MATH 340: Discrete Structures II. Winter 2017.

Assignment #5: Enumeration.

Due in class on Friday, April 7th.

1. *Combinatorial identities.*

a) Give an algebraic proof of the following identity:

$$\binom{n+1}{m+1} = \sum_{k=m}^{n} \binom{k}{m}$$

b) Give a combinatorial (bijective) proof of the identity in a).

2. *Labelled trees.*
Let $f : [n] \to [n]$ be a function, and let T_f be a labelled tree on n vertices, constructed from f using the procedure demonstrated in class. Suppose that T contains a vertex of degree at least k. Show that f takes at most $n - k + 2$ different values.

3. *Catalan numbers. I.*
Give a bijection to show that the following is counted by Catalan numbers. The number of orderings of numbers $\{1, 2, \ldots, 2n\}$, such that

- the numbers $\{1, 3, \ldots, 2n - 1\}$ appear in order,
- the numbers $\{2, 4, \ldots, 2n\}$ appear in order,
- $2k - 1$ precedes $2k$ for every $1 \leq k \leq n$.

4. *Catalan numbers. II.* Given a sequence $+ + + - - + - - + + --$, construct

a) a Dyck path,

b) a rooted plane tree on 7 vertices,

c) a decomposition of a 8-gon into triangles,

corresponding to this sequence via the bijections shown in class.

5. *Generating functions.* For the following recurrences, find the ordinary generating function $F(x)$ and use it to obtain a closed formula for $f(n)$.

a) $f(n) = 6f(n-1) - 8f(n-2)$ for $n \geq 2$, $f(0) = 3, f(1) = 10$,

b) $f(n) = 4f(n-1) - 4f(n-2)$ for $n \geq 2$, $f(0) = 0, f(1) = 2$.