Virtually Unipotent Curves in Some Non-NPC Graph Manifolds

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ABSTRACT

Let M be a graph manifold containing a single JSJ torus T and whose JSJ blocks are of the form $\Sigma \times S^1$, where Σ is a compact orientable surface with boundary. We show that if M does not admit a Riemannian metric of everywhere nonpositive sectional curvature, then there is an essential curve on T such that any finite-dimensional linear representation of $\pi_1(M)$ maps an element representing that curve to a matrix all of whose eigenvalues are roots of 1. In particular, this shows that $\pi_1(M)$ does not admit a faithful finite-dimensional unitary representation, and gives a new proof that $\pi_1(M)$ is not linear over any field of positive characteristic.

1. Introduction

A matrix $P \in \mathrm{GL}(n, \mathbb{F})$, where $\mathbb{F}$ is a field, is *unipotent* if 1 is the only eigenvalue of P over the algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$. We say a matrix $P \in \mathrm{GL}(n, \mathbb{F})$ is *virtually unipotent* if P^m is unipotent for some positive integer m, that is, if the eigenvalues of P are all roots of 1 in $\overline{\mathbb{F}}$. The only matrix in $\mathrm{GL}(n, \overline{\mathbb{F}})$ that is both unipotent and diagonalizable is the identity matrix; thus, a matrix in $\mathrm{GL}(n, \overline{\mathbb{F}})$ that is both virtually unipotent and diagonalizable has finite order.

We begin with an observation about a group consisting entirely of unipotent matrices: the integral Heisenberg group H, defined as the subgroup of $\mathrm{GL}(3, \mathbb{R})$ consisting of the upper unitriangular integer matrices. One might ask if H can be realized as a subgroup of $\mathrm{GL}(n, \mathbb{F})$, for some (possibly different) n and $\mathbb{F}$, that contains no nontrivial unipotent matrices. The following remark contains an elementary argument that this is impossible. See the work of Button for a proof of a more general result [But19, Theorem 3.2], and for a broader discussion on unipotent matrices in matrix groups.

Remark 1.1. Let

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and let $\rho : H \to \mathrm{GL}(n, \mathbb{F})$ be any representation. Up to replacing $\mathbb{F}$ with its algebraic closure and postconjugating ρ, we may assume that $\rho(z)$ has a block-diagonal structure

$$\rho(z) = \mathrm{diag}(Z_1, \dots, Z_k),$$

where $Z_r \in \mathrm{GL}(n_r, \mathbb{F})$ is upper triangular with a unique eigenvalue $\lambda_r \in \mathbb{F}^*$, and that $\lambda_1, \dots, \lambda_k$ are distinct. Since $\rho(x), \rho(y)$ commute with $\rho(z)$, each of the former preserves the generalized

eigenspaces of $\rho(z)$ and thus has a block-diagonal structure

$$\rho(x) = \mathrm{diag}(X_1, \ldots, X_k),$$
$$\rho(y) = \mathrm{diag}(Y_1, \ldots, Y_k),$$

where $X_r, Y_r \in \mathrm{GL}(n_r, \mathbb{F})$. Since $z = [x, y]$, we have $Z_r = [X_r, Y_r]$, and so

$$\lambda_r^{n_r} = \det Z_r = 1$$

for $r = 1, \ldots, k$. We conclude that $\rho(z)$ is a virtually unipotent matrix. Since z has infinite order (as does any nontrivial unipotent matrix with entries in a field of characteristic zero), it follows that some power of $\rho(z)$ is a nontrivial unipotent matrix if ρ is to be faithful.

Remark 1.1 motivates the following definition.

DEFINITION 1.2. An element γ of an arbitrary group Γ is *VU* if any finite-dimensional linear representation of Γ maps γ to a virtually unipotent matrix.

Remark 1.3. If Γ is a residually finite group, as are many groups of interest and, in particular, as is the fundamental group of any closed 3-manifold [Hem87], then for any nontrivial element $\gamma \in \Gamma$, there is a finite-dimensional unitary representation ρ of Γ such that $\rho(\gamma)$ is nontrivial and hence, by diagonalizability of unitary matrices, not unipotent. Thus, for our purposes, it is not sensible to omit the word "virtually" in Definition 1.2.

Remark 1.4. If γ is a VU element of a group Γ, then any element in the conjugacy class of γ is VU in Γ. Moreover, if Γ_0 is an abelian subgroup of Γ generated by VU elements of Γ, then any element of Γ_0 is VU in Γ. The latter follows from the fact that an abelian subgroup of $\mathrm{GL}(n, \mathbb{F})$, where $\mathbb{F}$ is an algebraically closed field, is conjugate to an upper triangular subgroup of $\mathrm{GL}(n, \mathbb{F})$ [RR00, Theorem 1.1.5].

Remark 1.5. Suppose Γ_0 is a finite-index normal subgroup of a group Γ, and that γ is a VU element of Γ. Then a generator γ_0 of $\langle \gamma \rangle \cap \Gamma_0$ is a VU element of Γ_0. Indeed, let ρ_0 be a finite-dimensional linear representation of Γ_0. Then ρ_0 is a direct summand of the restriction $\rho|_{\Gamma_0}$, where ρ is the representation induced by ρ_0 on Γ. Since $\rho(\gamma_0)$ is a virtually unipotent matrix, it follows that the same is true for $\rho_0(\gamma_0)$.

Remark 1.6. Lubotzky–Mozes–Raghunathan [LMR00, Prop. 2.4] showed that an element generating a distorted cyclic subgroup of a finitely generated group is VU.

Note that a finite-order element of any group is VU. From Remark 1.1 (or Remark 1.6), one observes that the integer Heisenberg group H, viewed as an abstract group, contains an *infinite-order* VU element (namely, a generator of the center of H), and hence by Remark 1.5 so does the fundamental group of any closed 3-manifold with Nil geometry. In fact, the argument in Remark 1.1 shows that if an element γ of a group Γ is contained in $[C_\Gamma(\gamma), C_\Gamma(\gamma)]$, where $C_\Gamma(\gamma)$ is the centralizer of γ in Γ, then γ is a VU element of Γ. Thus, for example, an element of $\pi_1(M)$ representing a Seifert fiber of a closed 3-manifold M with $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ geometry is VU in $\pi_1(M)$.

A manifold is said to be *nonpositively curved (NPC)* if it admits a Riemannian metric of everywhere nonpositive sectional curvature. Closed 3-manifolds locally modeled on Nil or $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ are not NPC [GW71, Yau71, Ebe82]. The purpose of this article is to exhibit nontrivial VU elements within fundamental groups of non-NPC 3-manifolds of a different nature.

