THIN RIGHT-ANGLED COXETER GROUPS IN SOME UNIFORM ARITHMETIC LATTICES

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Abstract. Using a variant of an unpublished argument due to Agol, we show that an irreducible right-angled Coxeter group on \( n \geq 3 \) vertices embeds as a thin subgroup of a uniform arithmetic lattice in an indefinite orthogonal group \( O(p, q) \) for some \( p, q \geq 1 \) satisfying \( p + q = n \).

Let \( G \) be a semisimple algebraic \( \mathbb{R} \)-group and \( \Gamma \) a lattice in \( G := G(\mathbb{R}) \). A subgroup \( \Delta \subset \Gamma \) is said to be thin if \( \Delta \) is Zariski-dense in \( G \) but of infinite index in \( \Gamma \). It follows from the Borel density theorem [Bor60, Corollary 4.3] and a classical result of Tits [Tit72, Theorem 3] that if \( G \) as above is nontrivial, connected, and without compact factors, then any lattice in \( G \) contains a thin nonabelian free subgroup. A famous construction of Kahn–Markovic [KM12] produces thin surface subgroups of all uniform lattices in \( SO(3, 1) \) (see [Ham15], [LR16], [CF19], [KLM18] for some other manifestations of surface groups as thin groups). In [BL20], Ballas–Long show that many arithmetic lattices in \( SO(n, 1) \) virtually embed as thin subgroups of lattices in \( SL_{n+1}(\mathbb{R}) \), and raise the question as to which groups arise as thin groups. In this note, we observe the following.

**Theorem 1.** An irreducible right-angled Coxeter group on \( n \geq 3 \) vertices embeds as a thin subgroup of a uniform arithmetic lattice in \( O(p, q) \) for some \( p, q \geq 1 \) satisfying \( p + q = n \).

To that end, let \( \Sigma_1 \) be a connected simplicial graph on \( n \geq 3 \) vertices; we think of \( \Sigma_1 \) as a Coxeter diagram in the sense of [FT14, Section 2.1] all of whose edges are bold. Fix an order \( v_1, \ldots, v_n \) on the vertices of \( \Sigma_1 \), and let \( W \) be the group given by the presentation with generators \( \gamma_1, \ldots, \gamma_n \) subject to the relations \( \gamma_i^2 = 1 \) for \( i = 1, \ldots, n \), and \( [\gamma_i, \gamma_j] = 1 \) for each distinct \( i, j \in \{1, \ldots, n\} \) such that \( v_i \) and \( v_j \) are not adjacent in \( \Sigma_1 \). The group \( W \) is the (right-angled) Coxeter group associated to the diagram \( \Sigma_1 \). Let \( W^+ \) be the index-2 subgroup of \( W \) consisting of all elements that can be expressed as a product of an even number of the \( \gamma_i \); that \( W^+ \) indeed constitutes an index-2 subgroup of \( W \) follows, for instance, from faithfulness of the representation \( \sigma_1 \) of \( W \) to be defined in the sequel.

For \( d \in \mathbb{R} \), let \( M_d = (m_{ij}) \in M_n(\mathbb{Z}[d]) \) be the symmetric matrix given by

\[
M_d = \begin{cases} 
1 & \text{if } i = j \\
-d & \text{if } i \neq j \text{ and } v_i, v_j \text{ are joined by an edge in } \Sigma_1 \\
0 & \text{otherwise}.
\end{cases}
\]

Let \( \epsilon > 0 \) be such that \( M_d \) is positive-definite for \( d \in [-\epsilon, \epsilon] \), and let \( D \geq 1 \) be such that \( M_D \) is nondegenerate and its signature constant as \( d \) varies within \( [D, \infty) \).

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Note that \( M_1 \) is the Gram matrix of the diagram \( \Sigma_1 \) (and the given order on the vertices of \( \Sigma_1 \)). In particular, we have that \( \epsilon < 1 \). For \( d > 1 \), the matrix \( M_d \) is the Gram matrix of the diagram \( \Sigma_d \) obtained from \( \Sigma_1 \) by replacing each edge with a dotted edge labeled by \( d \). (Here, we are again using the conventions employed by [FT14, Section 2.1].)

