# Problem Solving Seminar 2017.

# Problem Set 1. Proofs by contradiction.

## Classical results.

1. Prove that there are infinitely many prime numbers. (Recall that a number $p > 1$ is *prime* if and only its only divisors are $1$ and $p$.)

2. Recall that
$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots.$$
Show that $e$ is irrational.

3. Prove that there exists no polynomial $P$ with integer coefficients such that $P(n)$ is prime for every positive integer $n$.

## Problems.

1. **Put 1952. A1.** The polynomial $P(x)$ has all integral coefficients. The leading coefficient, the constant term, and $P(1)$ are all odd. Show that $P(x)$ has no rational roots.

2. **GA 7.** Show that there does not exist a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(2) = 3$ and $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$.

3. **GA 9.** Show that the interval $[0, 1]$ cannot be partitioned into two disjoint sets $A$ and $B$ such that $B = A + a$ for some real number $a$.

4. **Putnam 1995. A1.** Let $S$ be a set of real numbers which is closed under multiplication (that is, if $a$ and $b$ are in $S$, then so is $ab$). Let $T$ and $U$ be disjoint subsets of $S$ whose union is $S$. Given that the product of any *three* (not necessarily distinct) elements of $T$ is in $T$ and that the product of any three elements of $U$ is in $U$, show that at least one of the two subsets $T, U$ is closed under multiplication.

5. **Putnam 2014. B1.** A *base* 10 *over-expansion* of a positive integer $N$ is an expression of the form
$$N = d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_0 10^0$$
with $d_k \neq 0$ and $d_i \in \{0, 1, 2, \ldots, 10\}$ for all $i$. For instance, the integer $N = 10$ has two base 10 over-expansions: $10 = 10 \cdot 10^0$ and the usual base 10 expansion $10 = 1 \cdot 10^1 + 0 \cdot 10^0$. Which positive integers have a unique base 10 over-expansion?

6. **Hungary 1999.** Let $n > 1$ be an arbitrary positive integer, and let $k$ be the number of positive prime numbers less than or equal to $n$. Select $k + 1$ positive integers such that none of them divides the product of all the others. Prove that there exists a number among the chosen $k + 1$ that is bigger than $n$.

7. **Putnam 1958. B5.** $S$ is an infinite set of points in the plane. The distance between any two points of $S$ is integral. Prove that $S$ is a subset of a straight line.

8. **Putnam 2014. A5.** Let

$$P_n(x) = 1 + 2x + 3x^2 + \cdots + nx^{n-1}.$$

Prove that the polynomials $P_j(x)$ and $P_k(x)$ are relatively prime for all positive integers $j$ and $k$ with $j \neq k$.