THIN RIGHT-ANGLED COXETER GROUPS IN SOME UNIFORM ARITHMETIC LATTICES

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Abstract. Using a variant of an unpublished argument due to Agol, we show that an irreducible right-angled Coxeter group on \( n \geq 3 \) vertices embeds as a thin subgroup of a uniform arithmetic lattice in an indefinite orthogonal group \( O(p, q) \) for some \( p, q \geq 1 \) satisfying \( p + q = n \).

Let \( G \) be a semisimple algebraic \( \mathbb{R} \)-group and \( \Gamma \) a lattice in \( G := G(\mathbb{R}) \). A subgroup \( \Delta \subset \Gamma \) is said to be thin if \( \Delta \) is Zariski-dense in \( G \) but of infinite index in \( \Gamma \). It follows from the Borel density theorem [Bor60, Corollary 4.3] and a classical result of Tits [Tit72, Theorem 3] that if \( G \) as above is nontrivial, connected, and without compact factors, then any lattice in \( G \) contains a thin nonabelian free subgroup. A famous construction of Kahn–Markovic [KM12] produces thin surface subgroups of all uniform lattices in \( SO(3, 1) \) (see [Ham15], [LR16], [CF19], [KLM18] for some other manifestations of surface groups as thin groups). In [BL20], Ballas–Long show that many arithmetic lattices in \( SO(n, 1) \) virtually embed as thin subgroups of lattices in \( SL_{n+1}(\mathbb{R}) \), and raise the question as to which groups arise as thin groups. In this note, we observe the following.

**Theorem 1.** An irreducible right-angled Coxeter group on \( n \geq 3 \) vertices embeds as a thin subgroup of a uniform arithmetic lattice in \( O(p, q) \) for some \( p, q \geq 1 \) satisfying \( p + q = n \).

To that end, let \( \Sigma_1 \) be a connected simplicial graph on \( n \geq 3 \) vertices; we think of \( \Sigma_1 \) as a Coxeter scheme in the sense of [VS93, pg. 201, Def. 1.7] all of whose edges are bold. Fix an order \( v_1, \ldots, v_n \) on the vertices of \( \Sigma_1 \), and let \( W \) be the group given by the presentation with generators \( \gamma_1, \ldots, \gamma_n \) subject to the relations \( \gamma_i^2 = 1 \) for \( i = 1, \ldots, n \), and \( [\gamma_i, \gamma_j] = 1 \) for each distinct \( i, j \in \{1, \ldots, n\} \) such that \( v_i \) and \( v_j \) are not adjacent in \( \Sigma_1 \). The group \( W \) is the (right-angled) Coxeter group associated to the graph \( \Sigma_1 \). (This convention will be convenient for our purposes; however, in the literature, the right-angled Coxeter group associated to a graph \( \Sigma \) is often defined as the right-angled Coxeter group associated to the complement graph of \( \Sigma \) in our sense.) Let \( W^+ \) be the index-2 subgroup of \( W \) consisting of all elements that can be expressed as a product of an even number of the \( \gamma_i \); that \( W^+ \) indeed constitutes an index-2 subgroup of \( W \) follows, for instance, from faithfulness of the representation \( \sigma_1 \) of \( W \) to be defined in the sequel.

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For $d \in \mathbb{R}$, let $M_d = (m_{ij}) \in M_n(\mathbb{Z}[d])$ be the symmetric matrix given by

$$m_{ij} = \begin{cases} 
1 & \text{if } i = j \\
-d & \text{if } i \neq j \text{ and } v_i, v_j \text{ are joined by an edge in } \Sigma_1 \\
0 & \text{otherwise.}
\end{cases}$$

Let $\epsilon > 0$ be such that $M_d$ is positive-definite for $d \in [-\epsilon, \epsilon]$, and let $D \geq 1$ be such that $M_d$ is nondegenerate and its signature constant as $d$ varies within $[D, \infty)$. Note that $M_1$ is the Gram matrix of the Coxeter scheme $\Sigma_1$ (and the given order on the vertices of $\Sigma_1$). In particular, we have that $\epsilon < 1$. For $d > 1$, the matrix $M_d$ is the Gram matrix of the Coxeter scheme $\Sigma_d$ obtained from $\Sigma_1$ by replacing each edge with a dotted edge labeled by $d$. (Here, we are again using the conventions employed by [VS93].)