Theorem 1.7. *Let M be a connected closed orientable irreducible 3-manifold containing exactly one JSJ torus, and each of whose JSJ blocks is a product of S^1 with a surface. If M is not NPC, then $\pi_1(M)$ contains a nontrivial VU element.*

We use a necessary and sufficient condition (Theorem 2.2) for such 3-manifolds M to be NPC due to Buyalo–Kobel'skii [BK95], and independently Kapovich–Leeb [KL96] in the case that M has two JSJ blocks. Our argument is similar to Button's proof that Gersten's free-by-cyclic group contains a nontrivial VU element [But17a, Theorem 4.5]. We remark that if M is a 3-manifold as in the statement of Theorem 1.7 that is not the mapping torus of an Anosov homeomorphism of the 2-torus, then it follows from [KL98] that all cyclic subgroups of $\pi_1(M)$ are undistorted.

An example of a 3-manifold M as in the statement of Theorem 1.7 is the mapping torus of a Dehn twist about an essential simple closed curve on a closed orientable surface of genus at least 2 [KL96, Theorem 3.7]. In this case, our proof in fact shows that an element of $\pi_1(M)$ representing that curve is VU.

Since a 3-manifold M as in the statement of Theorem 1.7 is aspherical, any nontrivial element of $\pi_1(M)$ has infinite order. We are interested in infinite-order VU elements, since the existence of such an element within a group has interesting representation theoretic consequences for the group. The following definition is due to Button [But17a, Definition 2.2].

Definition 1.8. *A group Γ is NIU-linear if there is a faithful finite-dimensional linear representation ρ of Γ such that $\rho(\Gamma)$ does not contain unipotent matrices of infinite order.*

Since unitary matrices are diagonalizable, any group admitting a faithful finite-dimensional unitary representation is NIU-linear (in fact, the image of such a representation will contain no nontrivial unipotent matrices). Furthermore, if $\mathbb{F}$ is a field of positive characteristic, then every unipotent matrix over $\mathbb{F}$ has finite order [But17a, Proposition 2.1], and so any group admitting a faithful finite-dimensional linear representation over such $\mathbb{F}$ is NIU-linear. Since a group containing an infinite-order VU element is evidently not NIU-linear, such a group neither admits a faithful finite-dimensional unitary representation, nor a faithful finite-dimensional representation over a field of positive characteristic.

As a consequence of Theorem 1.7 and the work of several others, we obtain the following corollary.

Corollary 1.9. *Let M be as in the statement of Theorem 1.7 and let $\Gamma = \pi_1(M)$. Then the following are equivalent:*

(i) *M is NPC;*

(ii) *Γ virtually embeds in a finitely generated right-angled Artin group;*

(iii) *Γ admits a faithful finite-dimensional unitary representation;*

(iv) *Γ admits a faithful finite-dimensional linear representation over a field of positive characteristic;*

(v) *Γ is NIU-linear;*

(vi) *Γ does not contain a nontrivial VU element.*

We explain how Corollary 1.9 can be established using Theorem 1.7. For us, a *graph manifold* is a connected closed orientable irreducible non-Seifert 3-manifold all of whose JSJ blocks are Seifert. The manifolds described in the statement of Theorem 1.7 can be thought of as the simplest examples of graph manifolds. That (i) implies (ii) in Corollary 1.9 is due to Liu [Liu13],

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who showed that the fundamental group of any NPC graph manifold is virtually a subgroup of a finitely generated right-angled Artin group (RAAG). Agol [Ago18] showed that such a RAAG embeds in $\mathrm{U}(n)$ for some n, so that (ii) implies (iii). Moreover, the work of Berlai–de la Nuez González [BG19] implies that a finitely generated RAAG admits a faithful finite-dimensional linear representation over a field of positive characteristic (indeed, *any* prime characteristic), and so (ii) also implies (iv). It was discussed before the statement of Corollary 1.9 that each of (iii) and (iv) imply (v), and that (v) implies (vi). Finally, the fact that (vi) implies (i) is precisely the statement of Theorem 1.7.

Remark 1.10. Note that if γ is an infinite-order VU element of a group Γ, then $\langle \gamma \rangle \subset \Gamma$ is not a virtual retract of Γ by Remark 1.5. Thus, the fact that statement (ii) in Corollary 1.9 implies statement (vi) also follows from work of Minasyan [Min19].

Remark 1.11. It was already known that statement (iv) in Corollary 1.9 does not hold for the fundamental group Γ of any non-NPC graph manifold. Indeed, Button [But19] proved that any finitely generated group Γ satisfying (iv) acts properly by semisimple isometries on a complete CAT(0) metric space, and Leeb [Lee92] showed that if the fundamental group of a graph manifold M admits such an action, then M is NPC. At the time of writing of this article, it is not known if a single non-NPC graph manifold without Sol geometry admits a faithful finite-dimensional linear representation over a field of characteristic zero.

Remark 1.12. It follows from the Lie–Kolchin–Mal'cev theorem that if Γ is a polycyclic group then Γ is NIU-linear if and only if Γ is virtually abelian [But17b]. Thus, for example, closed 3-manifolds with Nil or Sol geometry do not have NIU-linear fundamental groups. However, this handy obstruction to NIU-linearity is not useful for dealing with most 3-manifolds M as in the statement of Theorem 1.7 since, for all such M apart from the mapping torus of an Anosov homeomorphism of the 2-torus, any subgroup of $\pi_1(M)$ lacking a nonabelian free subgroup is in fact abelian [FLS11].

Statements (i)-(v) in Corollary 1.9 were known to be equivalent for M a closed aspherical geometric 3-manifold and $\Gamma = \pi_1(M)$ [But17a]. It is also true that (vi) is equivalent to (i)-(v) in this case. Indeed, to argue this, it suffices to show that if M is a closed aspherical geometric non-NPC 3-manifold, that is, if M is a closed 3-manifold with Nil, $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, or Sol geometry, then $\pi_1(M)$ contains a nontrivial VU element. This was already justified for the first two geometries, and closed Sol 3-manifolds are handled by Theorem 1.7. Indeed, in a group Γ of the form $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ where $A \in \mathrm{SL}(2,\mathbb{Z})$ is hyperbolic, any nontrivial element of the $\mathbb{Z}^2$ factor generates a distorted cyclic subgroup of Γ [NS20, Lemma 5.2] and is thus VU (see Remark 1.6).

We conjecture the following.