For \( d \geq 1 \), let \( \sigma_d : W \to \text{GL}_n(\mathbb{R}) \) be the Tits–Vinberg representation associated to the Coxeter diagram \( \Sigma_d \) and the given order on its vertices; this is the representation given by

\[
\sigma_d(\gamma_i)(v) = v - 2(v^T M_d e_i)e_i
\]

for \( i = 1, \ldots, n \) and \( v \in \mathbb{R}^n \), where \((e_1, \ldots, e_n)\) is the standard basis for \( \mathbb{R}^n \). It follows from Vinberg’s theory of reflection groups that the representations \( \sigma_d, d \geq 1 \), are faithful [Vin71, Theorem 5] (see Lecture 1 in [Ben04] for an exposition). This family of representations was studied in [DGK20].

If \( M \in M_n(\mathbb{R}) \) is a symmetric matrix and \( A \) is a subdomain of \( \mathbb{C} \), we write

\[
O(M; A) = \{ g \in \text{GL}_n(A) : g^T M g = M \},
\]

\[
\text{SO}(M; A) = \{ g \in \text{SL}_n(A) : g^T M g = M \}.
\]

Note that we have \( W_d := \sigma_d(W) \subset O(M_d; \mathbb{R}) \) by design.

**Lemma 2.** The group \( W_d \) is Zariski-dense in \( O(M_d; \mathbb{R}) \) for \( d \geq D \).

**Proof.** The proof of the main theorem in [BdlH04] applies here, so we only sketch the argument provided there. Let \( d \geq D \) and let \( G_d \) be the Zariski-closure of \( W_d \) in \( O(M_d; \mathbb{R}) \). Denote by \( \mathfrak{g} \) and \( \mathfrak{h} \) the Lie algebras of \( O(M_d; \mathbb{R}) \) and \( G_d \), respectively. It is enough to show that \( \mathfrak{g} = \mathfrak{h} \), since the Zariski-closure of \( \text{SO}(M_d; \mathbb{R})^\circ \) is \( \text{SO}(M_d; \mathbb{R}) \) and since \( W_d \subsetneq \text{SO}(M_d; \mathbb{R}) \).

For each distinct pair \( i, j \in \{1, \ldots, n\} \), let \( E_{i,j} \) be the orthogonal complement of \((e_i, e_j)\) in \( \mathbb{R}^n \) with respect to \( M_d \). The subgroup of \( O(M_d; \mathbb{R}) \) consisting of all elements that fix each vector in \( E_{i,j} \) is a 1-dimensional closed subgroup of \( O(M_d; \mathbb{R}) \) whose identity component \( T_{i,j} \) corresponds to a subspace \((X_{i,j})\) of \( \mathfrak{g} \) for some \( X_{i,j} \in \mathfrak{g} \). Since \( M_d \) is nondegenerate, the elements \( X_{i,j} \) form a basis for \( \mathfrak{g} \) as a vector space [BdlH04, Lemme 7]. Thus, to show \( \mathfrak{g} = \mathfrak{h} \), it suffices to show that \( X_{i,j} \in \mathfrak{h} \) for each distinct pair \( i, j \in \{1, \ldots, n\} \).

To that end, let \( i, j \in \{1, \ldots, n\} \), \( i \neq j \), and suppose first that \( v_i \) and \( v_j \) are adjacent in \( \Sigma_1 \). Then \( \sigma_d(\gamma_i\gamma_j) \) generates an infinite cyclic subgroup of \( T_{i,j} \), so that \( T_{i,j} \subset G_d \). It follows that \( X_{i,j} \in \mathfrak{h} \) in this case. One now verifies that, since \( \Sigma_1 \) is connected, any Lie subalgebra of \( \mathfrak{g} \) that contains \( X_{i,j} \) for all \( i,j \) such that \( v_i, v_j \) are adjacent in \( \Sigma_1 \) contains \( X_{i,j} \) for each distinct pair \( i,j \in \{1, \ldots, n\} \) [BdlH04, Preuve du Théorème, second cas].

