Problem Solving Seminar Fall 2021. Problem set 5. Algebra.

Classical results.

1. **Vandermonde.** Let
$$V = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{bmatrix}.$$
Then
$$\det(V) = \prod_{1 \le i < j \le n} (x_j - x_i).$$

2. **Cayley-Hamilton.** Given an $n \times n$ matrix A *the characteristic polynomial* of A is defined by $P_A(\lambda) = \det(\lambda I_n - A)$, where I_n is the identity matrix. Then $P_A(A) = 0$ for every A.

3. In Oddtown there are n citizens and m clubs $A_1, A_2, \dots, A_m \subseteq \{1, 2, \dots, n\}$. The laws of Oddtown prescribe that
   - The clubs must have distinct memberships. ($A_i \neq A_j$ for $i \neq j$),
   - Every club has odd number of members,
   - Every two distinct clubs have an even number of members in common. ($|A_i \cap A_j|$ is even if $i \neq j$).

   Show that $m \le n$.

Problems.

1. **Putnam 1959. A1.** Prove that one can find a polynomial $P(y)$ with real coefficients such that $P(x - 1/x) = x^n - 1/x^n$ if and only if n is odd.

2. **Putnam 2007. B1.** Let f be a non-constant polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$.

3. **Putnam 1991. A2.** M and N are real unequal $n \times n$ matrices satisfying $M^3 = N^3$ and $M^2 N = N^2 M$. Can we choose M and N so that $M^2 + N^2$ is invertible?

4. **Putnam 2012. A2.** Let $*$ be a commutative and associative binary operation on a set S. Assume that for every x and y in S, there exists z in S such that $x * z = y$. (This z may depend on x and y.) Show that if a, b, c are in S and $a * c = b * c$, then $a = b$.

5. **Putnam 1994. A4.** Let A and B be 2×2 matrices with integer entries such that $A, A + B, A + 2B, A + 3B$, and $A + 4B$ are all invertible matrices whose inverses have integer entries. Show that $A + 5B$ is invertible and that its inverse has integer entries.

6. **Putnam 2006. B4.** Let Z denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. (Thus Z has 2^n elements, which are the vertices of a unit hypercube in $\mathbb{R}^n$.) Let k be given, $0 \le k \le n$. Find the maximum, over all vector subspaces $V \subseteq \mathbb{R}^n$ of dimension k, of the number of points in $V \cap Z$.

7. **MIT PS seminar.** A mansion has n rooms. Each room has a lamp and a switch connected to its lamp. However, switches may also be connected to lamps in other rooms, subject to the following condition: if the switch in room a is connected to the lamp in room b, then the switch in room b is also connected to the lamp in room a. Each switch, when flipped, changes the state (from on to off or vice versa) of each lamp connected to it. Suppose at some points the lamps are all off. Prove that no matter how the switches are wired, it is possible to flip some of the switches to turn all of the lamps on.