CONJECTURE 1.13. *Statements (i)-(vi) in Corollary 1.9 are equivalent for any closed aspherical 3-manifold M and $\Gamma = \pi_1(M)$.*

By work of Agol [AGM13] and Przytycki–Wise [PW18], and the aforementioned work of Liu [Liu13], the fundamental group Γ of any closed NPC 3-manifold satisfies (ii) and hence (iii)-(vi) in Corollary 1.9. Furthermore, a closed aspherical non-geometric non-NPC 3-manifold is (up to passing to its orientation cover) a graph manifold [Lee92]. Thus, Conjecture 1.13 amounts to the claim that the fundamental group of any non-NPC graph manifold contains a nontrivial VU element.

Organization

In Section 2, we fix the language in which we prove Theorem 1.7, and also present two lemmas that will be useful in the proof. The proof of Theorem 1.7 is contained in Section 3 and is divided into two cases: the case that there is a single block in the JSJ decomposition of the 3-manifold M (Theorem 3.1), and the case that there are two (Theorem 3.2). The two proofs are very similar and are somewhat technical, but involve only elementary linear algebra.

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2. Preliminaries

2.1 Definitions

If S is a (not necessarily connected) closed surface embedded in a 3-manifold M, we denote by $M\big| S$ the complement in M of a small open tubular neighborhood of S. If M is a connected closed orientable irreducible 3-manifold, then there is, up to isotopy, a unique minimal collection $\mathcal{E}$ of disjoint embedded incompressible tori such that each component of $M\big|\bigcup\mathcal{E}$ is either Seifert or atoroidal (see, for example, [Kap01, Thm 1.41] and the references therein). The decomposition of M into the components of $M\big|\bigcup\mathcal{E}$ is called the *Jaco–Shalen–Johannson (JSJ) decomposition* of M. If $\mathcal{E} = \emptyset$, we say M has *trivial* JSJ decomposition. Note that if M is the mapping torus of an Anosov homeomorphism of the 2-torus, then M has nontrivial JSJ decomposition.

Let $\mathfrak{G}$ denote the class of all connected closed orientable irreducible non-Seifert 3-manifolds M such that each component M_v of $M\big|\bigcup\mathcal{E}$, where $\mathcal{E}$ is the collection of JSJ tori in M, is a trivial S^1-bundle over a compact orientable surface Σ_v with boundary. The manifolds M_v are the *blocks* of M. The *underlying graph* $\mathcal{G} = \mathcal{G}(M)$ of M is the graph dual to the JSJ decomposition of M; the graph $\mathcal{G}$ is well-defined since the collection $\mathcal{E}$ is unique up to isotopy. We identify the vertex set $\mathcal{V}$ of $\mathcal{G}$ with the set of blocks of M, and the set of unoriented edges of $\mathcal{G}$ with $\mathcal{E}$. Denote by $\mathcal{W}$ the set of oriented edges of $\mathcal{G}$. We identify $\mathcal{W}$ with the set of boundary components of $M\big|\bigcup\mathcal{E}$ by assigning to each oriented edge $w \in \partial v \subset \mathcal{W}$ the corresponding boundary component T_w of M_v.

Choose an orientation of M, thereby inducing an orientation on each block M_v of M, and hence on each component of ∂M_v. For each $v \in \mathcal{V}$, choose an orientation of the fibers in M_v, as well as a *Waldhausen basis* for $H_1(\partial M_v; \mathbb{Z})$; that is, a basis $\{(f_w, z_w) \mid w \in \partial v\}$ for $H_1(\partial M_v; \mathbb{Z}) = \bigoplus_{w \in \partial v} H_1(T_w; \mathbb{Z})$ such that the elements f_w represent oriented fibers, the algebraic intersection number $\hat{i}(z_w, f_w)$ on T_w is $+1$, and the sum $\oplus_{w \in \partial v} z_w$ lies in the kernel of the map $H_1(\partial M_v; \mathbb{Z}) \to H_1(M_v; \mathbb{Z})$ induced by inclusion. We call the additional structure on M given by the choices made in this paragraph a *framing* of M.

An oriented edge $w \in \mathcal{W}$ corresponds to a gluing homeomorphism $T_{-w} \to T_w$, which induces an isomorphism $\phi_w : H_1(T_{-w}; \mathbb{Z}) \to H_1(T_w; \mathbb{Z})$. Define $B_w = \begin{pmatrix} a_w & b_w \\ c_w & d_w \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z})$ to be the matrix whose entries satisfy

$$\phi_w(f_{-w}) = a_w f_w + b_w z_w,$$
$$\phi_w(z_{-w}) = c_w f_w + d_w z_w.$$

Note that $\det B_w = -1$ since M is orientable, that $B_{-w} = B_w^{-1}$, and that $b_w \neq 0$ by minimality of $\mathcal{E}$.

This article is concerned with the subclasses $\mathfrak{E}, \mathfrak{L}$ of $\mathfrak{G}$ consisting of all manifolds M in $\mathfrak{G}$ whose underlying graph is a single edge (joining distinct vertices) or a loop, respectively. We call $B \in \mathrm{GL}(2, \mathbb{Z})$ a *gluing matrix* for such a manifold M if $B = B_w$ for an oriented edge w of $\mathcal{G}(M)$ with respect to some framing of M.

The fundamental group $\pi_1(M)$ of a manifold $M \in \mathfrak{E}$ with gluing matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and whose blocks $M_v, M_{v'}$ have base surfaces Σ, Σ' of genus g, g', respectively, is isomorphic to the group $\Gamma_{g,g',B}^{\mathfrak{E}}$ given by the presentation with generators

$$x_1, y_1, \ldots, x_g, y_g, z, f,$$
$$x_1', y_1', \ldots, x_{g'}', y_{g'}', z', f'$$

subject to the relations

(I) $z = \prod_{i=1}^g [x_i, y_i]$,

(II) $[x_i, f] = [y_i, f] = 1$ for $i = 1, \ldots, g$,

(III) $z' = \prod_{i=1}^{g'} [x_i', y_i']$,

(IV) $[x_i', f'] = [y_i', f'] = 1$ for $i = 1, \ldots, g'$,

(V) $f' = f^a z^b$,

(VI) $z' = f^c z^d$,

where the subgroup $\langle x_1, y_1, \ldots, x_g, y_g \rangle$ (resp., $\langle x_1', y_1', \ldots, x_{g'}', y_{g'}' \rangle$) is the image of the map $\pi_1(\Sigma) \to \pi_1(M)$ (resp., $\pi_1(\Sigma') \to \pi_1(M)$) induced by the inclusions $\Sigma \subset M_v \subset M$ (resp., $\Sigma' \subset M_{v'} \subset M$), and the element f (resp., f') represents an oriented fiber of M_v (resp., $M_{v'}$).

Remark 2.1. Note that if C is obtained from B by negating a row or a column of B, then $\Gamma_{g,g',B}^{\mathfrak{E}} \cong \Gamma_{g,g',C}^{\mathfrak{E}}$. Note also that $\Gamma_{g,g',B}^{\mathfrak{E}} \cong \Gamma_{g',g,B^{-1}}^{\mathfrak{E}}$.