Now let \( K \subset \mathbb{R} \) be a real quadratic extension of \( \mathbb{Q} \), let \( \tau : K \to K \) be the nontrivial element of \( \text{Gal}(K/\mathbb{Q}) \), and let \( \mathcal{O}_K \) be the ring of integers of \( K \). Then by Dirichlet’s unit theorem, there is a unit \( \alpha \in \mathcal{O}_K^\times \) such that \( \alpha \geq \max\{\frac{1}{2}, D\} \). Thus, we have

\[
\frac{\abs{\tau(\alpha)}}{\epsilon} \leq \abs{\alpha \cdot \tau(\alpha)} = \abs{\alpha} = 1,
\]

where the final equality holds because \( \alpha \in \mathcal{O}_K^\times \). We conclude that \( \abs{\tau(\alpha)} \leq \epsilon \), and so \( M_d(\alpha) \) is positive-definite. It follows that \( \Gamma := O(M_d; \mathcal{O}_K) \) is a uniform arithmetic lattice in \( O(M_d; \mathbb{R}) \) (for an efficient survey of the relevant facts, see, for instance, Section 2 of [GPS87]). Moreover, we have \( W_\alpha \subset O(M_\alpha; \mathbb{Z}[\alpha]) \subset \Gamma \).
Remark 3. Note that Galois conjugation by $\tau$ transports $\Gamma$ and hence $W_\alpha$ into the compact group $O(M_\tau; \mathbb{R})$. That right-angled Coxeter groups on finitely many vertices embed in compact Lie groups had already been observed by Agol [Agol18] using a similar trick to the one above. Indeed, Agol’s argument was the inspiration for this note.

Proof of Theorem 1. We show that $W_\alpha$ is a thin subgroup of $\Gamma \subset O(M_\alpha; \mathbb{R})$. By Lemma 2, it suffices to show that $W_\alpha$ is of infinite index in $\Gamma$. Indeed, suppose otherwise. Then $W_\alpha$ is a uniform lattice in $O(M_\alpha; \mathbb{R})$. If $n = 3$, then immediately we obtain a contradiction, since in this case $W_\alpha$ is virtually a closed hyperbolic surface group, whereas $W_\alpha$ is virtually free. Now suppose $n > 3$. There is some $\beta > \alpha$ and a path $[\alpha, \beta] \to GL_n(\mathbb{R})$, $d \to h_d$ such that $h_d^\alpha M_d h_d = M_\alpha$ for all $d \in [\alpha, \beta]$ (this follows, for example, from the fact that $GL_n(\mathbb{R})$ acts continuously and transitively on the set $\Omega \subset M_n(\mathbb{R})$ of symmetric matrices with the same signature as $M_\alpha$, and so the orbit map $GL_n(\mathbb{R}) \to \Omega, g \to g^T M_\alpha g$ is a fiber bundle). Setting $g_d = h_d h_\alpha^{-1}$ for $d \in [\alpha, \beta]$, we have that $g_\alpha = I_d$ and $g_\alpha^T M_d g_\alpha = M_\alpha$ for $d \in [\alpha, \beta]$. For $d \in [\alpha, \beta]$, let $\rho_d = g_d^{-1} \sigma_d g_d$, and note $\rho_d(\Omega) \subset g_d^{-1}O(M_d; \mathbb{R}) g_d = O(g_d^T M_d g_d; \mathbb{R}) = O(M_\alpha; \mathbb{R})$.

Let $\rho_d^+ = \rho_d|_{W^+}$ and $\sigma_d^+ = \sigma_d|_{W^+}$ for $d \in [\alpha, \beta]$. Then $\rho_d^+(W^+)$ is a uniform lattice in the connected non-compact simple Lie group $SO(2,1)^c$, and the latter is not locally isomorphic to $SO(2,1)^c$ by our assumption that $n > 3$. Thus, by Weil local rigidity [Wei60, Wei62], up to choosing $\beta$ closer to $\alpha$, we may assume that for each $d \in [\alpha, \beta]$ there is some $a_d \in SO(M_\alpha; \mathbb{R})^0$ such that

$$\rho_d^+ = a_d \rho_d^+ a_d^{-1} = a_d \sigma_d^+ a_d^{-1}. \tag{1}$$

But $\rho_d^+ = g_d^{-1} \sigma_d^+ g_d$, so we obtain from (1) that the trace $\text{tr}(\sigma_d(\gamma_i \gamma_j))$ remains constant as $d$ varies within $[\alpha, \beta]$, where $i, j \in \{1, \ldots, n\}$ are chosen so that the vertices $v_i, v_j$ are adjacent in $\Sigma_1$.

