MATH 340: ENUMERATION REVIEW.

Counting using bijections

Review of MATH 240. A function $f : X \to Y$ is

- *a surjection* (or *onto*) if for every $y \in Y$ there exists $x \in X$ such that $y = f(x)$,
- *an injection* if for every $y \in Y$ there exists at most one $x \in X$ such that $y = f(x)$,
- *a bijection* if for every $y \in Y$ there exists exactly one $x \in X$ such that $y = f(x)$.

Let $[n]$ denote $\{1, 2, \dots, n\}$

Theorem 1. (1) *There exist n^k sequences $s_1 s_2 \dots s_k$ of length k such that $s_1, s_2, \dots, s_k \in [n]$. Equivalently, there are n^k functions $f : [k] \to [n]$.*
(2) *There exist $n(n-1)\dots(n-k+1)$ sequences $s_1 s_2 \dots s_k$ of length n such that $s_1, s_2, \dots, s_k \in [n]$, and $s_i \neq s_j$ for $i \neq j$. Equivalently, there are $n(n-1)\dots(n-k+1)$ injections $f : [k] \to [n]$.*
(3) *There are $n!$ permutations of $[n]$, i.e. bijections $f : [n] \to [n]$.*

Lemma 2. *There are 2^n subsets of an n element set.*

Lemma 3. *There are*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

k element subsets of an n element set.

Theorem 4 (Binomial theorem)**.**

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Corollary 5.

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n},$$

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots$$

Theorem 6. *There exist $\binom{n+k-1}{k-1}$ solutions to the equation*

$$x_1 + x_2 + \dots + x_k = n,$$

such that $x_1, x_2, \dots, x_k \geq 0$ are integers.

There exist $\binom{n-1}{k-1}$ solutions to the above equation if we require that $x_1, x_2, \dots, x_k \geq 1$ instead.

Labelled trees.

Theorem 7. *There exist n^{n-2} trees on n vertices with vertices labelled $1, 2, \dots, n$.*

Catalan numbers. Let C_n denote the *nth Catalan number.*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Theorem 8. *C_n counts the number of the following objects:*

- *Sequences of n pluses and n minuses, such that each (initial) pratial sum is non-negative.*
- Dyck walks*: Paths from $(0,0)$ to $(2n,0)$ using steps $(1,1)$ and $(1,-1)$ and never going below the x axis.*
- *Rooted plane trees with $n+1$ vertices.*
- *Planted[1] trivalent[2] with $2n+2$ vertices*
- *Decompositions of an $(n+2)$-gon into n triangles.*

Generating functions

The formal power series

$$F(x) = \sum_{n\geq 0} f(n)x^n$$

is *the ordinary generating function for the sequence $f(n)$.*

Basic generating function method.

(1) Find a recurrence for $f(n)$
(2) Multiply both sides of the recurrence by x^n.
(3) Solve the resulting equation to find $F(x)$.
(4) Express $F(x)$ as power series again to find $f(n)$.

$$\frac{1}{1-ax} = \sum_{n\geq 0} a^n x^n.$$

Manipulating ordinary generating functions. Let $F(x) = \sum_{n\geq 0} f(n)x^n$ then

$$\sum_{n\geq 0} f(n+k)x^n = \frac{F(x) - f(0) - f(1)x - \ldots - f(k-1)x^{k-1}}{x^k},$$

$$x\frac{d}{dx}F(x) = \sum_{n\geq 0} nf(n)x^n.$$

If $G(x) = \sum_{n\geq 0} g(n)x^n$, $H(x) = \sum_{n\geq 0} h(n)x^n$ then

$$H(x)G(x) = \sum_{n\geq 0}\left(\sum_{k=0}^{n} h(k)g(n-k)\right)x^n.$$

Convolutions. If $F_i(x)$ is the generating function for selecting items from the set S_i for $i = 1, 2, \ldots, k$ then $F_1(x)F_2(x)\ldots F_k(x)$ is the generating functions for selecting items from $S_1 \cup S_2 \ldots \cup S_k$.

[1]the root has degree one

[2]every vertex has degree one or three

Exponential generating functions. The formal power series

$$\hat{F}(x) = \sum_{n\geq 0} f(n)\frac{x^n}{n!}$$

is *the exponential generating function for the sequence* $f(n)$.

If $\hat{F}(x) = \sum_{n\geq 0} f(n)\frac{x^n}{n!}$ then

$$\frac{d}{dx}\hat{F}(x) = \sum_{n\geq 0} f(n+1)\frac{x^n}{n!},$$

$$x\frac{d}{dx}\hat{F}(x) = \sum_{n\geq 0} nf(n)\frac{x^n}{n!},$$

If $G(x) = \sum_{n\geq 0} g(n)\frac{x^n}{n!}$, $\hat{H}(x) = \sum_{n\geq 0} h(n)\frac{x^n}{n!}$ then

$$\hat{H}(x)\hat{G}(x) = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}h(k)g(n-k)\right)\frac{x^n}{n!}.$$

$$e^{ax} = \sum_{n\geq 0} a^n\frac{x^n}{n!}.$$