The fundamental group $\pi_1(M)$ of a manifold $M \in \mathfrak{L}$ with gluing matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and the base surface Σ of whose unique block M_v has genus g is isomorphic to the group $\Gamma_{g,B}^{\mathfrak{L}}$ given by the presentation with generators

$$x_1, y_1, \ldots, x_g, y_g, z, z', f, t$$

subject to the relations

(1) $zz' = \prod_{i=1}^g [x_i, y_i]$,

(2) $[x_i, f] = [y_i, f] = [z, f] = 1$ for $i = 1, \ldots, g$,

(3) $tft^{-1} = f^a z^b$,

(4) $tz't^{-1} = f^c z^d$,

where the subgroup $\langle x_1, y_1, \ldots, x_g, y_g, z \rangle$ is the image of the map $\pi_1(\Sigma) \to \pi_1(M)$ induced by the inclusion $\Sigma \subset M_v \subset M$, and the element f represents an oriented fiber of M_v.

The following theorem is a special case of a result of Buyalo–Kobel'skii [BK95], and was proved independently by Kapovich–Leeb [KL96] in the case $M \in \mathfrak{E}$.

THEOREM 2.2. *Let $M \in \mathfrak{E}$ (resp., $M \in \mathfrak{L}$) and let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z})$ be a gluing matrix for M. Then M is NPC if and only if $a = d = 0$ (resp., if and only if $|a - d| < 2$).*

2.2 Basic lemmas

The following lemma will allow us to conjugate a representation ρ of the appropriate group Γ in a manner that makes the interactions between generalized eigenspaces of certain elements of $\rho(\Gamma)$ more apparent.

LEMMA 2.3. *Let $\mathbb{F}$ be an algebraically closed field, let $P, P', Q \in \mathrm{M}_{n \times n}(\mathbb{F})$, and let $\lambda_1, \ldots, \lambda_k$ (resp. $\lambda_1', \ldots, \lambda_\ell'$) be the distinct eigenvalues of P (resp. P'). If P, P', Q pairwise commute, then there is a single matrix $C \in \mathrm{GL}(n, \mathbb{F})$ such that*

$$CPC^{-1} = \mathrm{diag}\left(P_{1,1}, \ldots, P_{1,\ell}, \ldots, P_{k,1}, \ldots, P_{k,\ell}\right),$$
$$CP'C^{-1} = \mathrm{diag}\left(P'_{1,1}, \ldots, P'_{1,\ell}, \ldots, P'_{k,1}, \ldots, P'_{k,\ell}\right),$$
$$CQC^{-1} = \mathrm{diag}\left(Q_{1,1}, \ldots, Q_{1,\ell}, \ldots, Q_{k,1}, \ldots, Q_{k,\ell}\right),$$

where $P_{r,s}, P'_{r,s}, Q_{r,s}$ are (possibly empty) upper triangular matrices and the only eigenvalue of $P_{r,s}$ (resp., $P'_{r,s}$) is λ_r (resp., λ_s').

Proof. For $r = 1, \ldots, k$, let W_r be the generalized λ_r-eigenspace of P, and let $n_r = \dim W_r$. We index the standard ordered basis for $\mathbb{F}^n$ as follows:

$$(e_{1,1}, \ldots, e_{1,n_1}, \ldots, e_{k,1}, \ldots, e_{k,n_k}).$$

We may assume that $W_r = \mathrm{Span}(e_{r,1}, \ldots, e_{r,n_r})$. Since each of P', Q commutes with P, we have that P', Q preserve the generalized eigenspaces of P, so P, P', Q share a block-diagonal structure

$$P = \mathrm{diag}(P_1, \ldots, P_k),$$
$$P' = \mathrm{diag}(P'_1, \ldots, P'_k),$$
$$Q = \mathrm{diag}(Q_1, \ldots, Q_k),$$

where $P_r, P'_r \in \mathrm{M}_{n_r \times n_r}(\mathbb{F})$. We may also assume that for some indexing

$$(e_{r,1,1}, \ldots, e_{r,1,n_{r,1}}, \ldots, e_{r,\ell,1}, \ldots, e_{r,\ell,n_{r,\ell}})$$

of the ordered basis $(e_{r,1}, \ldots, e_{r,n_r})$ for W_r, where the $n_{r,s}$ are nonnegative integers satisfying $\sum_{s=1}^{\ell} n_{r,s} = n_r$, we have that $\mathrm{Span}(e_{r,s,1}, \ldots, e_{r,s,n_{r,s}})$ is the generalized λ_s'-eigenspace of P'_r. Since each of P_r, Q_r commutes with P'_r, we have that P_r, Q_r preserve the generalized eigenspaces of P'_r, so P_r, P'_r, Q_r share a block-diagonal structure

$$P_r = \mathrm{diag}(P_{r,1}, \ldots, P_{r,\ell}),$$
$$P'_r = \mathrm{diag}(P'_{r,1}, \ldots, P'_{r,\ell}),$$
$$Q_r = \mathrm{diag}(Q_{r,1}, \ldots, Q_{r,\ell}).$$

Now since $P_{r,s}, P'_{r,s}, Q_{r,s}$ pairwise commute, they are simultaneously upper triangularizable [RR00, Theorem 1.1.5], and Lemma 2.3 follows. $\square$

The following lemma will allow us to reduce systems of equations whose unknowns lie in $\mathbb{F}^*$, where $\mathbb{F}$ is some field, to systems of linear equations with integer unknowns. It is a step in the proof of Theorem 4.5 in [But17a]. We include Button's argument for the convenience of the reader.

Lemma 2.4. *Let M be an integer matrix with L columns and suppose there is a subset $I \subset$ $\{1,\ldots,L\}$ such that for any $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_L)^T \in \mathbb{Z}^L$ satisfying $M\boldsymbol{\alpha} = 0$, we have $\alpha_i = 0$ for $i \in I$. Let A be a torsion-free abelian group, and suppose $\mathbf{a} = (a_1,\ldots,a_L)^T \in A^L$ satisfies $M\mathbf{a} = 0$. Then $a_i = 0$ for $i \in I$.*

Proof. Let $A_0 = \langle a_1,\ldots,a_L \rangle \subset A$. Then A_0 is a finitely generated torsion-free abelian group, so there is an isomorphism $\varphi : A_0 \to \mathbb{Z}^K$ for some K. For each $j = 1,\ldots,K$, we have

$$M(\varphi_j(a_1),\ldots,\varphi_j(a_L))^T = 0,$$

where $\varphi_j = p_j \circ \varphi$ and $p_j : \mathbb{Z}^K \to \mathbb{Z}$ is the projection onto the jth coordinate, so that $\varphi_j(a_i) = 0$ for $i \in I$. We conclude that $\varphi(a_i) = 0$, and hence $a_i = 0$, for $i \in I$. $\qquad\square$

3. Proof of Theorem 1.7

We divide Theorem 1.7 into Theorem 3.1 (the loop case) and Theorem 3.2 (the edge case), and prove each separately.

