MATH 240: Discrete structures I. Fall 2011

Assignment #1: Sets and Logic. Solutions.

1. *Set Identities.* Prove the following set identities.

a) $(A - B) \cap C = (A \cap C) - (B \cap C)$,

b) $(A \oplus B) \oplus (A \oplus C) = B \oplus C$.

Solution a): Consider $x \in (A - B) \cap C$. We have $x \in A - B$ and $x \in C$. Therefore, $x \in A$ and $x \notin B$. It follows that $x \in A \cap C$ and $x \notin B \cap C$. Therefore, $x \in (A \cap C) - (B \cap C)$.
For the other inclusion consider $y \in (A \cap C) - (B \cap C)$. We have $y \in A \cap C$ and $y \notin B \cap C$. Therefore, $y \in A$ and $y \in C$. As $y \notin B \cap C$, it follows that $y \notin B$. Therefore $y \in (A - B) \cap C$.
We have shown that every element of $(A - B) \cap C$ is an element of $(A \cap C) - (B \cap C)$ and vice versa. Therefore the sets are equal.

b): The argument is similar to that in a). Consider $x \in (A \oplus B) \oplus (A \oplus C)$. It follows that x belongs to exactly one of the sets $A \oplus B$ and $A \oplus C$. Both sides of the formula are symmetric with respect to switching the roles of B and C, so without loss of generality we suppose that $x \in A \oplus B$ and $x \notin A \oplus C$. Therefore, x is in exactly one of the sets A and B. If $x \in A$ and $x \notin B$ then $x \in C$, as $x \notin A \oplus C$. In this case, $x \in B \oplus C$. In the remaining case $x \notin A$ and $x \in B$, we have $x \notin C$. Again, we conclude that $x \in B \oplus C$.
The proof of the other inclusion is analogous.

2. *Propositions.* Which of the following sentences are propositions?

a) It rained yesterday.

b) The last digit of the smallest prime number larger than 100^{100} is 1.

c) This sentence is false.

d) $6 + 5 = 10$.

Solution: All but c) are propositions as each of them is either true or false. The sentence in c) can be neither true nor false.

3. *Truth tables.* Use a truth table to verify the following equivalence

$$\neg(P \vee (Q \wedge (\neg R))) \leftrightarrow (\neg P) \wedge ((\neg Q) \vee R).$$

Solution:

P	Q	R	$Q \wedge (\neg R)$	$\neg(P \vee (Q \wedge (\neg R)))$	$(\neg Q) \vee R$	$(\neg P) \wedge ((\neg Q) \vee R))$
T	T	T	F	F	T	F
T	T	F	T	F	F	F
T	F	T	F	F	T	F
T	F	F	F	F	T	F
F	T	T	F	T	T	T
F	T	F	T	F	F	F
F	F	T	F	T	T	T
F	F	F	F	T	T	T

4. *Conditional equivalence.* Which of the following implications are true?

a) If $2 + 2 = 5$ then $2 + 2 = 6$.

b) If $2 + 2 = 4$ then the world is flat.

c) If both of the previous statements are true then $2 + 2 = 7$.

Solution: a) True. b) False. c) True.

5. *Tautologies.* Which of the following are tautologies? If the statement is a tautology give a proof using the appropriate rules of logic demonstrated in class at each step of the proof. (Avoid using truth tables if possible.) If not, then justify your answer by giving a counter-example, i.e., a truth assignment which results in a false value.

a) $p \to (p \vee q)$.

b) $((p \vee q \vee r) \wedge (p \to r) \wedge (q \to r)) \to r$.

c) $(p \to (q \to r)) \leftrightarrow ((p \to q) \to r)$.

Solution a):

$$
\begin{aligned}
&p \to (p \vee q) \leftrightarrow \\
&(\neg p) \vee (p \vee q) \leftrightarrow \\
&(\neg p \vee p) \vee q \leftrightarrow \\
&1 \vee q \leftrightarrow 1.
\end{aligned}
$$

Solution b): If r is true then the implication is true so assume r is false. The formula is now true as long as

$$(p \vee q \vee 0) \wedge (p \to 0) \wedge (q \to 0) \leftrightarrow 0.$$

But

$$
\begin{aligned}
&(p \vee q \vee 0) \wedge (p \to 0) \wedge (q \to 0) \leftrightarrow \\
&(p \vee q) \wedge (\neg p) \wedge (\neg q) \leftrightarrow \\
&(p \vee q) \wedge (\neg(p \vee q)) \leftrightarrow \\
&0,
\end{aligned}
$$

as desired.

Solution c): Let p and r be false and let q be true. Then $q \to r$ is false, and $p \to q$ is true. The left side of the equivalence is then true and the right side is false. Thus the logic formula is not a tautology.

6. *Circuits.* Suppose we have a committee of four people. Design a circuit which determines if exactly two of them vote yes on an issue.

Solution : There are many solutions. The most straightforward one is the circuit corresponding to the following logic formula:

$$
\begin{aligned}
&(p \wedge q \wedge (\neg r) \wedge (\neg s)) \vee (p \wedge (\neg q) \wedge r \wedge (\neg s)) \vee (p \wedge (\neg q) \wedge (\neg r) \wedge s) \\
&\quad \vee ((\neg p) \wedge q \wedge r \wedge (\neg s)) \vee ((\neg p) \wedge q \wedge (\neg r) \wedge s) \vee ((\neg p) \wedge (\neg q) \wedge r \wedge s).
\end{aligned}
$$