For $d \geq 1$, let $\sigma_d : W \to \text{GL}_n(\mathbb{R})$ be the Tits–Vinberg representation associated to the Coxeter scheme $\Sigma_d$ and the given order on its vertices; this is the representation given by

$$\sigma_d(\gamma_i)(v) = v - 2(v^T M_d e_i) e_i$$

for $i = 1, \ldots, n$ and $v \in \mathbb{R}^n$, where $(e_1, \ldots, e_n)$ is the standard basis for $\mathbb{R}^n$. It follows from Vinberg’s theory of reflection groups that the representations $\sigma_d, d \geq 1$, are faithful [Vin71, Theorem 5] (see Lecture 1 in [Ben04] for an exposition). This family of representations was studied in [DGK20].

If $M \in M_n(\mathbb{R})$ is a symmetric matrix and $A$ is a subdomain of $\mathbb{C}$, we write

$$\text{O}(M; A) = \{g \in \text{GL}_n(A) : g^T M g = M\},$$

$$\text{SO}(M; A) = \{g \in \text{SL}_n(A) : g^T M g = M\}.$$ 

Note that we have $W_d := \sigma_d(W) \subset \text{O}(M_d; \mathbb{R})$ by design.

**Lemma 2.** The group $W_d$ is Zariski-dense in $\text{O}(M_d; \mathbb{R})$ for $d \geq D$.

**Proof.** The proof of the main theorem in [BdlH04] applies here, so we only sketch the argument provided there. Let $d \geq D$ and let $G_d$ be the Zariski-closure of $W_d$ in $\text{O}(M_d; \mathbb{R})$. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of $\text{O}(M_d; \mathbb{R})$ and $G_d$, respectively. It is enough to show that $\mathfrak{g} = \mathfrak{h}$, since the Zariski-closure of $\text{SO}(M_d; \mathbb{R})$ is $\text{SO}(M_d; \mathbb{R})$ and since $W_d \not\subset \text{SO}(M_d; \mathbb{R})$.

For each distinct pair $i, j \in \{1, \ldots, n\}$, let $E_{i,j}$ be the orthogonal complement of $(e_i, e_j)$ in $\mathbb{R}^n$ with respect to $M_d$. The subgroup of $\text{O}(M_d; \mathbb{R})$ consisting of all elements that fix each vector in $E_{i,j}$ is a 1-dimensional closed subgroup of $\text{O}(M_d; \mathbb{R})$ whose identity component $T_{i,j}$ gives rise to a subspace $\langle X_{i,j} \rangle$ of $\mathfrak{g}$ for some $X_{i,j} \in \mathfrak{g}$. Since $M_d$ is nondegenerate, the elements $X_{i,j}$ form a basis for $\mathfrak{g}$ as a vector space [BdlH04, Lemme 7]. Thus, to show $\mathfrak{g} = \mathfrak{h}$, it suffices to show that $X_{i,j} \in \mathfrak{h}$ for each distinct pair $i, j \in \{1, \ldots, n\}$.

To that end, let $i, j \in \{1, \ldots, n\}$, $i \neq j$, and suppose first that $v_i$ and $v_j$ are adjacent in $\Sigma_1$. Then $\sigma_d(\gamma_i \gamma_j)$ generates an infinite cyclic subgroup of $T_{i,j}$, so that $T_{i,j} \subset G_d$. It follows that $X_{i,j} \in \mathfrak{h}$ in this case. One now verifies that, since $\Sigma_1$ is connected, any Lie subalgebra of $\mathfrak{g}$ that contains $X_{i,j}$ for all $i, j$ such that $v_i, v_j$ are adjacent in fact contains $X_{i,j}$ for each distinct pair $i, j \in \{1, \ldots, n\}$ [BdlH04, Preuve du Théorème, second cas].