Theorem 3.1. *Suppose $M \in \mathfrak{L}$ is not NPC, and let $\Gamma = \pi_1(M)$. Then Γ contains a nontrivial VU element.*

Proof. By Theorem 2.2, we have $\Gamma = \Gamma_{g,B}^{\mathfrak{L}}$ for some $g \geqslant 0$ and $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z})$ with $\det B = -1$, $b \neq 0$, and $|a - d| \geqslant 2$. We show that $f^{a-1}z^b \in \Gamma$ is VU if $a - d \geqslant 2$, and that $f^{a+1}z^b$ is VU if $a - d \leqslant -2$.

Let $\mathbb{F}$ be an algebraically closed field, $n \geqslant 1$, and $\rho : \Gamma \to \mathrm{GL}(n,\mathbb{F})$ any representation. We may assume that ρ is indecomposable. Let $\lambda_1,\ldots,\lambda_k \in \mathbb{F}^*$ be the distinct eigenvalues of $\rho(f)$, and let $f' = tft^{-1}$. By Lemma 2.3 and relation (3) in the presentation of Γ, we may assume further that

$$\rho(f) = \mathrm{diag}(F_{1,1},\ldots,F_{1,k},\ldots,F_{k,1},\ldots,F_{k,k}),$$
$$\rho(f') = \mathrm{diag}(F'_{1,1},\ldots,F'_{1,k},\ldots,F'_{k,1},\ldots,F'_{k,k}),$$
$$\rho(z) = \mathrm{diag}(D_{1,1}Z_{1,1},\ldots,D_{1,k}Z_{1,k},\ldots,D_{k,1}Z_{k,1},\ldots,D_{k,k}Z_{k,k}),$$

where $F_{r,s}, F'_{r,s}, Z_{r,s} \in \mathrm{GL}(n_{r,s},\mathbb{F})$ are (possibly empty) upper triangular matrices, $D_{r,s}$ is a (possibly empty) diagonal matrix in $\mathrm{GL}(n_{r,s},\mathbb{F})$ whose diagonal entries are $|b|^{\mathrm{th}}$ roots of 1 in $\mathbb{F}$, and the only eigenvalue of $F_{r,s}$ (resp., $F'_{r,s}, Z_{r,s}$) is λ_r (resp., $\lambda_s, \mu_{r,s}$), with $\mu_{r,s} \in \mathbb{F}^*$ satisfying

$$\lambda_s = \lambda_r^a \mu_{r,s}^b. \tag{3.1}$$

Since, by relation (2) in the presentation of Γ, each of $\rho(z'), \rho(x_1), \rho(y_1),\ldots,\rho(x_g), \rho(y_g)$ commutes with $\rho(f)$, each preserves the generalized eigenspaces of $\rho(f)$, and so

$$\rho(z') = \mathrm{diag}(Z'_1,\ldots,Z'_k),$$
$$\rho(x_i) = \mathrm{diag}(X_1^{(i)},\ldots,X_k^{(i)}),$$
$$\rho(y_i) = \mathrm{diag}(Y_1^{(i)},\ldots,Y_k^{(i)})$$

for some $Z'_r, X_r^{(i)}, Y_r^{(i)} \in \mathrm{GL}(n_r,\mathbb{F})$, where $n_r = \sum_{s=1}^k n_{r,s}$ is the dimension of the generalized λ_r-eigenspace of $\rho(f)$.

Let V_r be the generalized λ_r-eigenspace of $\rho(f')$. Then $\rho(t)^{-1}V_r$ is the generalized λ_r-eigenspace

of $\rho(t)^{-1}\rho(f')\rho(t) = \rho(f)$, and the characteristic polynomial of

$$\rho(z')\big|_{\rho(t)^{-1}V_r} = \rho(t)^{-1}\rho(f^ch^d)\rho(t)\big|_{\rho(t)^{-1}V_r}$$

coincides with the characteristic polynomial of $\rho(f^cz^d)\big|_{V_r}$. Thus, up to multiplying each root by a root of 1, the characteristic polynomial of the block Z_r' is

$$(x - \lambda_1^c\mu_{1,r}^d)^{n_{1,r}} \ldots (x - \lambda_k^c\mu_{k,r}^d)^{n_{k,r}}.$$

Now let $Z_r = \mathrm{diag}(D_{r,1}Z_{r,1}, \ldots, D_{r,k}Z_{r,k})$. Then, by relation (1) in the presentation of Γ, we have $Z_rZ_r' = \prod_{i=1}^g [X_r^{(i)}, Y_r^{(i)}]$, so that $\det(Z_rZ_r') = 1$. It follows that

$$\prod_{s=1}^k \mu_{r,s}^{n_{r,s}}(\lambda_s^c\mu_{s,r}^d)^{n_{s,r}} = 1 \tag{3.2}$$

in the quotient A of the group of units $\mathbb{F}^*$ by its torsion subgroup. Viewing (3.1) also as equations in A and switching to additive notation within A, we obtain the equations

$$\lambda_s = a\lambda_r + b\mu_{r,s}, \tag{3.3}$$

$$\sum_{s=1}^k (n_{r,s}\mu_{r,s} + n_{s,r}(c\lambda_s + d\mu_{s,r})) = 0. \tag{3.4}$$

Multiplying (3.4) by b and substituting $\lambda_s - a\lambda_r$ for $b\mu_{r,s}$, we have

$$\sum_{s=1}^k \Big(n_{r,s}(\lambda_s - a\lambda_r) + n_{s,r}\big(bc\lambda_s + d(\lambda_r - a\lambda_s)\big)\Big) = 0$$

and so

$$\sum_{s=1}^k \big(n_{r,s} + (bc - ad)n_{s,r}\big)\lambda_s = \lambda_r \sum_{s=1}^k (an_{r,s} - dn_{s,r}). \tag{3.5}$$

Since $bc - ad = -\det B = 1$, the left-hand side of (3.5) is equal to $\sum_{s=1}^k (n_{r,s} + n_{s,r})\lambda_s$. On the other hand, since $\sum_{s=1}^k n_{s,r} = \sum_{s=1}^k n_{r,s} = n_r$, the right-hand side of (3.5) is equal to $(a-d)n_r\lambda_r$.

