Problem Solving Seminar. Fall 2019.

Problem Set 3. Algebra.

Classical results.

1. **Hilbert.** Let
$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{bmatrix}.$$
Then $\det(H) \neq 0$.

2. Let p be a prime. Show that the polynomial $x^{p-1} + x^{p-2} + \ldots + x + 1$ can not be expressed as a product of two non-constant polynomials with integer coefficients.

3. In Oddtown there are n citizens and m clubs $A_1, A_2, \ldots, A_m \subseteq \{1, 2, \ldots, n\}$. The laws of Oddtown prescribe that
   - The clubs must have distinct memberships. ($A_i \neq A_j$ for $i \neq j$),
   - Every club has odd number of members,
   - Every two distinct clubs have an even number of members in common. ($|A_i \cap A_j|$ is even if $i \neq j$).

   Show that $m \leq n$.

Problems.

1. **Putnam 1959. A1.** Prove that one can find a polynomial $P(y)$ with real coefficients such that $P(x - 1/x) = x^n - 1/x^n$ if and only if n is odd.

2. **Putnam 1991. A2.** M and N are real unequal $n \times n$ matrices satisfying $M^3 = N^3$ and $M^2N = N^2M$. Can we choose M and N so that $M^2 + N^2$ is invertible?

3. **Putnam 2012. A2.** Let $*$ be a commutative and associative binary operation on a set S. Assume that for every x and y in S, there exists z in S such that $x * z = y$. (This z may depend on x and y.) Show that if a, b, c are in S and $a * c = b * c$, then $a = b$.

4. **Putnam 2008. A2.** Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

5. **Putnam 1994. A4.** Let A and B be 2×2 matrices with integer entries such that $A, A + B, A + 2B, A + 3B$, and $A + 4B$ are all invertible matrices whose inverses have integer entries. Show that $A + 5B$ is invertible and that its inverse has integer entries.

6. **Putnam 2006. B4.** Let Z denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. (Thus Z has 2^n elements, which are the vertices of a unit hypercube in $\mathbb{R}^n$.) Let k be given, $0 \leq k \leq n$. Find the maximum, over all vector subspaces $V \subseteq \mathbb{R}^n$ of dimension k, of the number of points in $V \cap Z$.

7. **Putnam 2014. A6.** Let n be a positive integer. What is the largest k for which there exist $n \times n$ matrices $M_1, \ldots, M_k$ and $N_1, \ldots, N_k$ with real entries such that for all i and j, the matrix product $M_i N_j$ has a zero entry somewhere on its diagonal if and only if $i \neq j$?

8. **Putnam 1996. B6.** The origin lies inside a convex polygon whose vertices have coordinates (a_i, b_i) for $i = 1, 2, \ldots, n$. Show that we can find $x, y > 0$ such that

$$a_1 x^{a_1} y^{b_1} + a_2 x^{a_2} y^{b_2} + \ldots + a_n x^{a_n} y^{b_n} = 0$$

and

$$b_1 x^{a_1} y^{b_1} + b_2 x^{a_2} y^{b_2} + \ldots + b_n x^{a_n} y^{b_n} = 0.$$