MATH 240: Discrete structures I. Fall 2011

Assignment #5: Combinatorics and Graph Theory. Solutions.

1. *Fibonacci Numbers.*
Show that for every positive integer n the Fibonacci number F_{5n} is divisible by 5.

Solution: By induction on n. Base case ($n = 1$): $F_5 = 5$.
Induction step ($n \to n + 1$):

$$\begin{aligned} F_{5(n+1)} &= F_{5n+4} + F_{5n+3} = (F_{5n+3} + F_{5n+2}) + F_{5n+3} = 2F_{5n+3} + F_{5n+2} = \\ &= 2(F_{5n+2} + F_{5n+1}) + F_{5n+2} = 3F_{5n+2} + 2F_{5n+1} = 3(F_{5n+1} + F_{5n}) + 2F_{5n+1} \\ &= 5F_{5n+1} + 3F_{5n}. \end{aligned}$$

As F_{5n} is divisible by 5 by the induction hypothesis we conclude that so is $F_{5(n+1)}$.

2. *Recurrence relations.*

(a) Solve the recurrence relation

$$p(n) = 4p(n-1) + 5$$

with initial conditions $p(0) = 1, p(1) = 9$.

(b) Let f_n be the number of subsets of $\{1, 2, \ldots, n\}$ that contain no three consecutive integers. Find a recurrence for f_n.

Solution: (a) We search for a solution in the form $p(n) = a4^n + b$ for constants a and b. The recurrence relation is satisfied when

$$a4^n + b = 4(a4^{n-1} + b) + 5 = a4^n + (4b + 5),$$

that is when $b = 4b + 5$. Therefore $b = -5/3$. Plugging in the initial condition we have $a + b = 1$, and therefore $a = 8/3$. The final answer is

$$p(n) = \frac{8}{3}4^n - \frac{5}{3}.$$

(b) Consider subsets counted by f_n. The number of subsets not containing n as an element is f_{n-1}. The number of subsets which contain n, but do not contain $n - 1$, is f_{n-2}. (They correspond exactly to subsets of $\{1, 2, \ldots, n-2\}$ containing no 3 consecutive integers.) Finally, if a subset contains n and $n - 1$ then it can not contain $n - 2$ and therefore there are f_{n-3} such subsets. This gives a recurrence

$$f_n = f_{n-1} + f_{n-2} + f_{n-3}.$$

3. *Inclusion-Exclusion.*

(a) An integer n is called *square free* if it does not have a divisor of the form k^2 where $k \in \{2, 3, \ldots, n\}$. Find the number of square-free integers between 1 and 120.

(b) In how many permutations of the set $\{0, 1, 2, \ldots, 9\}$ do either of 0 and 1, or 2 and 0, or 3 and 2 appear consecutively? (For example, we do not count

$$(5, 6, 0, 4, 9, 2, 3, 7, 8, 1),$$

as we want 3 and 2 to appear consecutively in that order. We count

$$(3, 5, 7, 2, 0, 1, 9, 8, 4, 6),$$

both 0 and 1, and 2 and 0 appear consecutively in it.)

Solution: (a) An integer is square free if and only if it does not have a divisor of the form p^2 for some prime p. Let A_p be the set of all integers between 1 and 120 divisible by p^2. Then the number we are interested in is $120 - |A_2 \cup A_3 \cup A_5 \cup A_7|$.
By inclusion-exclusion

$$\begin{aligned}|A_2 \cup A_3 \cup A_5 \cup A_7| &= |A_2| + |A_3| + |A_5| + |A_7| - |A_2 \cap A_3| - |A_2 \cap A_5| \\ &= \frac{120}{2^2} + \lfloor \frac{120}{3^2} \rfloor + \lfloor \frac{120}{5^2} \rfloor + \lfloor \frac{120}{7^2} \rfloor - \lfloor \frac{120}{2^2 3^2} \rfloor - \lfloor \frac{120}{2^2 5^2} \rfloor \\ &= 30 + 13 + 4 + 2 - 3 - 1 = 45.\end{aligned}$$

(All the other summands in the inclusion-exclusion formula are equal to zero. $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.) The final answer is $120 - 45 = 75$.