In summary, $\lambda_1, \ldots, \lambda_k$ satisfy

$$\sum_{s=1}^k (n_{r,s} + n_{s,r})\lambda_s = (a - d)n_r\lambda_r \tag{3.6}$$

as elements of A.

Consider the case that $a - d \geqslant 2$. We show that, in this case, if we set $A = \mathbb{Z}$, then (3.6) implies $\lambda_1 = \ldots = \lambda_k$, so that

$$(a - 1)\lambda_r + b\mu_{r,s} = a\lambda_r - \lambda_s + b\mu_{r,s} = 0,$$

where the second equality follows from (3.3). By Lemma 2.4, it will follow that $(a-1)\lambda_r + b\mu_{r,s} = 0$ in the original torsion-free abelian group A, i.e., that $\rho(f^{a-1}z^b)$ is virtually unipotent, and thus completing the proof of the $a - d \geqslant 2$ case.

To that end, suppose for a contradiction that the integers $\lambda_1, \ldots, \lambda_k$ are not all equal. Then we may assume

$$\lambda_1 = \ldots = \lambda_{r_0} > \lambda_{r_0+1}, \ldots, \lambda_k$$

for some $r_0 \in \{1, \ldots, k-1\}$. Thus, for $r = 1, \ldots, r_0$, either we have $n_{r,s} + n_{s,r} = 0$ for all $s > r_0$, or we obtain the contradiction

$$2n_r\lambda_r = \sum_{s=1}^{k}(n_{r,s} + n_{s,r})\lambda_r > \sum_{s=1}^{k}(n_{r,s} + n_{s,r})\lambda_s = (a-d)n_r\lambda_r \geqslant 2n_r\lambda_r.$$

We conclude that $n_{r,s} = n_{s,r} = 0$ for $r \leqslant r_0$ and $s > r_0$, so that $\rho(t)$ preserves the span of the first $\sum_{r=1}^{r_0} n_r$ standard basis vectors and the span of the last $\sum_{r=r_0+1}^{k} n_r$ standard basis vectors of $\mathbb{F}^n$. But then $\rho(\Gamma)$ also preserves each of these subspaces, contradicting the indecomposability of ρ. This completes the proof for the case $a - d \geqslant 2$.

We now consider the case $a - d \leqslant -2$. We show in this case that, considered over $\mathbb{Z}$, equations (3.3) and (3.6) imply that $(a+1)\lambda_r + b\mu_{r,s} = 0$ for all r, s with $n_{r,s} \neq 0$. One then applies Lemma 2.4 to conclude that $\rho(f^{a+1}z^b)$ is virtually unipotent.

We first observe that, setting $A = \mathbb{Z}$, equation (3.6) implies that $|\lambda_1| = \ldots = |\lambda_k|$. Indeed, suppose otherwise. Then we may assume $|\lambda_1| = \ldots = |\lambda_{r_0}| > |\lambda_{r_0+1}|, \ldots, |\lambda_k|$ for some $r_0 \in \{1, \ldots, k-1\}$. Thus, for $r \leqslant r_0$, either we have $n_{r,s} + n_{s,r} = 0$ when $s > r_0$, or we obtain the contradiction

$$2n_r|\lambda_r| = \sum_{s=1}^{k}(n_{r,s} + n_{s,r})|\lambda_r| > \sum_{s=1}^{k}(n_{r,s} + n_{s,r})|\lambda_s| \geqslant \left|\sum_{s=1}^{k}(n_{r,s} + n_{s,r})\lambda_s\right| = |a-d|n_r|\lambda_r|$$
$$\geqslant 2n_r|\lambda_r|.$$

We conclude that indeed $n_{r,s} = n_{s,r} = 0$ for $r \leqslant r_0$ and $s > r_0$, so that $\rho(t)$ preserves the span of the first $\sum_{r=1}^{r_0} n_r$ standard basis vectors and that of the last $\sum_{r=r_0+1}^{k} n_r$ standard basis vectors of $\mathbb{F}^n$. But then $\rho(\Gamma)$ also preserves each of these subspaces, contradicting the indecomposability of ρ. This establishes our claim that $|\lambda_1| = \ldots = |\lambda_k|$.

Now suppose for a contradiction that there is a solution to equation (3.6) with $\lambda_1 = \ldots = \lambda_k = \lambda$, where λ is a nonzero integer. Then we have

$$2n_1\lambda = \sum_{s=1}^{k}(n_{1,s} + n_{s,1})\lambda = (a-d)n_1\lambda,$$

and hence $a - d = 2$, contradicting our assumption that $a - d \leqslant -2$.

Thus, if the λ_r are to satisfy equation (3.6), then either $\lambda_1 = \ldots = \lambda_k = 0$, or, up to reordering the λ_r, we have $\lambda_1 = \ldots = \lambda_{r_0} = -\lambda_{r_0+1} = \ldots = -\lambda_k = \lambda$ for some nonzero integer λ and some $r_0 \in \{1, \ldots, k-1\}$. In either case, we have by (3.3) that

$$(a+1)\lambda_r + b\mu_{r,s} = a\lambda_r - \lambda_s + b\mu_{r,s} = 0$$

whenever $r \leqslant r_0$ and $s > r_0$, and whenever $r > r_0$ and $s \leqslant r_0$. This is in fact true for all r and s in the first case. In the second case, we have by (3.6) that for $r > r_0$,

$$(a-d)n_r(-\lambda) = \sum_{s=1}^{r_0}(n_{r,s} + n_{s,r})\lambda - \sum_{s=r_0+1}^{k}(n_{r,s} + n_{s,r})\lambda = 2n_r\lambda - 2\sum_{s=r_0+1}^{k}(n_{r,s} + n_{s,r})\lambda,$$

and so

$$2\sum_{s=r_0+1}^{k}(n_{r,s} + n_{s,r}) = (2 + (a-d))n_r \leqslant 0$$

since $a - d \leqslant -2$, from which we conclude that $n_{r,s} = 0$ for $r, s > r_0$, and hence also for $r, s \leqslant r_0$ (in other words, the matrix $\rho(t)$ interchanges the span of the first $\sum_{r=1}^{r_0} n_r$ standard basis vectors with that of the last $\sum_{r=r_0+1}^{k} n_r$ standard basis vectors of $\mathbb{F}^n$). This completes the proof for the case $a - d \leqslant -2$. □

THEOREM 3.2. *Suppose $M \in \mathfrak{E}$ is not NPC, and let $\Gamma = \pi_1(M)$. Then Γ contains a nontrivial VU element.*

Proof. We have $\Gamma = \Gamma_{g,g',B}^{\mathfrak{E}}$ for some $g, g' \geqslant 1$, where $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a gluing matrix for M. Note that $b \neq 0$, and that, by Theorem 2.2, one of a, d is nonzero. By Remark 2.1, up to replacing B with its inverse, we may assume $a \neq 0$. Furthermore, by Remark 2.1 and the fact that $|\det B| = 1$, up to negating rows and columns of B, we may assume $a, b, c, d \geqslant 0$. We show that if $c = 0$ (resp., $c > 0$) then z (resp., f) is a VU element of Γ.

