MATH 240: Discrete structures I. Fall 2011

Assignment #2: Proofs. Due Friday, October 12th.

1. *Problems in NP.* Show that the following problems are in NP.

a) `The Knapsack problem.`

Given n items numbered 1 through n so that i-th item has weight w_i and value v_i, determine whether it is possible to select some of these items with total weight at most W and total value at least V.

b) `Quadratic Diophantine equations.`

Given three natural number a, b and c do there exist integers x and y so that $ax^2 + by = c$? (For this problem assume that the numbers are given to you in binary. The size of the input is $n = \log_2(a) + \log_2(b) + \log_2(c)$. You need to show that there exists a certificate for the YES answer, which can be checked in a time polynomial in n.)

2. *Order notation.* For each of the following pairs of functions indicate whether $f = O(g)$ or $f = \Omega(g)$, or both. (All logarithms may be assumed to be natural.) In each case, briefly justify your answer.

1. $f(n) = 2n^2 + 5n, g(n) = 5n^2 + 2n$;
2. $f(n) = \log(n), g(n) = \log(n^2)$;
3. $f(n) = \log(n), g(n) = (\log n)^2$;
4. $f(n) = n^3 2^n, g(n) = n^2 3^n$;
5. $g(n) = (\log n)^n, g(n) = n^{\log n}$;
6. $f(n) = n!, g(n) = n^n$.

3. *Predicate calculus.*

a) Write down in the predicate calculus the negation of the following statement without using $\neg(\forall n(\ldots))$ construction.

$$\forall n \in \mathbb{N}(\exists m \in \mathbb{N}((n^2 = 3m) \vee (n^2 = 3m - 2))).$$

b) Is the statement in a) true or is its negation true?

4. *Social choice functions.* Does there exist a social choice function f satisfying the following property: For any pair of candidates α and β, if at least 60% of all the voters prefer α to β then f ranks α above β.

5. *Mathematical induction.*

a) Show that for all positive integers n

$$1^2 + 2^2 + \dots n^2 = \frac{n(n+1)(2n+1)}{6}.$$

b) A sequence a_n is defined recursively by $a_0 = 0, a_1 = 1$, and $a_{n+2} = 7a_{n+1} - 12a_n$ for $n \geq 0$. Show that $a_n = 4^n - 3^n$ for all non-negative integers n.

c) Show that $1 + hn \leq (1+h)^n$ for all real $h \geq -1$ and all positive integers n.