Now let $K \subset \mathbb{R}$ be a real quadratic extension of $\mathbb{Q}$, let $\tau : K \to K$ be the nontrivial element of $\text{Gal}(K/\mathbb{Q})$, and let $\mathcal{O}_K$ be the ring of integers of $K$. Then by
Dirichlet’s unit theorem, there is a unit \( \alpha \in \mathcal{O}_K^{*} \) such that \( \alpha \geq \max \{ \frac{1}{\epsilon}, D \} \). Thus, we have

\[
\frac{1}{\epsilon} |\tau(\alpha)| \leq \alpha |\tau(\alpha)| = |\alpha \cdot \tau(\alpha)| = 1,
\]

where the final equality holds because \( \alpha \in \mathcal{O}_K^{*} \). We conclude that \( |\tau(\alpha)| \leq \epsilon \), and so \( M_{\tau(\alpha)} \) is positive-definite. It follows that \( \Gamma := O(M_\alpha; \mathcal{O}_K) \) is a uniform arithmetic lattice in \( O(M_\alpha; \mathbb{R}) \) (for an efficient survey of the relevant facts, see, for instance, Section 2 of [GPS87]). Moreover, we have \( W_\alpha \subset O(M_\alpha; \mathbb{Z}[\alpha]) \subset \Gamma \).

**Remark 3.** Note that Galois conjugation by \( \tau \) transports \( \Gamma \) and hence \( W_\alpha \) into the compact group \( O(M_{\tau(\alpha)}; \mathbb{R}) \). That finitely generated right-angled Coxeter groups embed in compact Lie groups had already been observed by Agol [Ago18] using a similar trick to the one above. Indeed, Agol’s argument was the inspiration for this note.

**Proof of Theorem 1.** We show that \( W_\alpha \) is a thin subgroup of \( \Gamma \subset O(M_\alpha; \mathbb{R}) \). By Lemma 2, it suffices to show that \( W_\alpha \) is of infinite index in \( \Gamma \). Indeed, suppose otherwise. Then \( W_\alpha \) is a uniform lattice in \( O(M_\alpha; \mathbb{R}) \). If \( n = 3 \), then immediately we obtain a contradiction, since in this case \( W_\alpha \) is virtually a closed hyperbolic surface group, whereas \( W \) is virtually free. If \( M_\alpha \) has signature \((2,2)\) (the one case under consideration in which \( \text{SO}(M_\alpha; \mathbb{R})^0 \) is not simple), then we again obtain a contradiction as \( W \) has virtual cohomological dimension at most 3 (for instance, since the latter is an upper bound for the dimension of the Davis complex associated to the infinite right-angled Coxeter group \( W \); see [Dav08, Chapter 1]), while the symmetric space associated to \( O(M_\alpha; \mathbb{R}) \) has dimension 4. Now suppose that \( n > 3 \) and that the signature of \( M_\alpha \) is not \((2,2)\). There is some \( \beta > \alpha \) and a path \([\alpha, \beta] \rightarrow \text{GL}_n(\mathbb{R}) \), \( d \mapsto h_d \) such that \( h_d^T M_d h_d = M_\alpha \) for all \( d \in [\alpha, \beta] \) (this follows, for example, from the fact that \( \text{GL}_n(\mathbb{R}) \) acts continuously and transitively on the set \( \Omega \subset M_\alpha(\mathbb{R}) \) of symmetric matrices with the same signature as \( M_\alpha \), and so the orbit map \( \text{GL}_n(\mathbb{R}) \rightarrow \Omega, g \mapsto g^T M_\alpha g \) is a fiber bundle). Setting \( g_d = h_d h_{\alpha}^{-1} \) for \( d \in [\alpha, \beta] \), we have that \( g_d = I_\alpha \) and \( g_d^T M_d g_d = M_\alpha \) for \( d \in [\alpha, \beta] \). For \( d \in [\alpha, \beta] \), let \( \rho_d = g_d^{-1} \sigma_d g_d \), and note

\[
\rho_d(W') \subset g_d^{-1} O(M_\alpha; \mathbb{R}) g_d = O(g_d^T M_d g_d; \mathbb{R}) = O(M_\alpha; \mathbb{R}).
\]

Let \( \rho_+^d = \rho_d|_{W^+} \) and \( \sigma_+^d = \sigma_d|_{W^+} \) for \( d \in [\alpha, \beta] \). Then \( \rho_+^d(W^+) \) is a uniform lattice in the connected non-compact simple Lie group \( \text{SO}(M_\alpha; \mathbb{R})^0 \), and the latter is not locally isomorphic to \( \text{SO}(2,1)^0 \) by our assumption that \( n > 3 \). Thus, by Weil local rigidity [Wei00, Wei02], up to choosing \( \beta \) closer to \( \alpha \), we may assume that for each \( d \in [\alpha, \beta] \) there is some \( a_d \in \text{SO}(M_\alpha; \mathbb{R})^0 \) such that

\[
(1) \quad \rho_d^+ = a_d \rho_+^d a_d^{-1} = a_d \sigma_+^d a_d^{-1}.
\]

But \( \rho_d^+ = g_d^{-1} \sigma_d^+ g_d \), so we obtain from (1) that the trace \( \text{tr}(\sigma_d(\gamma_i \gamma_j)) \) remains constant as \( d \) varies within \([\alpha, \beta]\), where \( i, j \in \{1, \ldots, n\} \) are chosen so that the vertices \( v_i, v_j \) are adjacent in \( \Sigma_1 \).