Let $\mathbb{F}$ be an algebraically closed field, $n \geqslant 1$, and $\rho : \Gamma \to \mathrm{GL}(n, \mathbb{F})$ any representation. Let $\lambda_1, \ldots, \lambda_k \in \mathbb{F}^*$ (resp., $\lambda_1', \ldots, \lambda_\ell' \in \mathbb{F}^*$) be the distinct eigenvalues of $\rho(f)$ (resp., $\rho(f')$). By Lemma 2.3 and relation (V) in the presentation of Γ, we may assume that

$$\rho(f) = \mathrm{diag}(F_{1,1}, \ldots, F_{1,\ell}, \ldots, F_{k,1}, \ldots, F_{k,\ell}),$$
$$\rho(f') = \mathrm{diag}(F_{1,1}', \ldots, F_{1,\ell}', \ldots, F_{k,1}', \ldots, F_{k,\ell}'),$$
$$\rho(z) = \mathrm{diag}(D_{1,1}Z_{1,1}, \ldots, D_{1,\ell}Z_{1,\ell}, \ldots, D_{k,1}Z_{k,1}, \ldots, D_{k,\ell}Z_{k,\ell}),$$

where $F_{r,s}, F_{r,s}', Z_{r,s} \in \mathrm{GL}(n_{r,s}, \mathbb{F})$ are (possibly empty) upper triangular matrices, $D_{r,s}$ is a diagonal matrix in $\mathrm{GL}(n_{r,s}, \mathbb{F})$ whose diagonal entries are b^{th} roots of 1 in $\mathbb{F}$, and the only eigenvalue of $F_{r,s}$ (resp., $F_{r,s}', Z_{r,s}$) is λ_r (resp., $\lambda_s', \mu_{r,s}$), with $\mu_{r,s} \in \mathbb{F}^*$ satisying

$$\lambda_s' = \lambda_r^a \mu_{r,s}^b. \tag{3.7}$$

Since, by relation (II) in the presentation of Γ, the $\rho(x_i), \rho(y_i)$ commute with $\rho(f)$, each of the former preserves the generalized eigenspaces of $\rho(f)$. Thus, we have

$$\rho(x_i) = \mathrm{diag}(X_1^{(i)}, \ldots, X_k^{(i)}),$$
$$\rho(y_i) = \mathrm{diag}(Y_1^{(i)}, \ldots, Y_k^{(i)}),$$

for some $X_r^{(i)}, Y_r^{(i)} \in \mathrm{GL}(n_r, \mathbb{F})$, where $n_r = \sum_{s=1}^{k} n_{r,s}$ is the dimension of the generalized λ_r-eigenspace of $\rho(f)$. Letting $Z_r = \mathrm{diag}(D_{r,1}Z_{r,1}, \ldots, D_{r,\ell}Z_{r,\ell})$, we have by relation (I) in the presentation of Γ that

$$Z_r = \prod_{i=1}^{g} [X_r^{(i)}, Y_r^{(i)}]$$

for $r = 1, \ldots, k$. Thus, $\det Z_r = 1$, and so

$$\prod_{s=1}^{\ell} \mu_{r,s}^{n_{r,s}} = 1 \tag{3.8}$$

in the quotient A of $\mathbb{F}^*$ by its torsion subgroup.

Since, by relation (IV) in the presentation of Γ, the $\rho(x_i'), \rho(y_i')$ commute with $\rho(f')$, each of the former preserves the eigenspaces of $\rho(f')$. Thus, by a similar argument to the one given

above, and by relation (VI) in the presentation of Γ, we have

$$\prod_{r=1}^{k} (\lambda_r^c \mu_{r,s}^d)^{n_{r,s}} = 1 \tag{3.9}$$

in A for $s = 1, \ldots, \ell$. Switching to additive notation within A, we obtain from (3.7), (3.8), (3.9) the equations

$$a\lambda_1 + b\mu_{1,s} = \ldots = a\lambda_k + b\mu_{k,s} \text{ for } s = 1, \ldots, \ell, \tag{3.10}$$

$$\sum_{s=1}^{\ell} n_{r,s}\mu_{r,s} = 0 \text{ for } r = 1, \ldots, k, \tag{3.11}$$

$$\sum_{r=1}^{k} n_{r,s}(c\lambda_r + d\mu_{r,s}) = 0 \text{ for } s = 1, \ldots, \ell. \tag{3.12}$$

We now set $A = \mathbb{Z}$ and show that, in this context, equations (3.11), (3.12), and (3.10) imply that if $c = 0$ (resp., $c > 0$) then $\mu_{r,s} = 0$ whenever $n_{r,s} > 0$ (resp., then $\lambda_r = 0$ for $r = 1, \ldots, k$). By Lemma 2.4, the same statements will hold in the original torsion-free abelian group A, thus completing the proof.

Suppose first that $c = 0$. Note that since $|\det B| = 1$, this implies that $a = d = 1$, so that equations (3.10), (3.12) are reduced to

$$\lambda_1 + b\mu_{1,s} = \ldots = \lambda_k + b\mu_{k,s} \text{ for } s = 1, \ldots, \ell, \tag{3.13}$$

$$\sum_{r=1}^{k} n_{r,s}\mu_{r,s} = 0 \text{ for } s = 1, \ldots, \ell. \tag{3.14}$$

