

Problem Seminar. Fall 2016.

Problem Set 7. Calculus.

Classical results.

1. Every continuous mapping of a circle into a line carries some pair of diametrically opposite points to the same point.
2. **Mean value theorem.** If $f : [a, b] \rightarrow \mathbb{R}$ is a differentiable function then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

3. **Leibniz formula.**

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

4. **Gaussian integral.**

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

Problems.

1. **Putnam 1964. B1.** Let $a_1, a_2, \dots$ be positive integers such that $\sum_{i=1}^{\infty} \frac{1}{a_i}$ converges. For each n , let b_n denote the number of positive integers i for which $a_i \leq n$. Prove that $\lim_{n \rightarrow \infty} \frac{b_n}{n} = 0$.

2. **Putnam 1987. B1.** Evaluate

$$\int_2^4 \frac{\sqrt{\ln(9-x)} dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}.$$

3. **Putnam 2012. B1.** Let S be a class of functions from $[0, \infty)$ to $[0, \infty)$ that satisfies:

- (i) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x+1)$ are in S ;
- (ii) If $f(x)$ and $g(x)$ are in S , the functions $f(x) + g(x)$ and $f(g(x))$ are in S ;
- (iii) If $f(x)$ and $g(x)$ are in S and $f(x) \geq g(x)$ for all $x \geq 0$, then the function $f(x) - g(x)$ is in S .

Prove that if $f(x)$ and $g(x)$ are in S , then the function $f(x)g(x)$ is also in S .

4. **Putnam 1995. A2.** For what pairs (a, b) of positive real numbers does the improper integral

$$\int_b^{\infty} \left(\sqrt{\sqrt{x+a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-b}} \right) dx$$

converge?

5. **Putnam 2007. B2.** Suppose that $f : [0, 1] \rightarrow \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x) dx = 0$. Prove that for every $\alpha \in (0, 1)$,

$$\left| \int_0^\alpha f(x) dx \right| \leq \frac{1}{8} \max_{0 \leq x \leq 1} |f'(x)|.$$

6. **Putnam 2013. A3.** Suppose that the real numbers $a_0, a_1, \dots, a_n$ and x , with $0 < x < 1$, satisfy

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \dots + \frac{a_n}{1-x^{n+1}} = 0.$$

Prove that there exists a real number y with $0 < y < 1$ such that

$$a_0 + a_1 y + \dots + a_n y^n = 0.$$

7. **Putnam 2000. A4.** Show that the improper integral

$$\lim_{B \rightarrow \infty} \int_0^B \sin(x) \sin(x^2) dx$$

converges.

8. **Putnam 2008. A4.** Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \leq e \\ x f(\ln x) & \text{if } x > e. \end{cases}$$

Does $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converge?