collection $\mathcal{F}$ of subsets of S has the property that any two members of $\mathcal{F}$ have at least one element in common, but $\mathcal{F}$ cannot be extended (while keeping this property). Prove that $\mathcal{F}$ contains exactly half of the subsets of S.

2. **Putnam 1992. B1.** Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S. For a given $n \geq 2$, what is the smallest possible number of distinct elements in A_S?

3. **Putnam 1958. B6.** A graph has n vertices $\{1, 2, \ldots, n\}$. Every pair of vertices is joined by a directed edge , either $i \to j$ or $j \to i$. Show that we can find a permutation of the vertices a_i so that $a_1 \to a_2 \to a_3 \to \ldots \to a_n$.

4. **Putnam 1993. A3.** Let $\mathcal{P}_n$ be the set of subsets of $\{1, 2, \ldots, n\}$. Let $c(n, m)$ be the number of functions $f : \mathcal{P}_n \to \{1, 2, \ldots, m\}$ such that $f(A \cap B) = \min\{f(A), f(B)\}$. Prove that
$$c(n, m) = \sum_{j=1}^{m} j^n$$

5. **Putnam 1997. A5.** Let N_n denote the number of ordered n-tuples of positive integers $(a_1, a_2, \ldots, a_n)$ such that $1/a_1 + 1/a_2 + \ldots + 1/a_n = 1$. Determine whether N_{10} is even or odd.

6. **Putnam 1996. B5.** We call a finite string of the symbols X and O *balanced* if every substring of consecutive symbols has a difference of at most 2 between the number of X's and the number of O's. For example, $XOOXOOX$ is not balanced, because the substring $OOXOO$ has a difference of 3. Find the number of balanced strings of length n.

7. **Putnam 2018. B6.** Let S be the set of sequences of length 2018 whose terms are in the set $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860. Prove that the cardinality of S is at most
$$2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018}.$$