We show by induction on $k + \ell$ that, in this case, $\mu_{r,s} = 0$ if $n_{r,s} > 0$. The base case $k + \ell = 2$ is trivial. By the symmetry of equations (3.11), (3.14), (3.13), we may assume that $\mu_{k,1} \geqslant \mu_{r,s}$ for all r and s, and that $\mu_{k,1} \geqslant \ldots \geqslant \mu_{k,\ell}$. Note that the former implies that in particular $\mu_{k,1} \geqslant \mu_{r,1}$, so we obtain from (3.13) that $\mu_{k,\ell} \geqslant \mu_{r,\ell}$ for $r = 1, \ldots, k$. If $\mu_{k,\ell} \geqslant 0$, then since $\sum_{s=1}^{\ell} n_{k,s}\mu_{k,s} = 0$, we must have $n_{k,1}\mu_{k,1} = \ldots = n_{k,\ell}\mu_{k,\ell} = 0$. This implies that $\mu_{k,s} = 0$ if $n_{k,s} > 0$, so we may apply the induction hypothesis to the system of equations

$$\lambda_1 + b\mu_{1,s} = \ldots = \lambda_{k-1} + b\mu_{k-1,s} \text{ for } s = 1, \ldots, \ell,$$

$$\sum_{s=1}^{\ell} n_{r,s}\mu_{r,s} = 0 \text{ for } r = 1, \ldots, k-1,$$

$$\sum_{r=1}^{k-1} n_{r,s}\mu_{r,s} = 0 \text{ for } s = 1, \ldots, \ell.$$

Now suppose that $\mu_{k,\ell} < 0$. Since $\mu_{k,\ell} \geqslant \mu_{r,\ell}$ for $r = 1, \ldots, k$ and $\sum_{r=1}^{k} n_{r,\ell}\mu_{r,\ell} = 0$, we have that $n_{1,\ell}\mu_{1,\ell} = \ldots = n_{k,\ell}\mu_{k,\ell} = 0$. This implies that $\mu_{r,\ell} = 0$ if $n_{r,\ell} > 0$, so we may apply the

induction hypothesis to the system of equations

$$\lambda_1 + b\mu_{1,s} = \ldots = \lambda_k + b\mu_{k,s} \text{ for } s = 1, \ldots, \ell - 1,$$

$$\sum_{s=1}^{\ell-1} n_{r,s}\mu_{r,s} = 0 \text{ for } r = 1, \ldots, k,$$

$$\sum_{r=1}^{k} n_{r,s}\mu_{r,s} = 0 \text{ for } s = 1, \ldots, \ell - 1.$$

This completes the proof for the case $c = 0$.

We assume for the remainder of the proof that $c > 0$. Define

$$N = \begin{pmatrix} n_{1,1} & \cdots & n_{k,1} \\ \vdots & \ddots & \vdots \\ n_{1,\ell} & \cdots & n_{k,\ell} \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} a\lambda_1 + b\mu_{1,1} \\ \vdots \\ a\lambda_1 + b\mu_{1,\ell} \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} c\lambda_1 \\ \vdots \\ c\lambda_k \end{pmatrix},$$

and let N_r be the rth column of N. We have

$$\begin{aligned}
\mathbf{u}^T N_r &= (a\lambda_r + b\mu_{r,1}, \ldots, a\lambda_r + b\mu_{r,\ell})N_r \\
&= (a\lambda_r, \ldots, a\lambda_r)N_r + b(\mu_{r,1}, \ldots, \mu_{r,\ell})N_r \\
&= (a\lambda_r, \ldots, a\lambda_r)N_r \\
&= \sum_{s=1}^{\ell} n_{r,s}a\lambda_r,
\end{aligned}$$

where the first equality follows from (3.10) and the third follows from (3.11). Thus,

$$\mathbf{u}^T N\mathbf{w} = \left(\sum_{s=1}^{\ell} n_{1,s}a\lambda_1, \ldots, \sum_{s=1}^{\ell} n_{k,s}a\lambda_k \right) \mathbf{w} = \sum_{r,s} n_{r,s}ac\lambda_r^2. \tag{3.15}$$

On the other hand, we have

$$N\mathbf{w} = \begin{pmatrix} \sum_{r=1}^{k} n_{r,1}c\lambda_r \\ \vdots \\ \sum_{r=1}^{k} n_{r,\ell}c\lambda_r \end{pmatrix} = -\begin{pmatrix} \sum_{r=1}^{k} n_{r,1}d\mu_{r,1} \\ \vdots \\ \sum_{r=1}^{k} n_{r,\ell}d\mu_{r,\ell} \end{pmatrix},$$

where the second equality follows from (3.12). It follows that

$$-\mathbf{u}^T N\mathbf{w} = \mathbf{u}^T \begin{pmatrix} \sum_{r=1}^{k} n_{r,1}d\mu_{r,1} \\ \vdots \\ \sum_{r=1}^{k} n_{r,\ell}d\mu_{r,\ell} \end{pmatrix} = \sum_{r,s} n_{r,s}(a\lambda_1 + b\mu_{1,s})d\mu_{r,s} = \sum_{r,s} n_{r,s}(a\lambda_r + b\mu_{r,s})d\mu_{r,s}, \tag{3.16}$$

where the last equality follows from (3.10). Combining (3.15) and (3.16), we obtain

$$0 = \sum_{r,s} n_{r,s}ac\lambda_r^2 + \sum_{r,s} n_{r,s}(a\lambda_r + b\mu_{r,s})d\mu_{r,s} = \sum_{r,s} n_{r,s}(bd\mu_{r,s}^2 + ad\lambda_r\mu_{r,s} + ac\lambda_r^2). \tag{3.17}$$

We claim that $bd\mu_{r,s}^2 + ad\lambda_r\mu_{r,s} + ac\lambda_r^2 \geqslant 0$ for any r and s. If $d = 0$, this is clear. Otherwise, we may view $bd\mu_{r,s}^2 + ad\lambda_r\mu_{r,s} + ac\lambda_r^2$ as a quadratic polynomial in $\mu_{r,s}$ with positive leading coefficient bd and discriminant

$$\Delta_r = (ad - 4bc)ad\lambda_r^2 = (\det B - 3bc)ad\lambda_r^2.$$

Since $|\det B| = 1$, we have that $\Delta_r \leqslant 0$, and so $bd\mu_{r,s}^2 + ad\lambda_r\mu_{r,s} + ac\lambda_r^2 \geqslant 0$.

Now let $r \in \{1, \ldots, k\}$. We show that $\lambda_r = 0$. Indeed, we have $n_{r,s} > 0$ for some s since $\sum_{s=1}^k n_{r,s} = n_r > 0$. Thus, by (3.17) and the previous paragraph, we have

$$bd\mu_{r,s}^2 + ad\lambda_r\mu_{r,s} + ac\lambda_r^2 = 0.$$

If $d = 0$, this immediately implies that $\lambda_r = 0$. Now suppose $d, \lambda_r > 0$. Then $\Delta_r < 0$ and so $bd\mu_{r,s}^2 + ad\lambda_r\mu_{r,s} + ac\lambda_r^2 > 0$, a contradiction. □

Remark 3.3. Note that if $c, d > 0$, we also obtain that $\mu_{r,s} = 0$ whenever $n_{r,s} > 0$, so that $\rho(z)$ is also a virtually unipotent matrix. Thus, if all the entries of a gluing matrix for a manifold $M \in \mathfrak{E}$ are nonzero, then any element of $\pi_1(M)$ representing a curve on the JSJ torus of M is VU.

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