We claim, however, that \( \text{tr}(\sigma_d(\gamma_i \gamma_j)) = 4d^2 - 4 + n \) for \( d \geq D \). Indeed, let \( d \geq D \). Then \( M_d \) is nondegenerate, so that \( \mathbb{R}^d \) splits as a direct sum of the 2-dimensional subspace \( \langle e_i, e_j \rangle \subset \mathbb{R}^n \) and its orthogonal complement \( E_{i,j} \) with respect to \( M_d \). Each of \( \gamma_i \) and \( \gamma_j \) acts as the identity on \( E_{i,j} \), so our claim is equivalent to the assertion that

\[
\text{tr} \left( \sigma_d(\gamma_i \gamma_j) \right)_{\langle e_i, e_j \rangle} = 4d^2 - 2,
\]

and the latter follows from the fact that,
with respect to the basis \((e_i, e_j)\) of \((e_i, e_j)\), the matrices representing \(\sigma_d(\gamma_i), \sigma_d(\gamma_j)\) are \(
abla = \begin{pmatrix} -1 & 2d \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2d & -1 \end{pmatrix}\), respectively. \(\Box\)

**Example 4.** We consider the case that \(n \geq 5\) and the complement graph of \(\Sigma_1\) is the cycle \(v_1v_2\ldots v_n\). In this case, the group \(W\) may be realized as the subgroup of \(\mathrm{Isom}(\mathbb{H}^2)\) generated by the reflections in the sides of a right-angled hyperbolic \(n\)-gon, so that \(W\) is virtually the fundamental group of a closed hyperbolic surface. We have

\[
M_d = (1 + d)I_n + d(J_n + J_n^{-1}) - d(I_n + J_n + \ldots + J_n^{-1})
\]

where \(J_n \in M_n(\mathbb{C})\) is the matrix

\[
J_n = \begin{pmatrix} e_2 & e_3 & \ldots & e_n & e_1 \end{pmatrix}.
\]

There is some \(C \in \mathrm{GL}_n(\mathbb{C})\) such that

\[
CJ_nC^{-1} = \text{diag}(1, \zeta_n, \zeta_n^2, \ldots, \zeta_n^{n-1})
\]

where \(\zeta_n = e^{2\pi i/n}\). Observe that

\[
C(I_n + J_n + \ldots + J_n^{-1})C^{-1} = \text{diag}(n, 0, \ldots, 0)
\]

\[
C(J_n + J_n^{-1})C^{-1} = \text{diag}\left(2, 2\cos\frac{2\pi}{n}, 2\cos\frac{2\pi}{n}, \ldots, 2\cos\frac{2\pi(n-1)}{n}\right).
\]

It follows from (2) that, counted with multiplicity, the eigenvalues of \(M_d\) are \(1 - d(n - 3)\) and \(1 + d\left(1 + 2\cos\frac{2\pi k}{n}\right)\), where \(k = 1, \ldots, n - 1\). Note that for \(d\) sufficiently large, we have that \(1 - d(n - 3) < 0\), and that \(1 + d\left(1 + 2\cos\frac{2\pi k}{n}\right) \geq 0\) if and only if \(\cos\frac{2\pi k}{n} \geq -\frac{1}{2}\). We conclude that the signature of \(M_d\) is \(\left(\left\lfloor \frac{n}{4} \right\rfloor, n - 2\left\lfloor \frac{n}{4} \right\rfloor\right)\) for all \(d\) sufficiently large. In particular, if \(n = 3m, m \geq 2\), then the signature of \(M_d\) is \((2m, m)\) for all \(d\) sufficiently large. The above discussion yields thin surface subgroups of uniform arithmetic lattices in \(\mathrm{SO}(2n + 1, \mathbb{R})\) for \(n \geq 5\).

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**References**


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