We claim, however, that $\text{tr}(\sigma_d(\gamma_i \gamma_j)) = 4d^2 - 4 + n$ for $d \geq 3$. Indeed, let $d \geq 3$. Then $M_d$ is nondegenerate, so that $\mathbb{R}^d$ splits as a direct sum of the 2-dimensional subspace $\langle e_i, e_j \rangle \subset \mathbb{R}^n$ and its orthogonal complement $E_{i,j}$ with respect to $M_d$. Each of $\gamma_i$ and $\gamma_j$ acts as the identity on $E_{i,j}$, so our claim is equivalent to the assertion that $\text{tr}(\sigma_d(\gamma_i \gamma_j)|_{E_{i,j}}) = 4d^2 - 2$, and the latter follows from the fact that, with respect to the basis $\langle e_i, e_j \rangle$ of $\langle e_i, e_j \rangle$, the matrices representing $\sigma_d(\gamma_i), \sigma_d(\gamma_j)$ are

$$\begin{pmatrix} -1 & 2d \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2d & -1 \end{pmatrix},$$

respectively. \hfill \Box

Example 4. We consider the case that $n \geq 5$ and the complement graph of $\Sigma_1$ is the cycle $v_1 v_2 \ldots v_n$. In this case, the group $W$ may be realized as the subgroup of $\text{Isom}(\mathbb{R}^2)$ generated by the reflections in the sides of a regular right-angled hyperbolic $n$-gon, so that $W$ is virtually the fundamental group of a closed hyperbolic surface. We have

$$M_d = (1 + d)I_n + d(J_n + J_n^{-1}) - d(I_n + J_n + \ldots + J_n^{-1}) \tag{2}$$

where $J_n \in M_n(\mathbb{C})$ is the matrix

$$J_n = \begin{pmatrix} e_2 & e_3 & \ldots & e_n & e_1 \end{pmatrix}.$$ 

There is some $C \in GL_n(\mathbb{C})$ such that

$$C J_n C^{-1} = \text{diag}(1, \zeta_n, \zeta_n^2, \ldots, \zeta_n^{n-1})$$
where $\zeta_n = e^{2\pi i/n}$. Observe that
\[
C(I_n + J_n + \ldots + J_n^{-1})C^{-1} = \text{diag}(n, 0, \ldots, 0)
\]
\[
C(J_n + J_n^{-1})C^{-1} = \text{diag}\left(2, 2 \cos \frac{2\pi}{n}, 2 \cos \frac{2\pi}{n}, \ldots, 2 \cos \frac{2\pi(n-1)}{n}\right).
\]
It follows from (2) that, counted with multiplicity, the eigenvalues of $M_d$ are
\[1 - d(n-3) \text{ and } 1 + d \left(1 + 2 \cos \frac{2\pi k}{n}\right)\]
where $k = 1, \ldots, n-1$. Note that for $d$ sufficiently large, we have that $1 - d(n-3) < 0$, and that $1 + d \left(1 + 2 \cos \frac{2\pi k}{n}\right) \geq 0$ if and only if $\cos \frac{2\pi k}{n} \geq -\frac{1}{2}$. We conclude that the signature of $M_d$ is $(2 \left\lfloor \frac{n}{3} \right\rfloor, n - 2 \left\lfloor \frac{n}{3} \right\rfloor)$ for all $d$ sufficiently large. In particular, if $n = 3m$, $m \geq 2$, then the signature of $M_d$ is $(2m, m)$ for all $d$ sufficiently large. The above discussion yields thin surface subgroups of uniform arithmetic lattices in $SO(2 \left\lfloor \frac{n}{3} \right\rfloor, n - 2 \left\lfloor \frac{n}{3} \right\rfloor)$ for each $n \geq 5$.

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References