(b) Let A be the set of permutation where 0 and 1 appear consecutively, B be the set of permutations, where 2 and 0 appear consecutively, and let C be the set of permutations, where 3 and 2 do. Then $|A| = |B| = |C| = 9!$. For example, permutations in A correspond to permutations of the alphabet $\{01, 2, 3, 4, \ldots, 9\}$, where 01 is considered as a single symbol. Similarly, $|A \cap B| = |B \cap C| = |A \cap C| = 8!$. (Permutations in $A \cap B$ correspond to permutations of the alphabet with symbol 201 replacing 0, 1 and 2.) Finally, $|A \cap B \cap C| = 7!$. Using the inclusion-exclusion formula we have

$$\begin{aligned}|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C| \\ &= 3 \cdot 9! - 3 \cdot 8! + 7!\end{aligned}$$

4. *Counting integer solutions.*

(a) How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 30,$$

such that $3 \leq x_i \leq 10$ for every $1 \leq i \leq 4$.

(b) How many non-negative integer solutions are there to the inequality

$$x_1 + x_2 + \ldots + x_k \leq n.$$

Solution: (a) First, let $y_i = x_i - 3$ for $1 \leq i \leq 4$. We will count integer solutions of the equation $y_1 + y_2 + y_3 + y_4 = 18$, with $0 \leq y_i \leq 7$, as there is a straightforward bijection between such solutions and the solutions of the original equation. There are

$$\binom{18+4-1}{4-1} = \frac{21 \cdot 20 \cdot 19}{6} = 1330$$

non-negative solutions to this equation, when we ignore the upper bounds. Let A_i be the set of solutions with $y_i \geq 8$. Then we are interested in $1330 - |A_1 \cup A_2 \cup A_3 \cup A_4|$. Applying inclusion-exclusion, we have

$$\begin{aligned} |A_1 \cup A_2 \cup A_3 \cup A_4| &= \sum_{i=1}^{4} |A_i| - \sum_{1 \leq i < j \leq 4} |A_i \cap A_j| \\ &= 4\binom{(18-8)+4-1}{3} - 6\binom{(18-2\cdot 8)+4-1}{3} = 4\frac{13 \cdot 12 \cdot 11}{6} - 6\frac{5 \cdot 4 \cdot 3}{6} \\ &= 4 \cdot 286 - 6 \cdot 10 = 1084. \end{aligned}$$

To compute $|A_1|$, for example, we used the fact that solutions in A_1 correspond to non-negative integer solutions of $z_1 + y_2 + y_3 + y_4 = 18 - 8$ after substitution $z_1 = y_1 - 8$. The final answer is $1330 - 1084 = 246$.

(b) Such solutions are in bijection with the non-negative integer solutions of the equation

$$x_1 + x_2 + \ldots + x_k + x_{k+1} = n.$$

There are

$$\binom{n + (k+1) - 1}{(k+1) - 1} = \binom{n+k}{k}$$

such solutions.

5. *Graph Degrees.*

(a) Does there exist a simple graph with 7 vertices and the following degrees: $\{0, 1, 2, 2, 2, 3, 6\}$?

(b) How many simple graphs are there with the vertex set $\{A, B, C, D\}$ such that two of the vertices have degree one and the remaining two vertices have degree two?

Solution: (a) No, as the vertex of degree six would have to be adjacent to every other vertex, but the vertex of degree zero has no neighbors.

(b) It is not hard to check that the only simple graph upto an isomorphism with these degrees is a path with four vertices. There are $4! = 24$ ways to label the vertices of the path with labels $\{A, B, C, D\}$ starting with a particular end. But using this method we count every graph twice, as we could have also started labeling from the other end of the path. Thus there are $24/2 = 12$ different graphs.