Proper CAT(0) Actions of Unipotent-Free Linear Groups

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Abstract

Let Γ be a finitely generated group of matrices over $\mathbb{C}$. We construct an isometric action of Γ on a complete CAT(0) space such that the restriction of this action to any subgroup of Γ containing no nontrivial unipotent elements is well behaved. As an application, we show that if M is a graph manifold that does not admit a nonpositively curved Riemannian metric, then any finite-dimensional $\mathbb{C}$-linear representation of $\pi_1(M)$ maps a nontrivial element of $\pi_1(M)$ to a unipotent matrix. In particular, the fundamental groups of such 3-manifolds do not admit any faithful finite-dimensional unitary representations.

1. Introduction

Let F be a field and n a positive integer. An element of $\mathrm{SL}_n(F)$ is *unipotent* if it has the same characteristic polynomial as the identity matrix. In [But17, But19], Button demonstrated that finitely generated subgroups of $\mathrm{SL}_n(F)$ containing no infinite-order unipotent elements share some properties with groups acting properly by semisimple isometries on complete CAT(0) spaces. Indeed, Button showed that if F has positive characteristic (in which case any unipotent element of $\mathrm{SL}_n(F)$ has finite order), then any finitely generated subgroup of $\mathrm{SL}_n(F)$ admits such an action [But19, Theorem 2.3]. The main theorem of this article is intended to serve as an analogue of the latter result in the characteristic-zero setting.

Theorem 1.1. *Let Γ be a finitely generated subgroup of $\mathrm{SL}_n(\mathbb{C})$, $n > 0$. Then Γ acts on a complete CAT(0) space X such that*

(i) for any subgroup $H < \Gamma$ containing no nontrivial unipotent matrices, the induced action of H on X is proper;

(ii) if such a subgroup H is free abelian of finite rank, then H preserves and acts as a lattice of translations on a thick flat in X; in particular, any infinite-order element of such a subgroup H acts ballistically on X;

(iii) if $g \in \Gamma$ is a diagonalizable, then g acts as a semisimple isometry of X.

See Section 2 for the relevant definitions. The space X is a finite product $\prod_i X_i$ of symmetric spaces of non-compact type and (possibly locally infinite) Euclidean buildings, and Γ acts on X via a product $\prod_i \mathrm{SL}_n(K_i)$, where the K_i are completions of the entry field E of Γ with respect to various absolute values on E. The technique of extracting information about a linear group by varying the absolute value on its entry field is credited to Tits [Tit72] and was employed by Margulis in the latter's proof of arithmeticity of higher-rank lattices [Mar84].

2010 Mathematics Subject Classification Primary: 20F67; Secondary: 20F65

The author was partially supported by the National Science Centre, Poland UMO-2018/30/M/ST1/00668.

Since an element of $\mathrm{SL}_n(\mathbb{C})$ that is both diagonalizable and unipotent must be trivial, the following corollary is immediate.

Corollary 1.2. *Any finitely generated subgroup of* $\mathrm{SL}_n(\mathbb{C})$ *consisting entirely of diagonalizable matrices acts properly by semisimple isometries on a complete* $\mathrm{CAT}(0)$ *space.*

Precompact subgroups of $\mathrm{SL}_n(\mathbb{C})$ are conjugate into $\mathrm{SU}(n)$ and thus consist entirely of diagonalizable matrices. Furthermore, by the Peter–Weyl theorem, any compact Lie group can be realized as a compact subgroup of $\mathrm{SL}_n(\mathbb{C})$ for some n [BTD85, Theorem III.4.1]. Thus, by Corollary 1.2, any finitely generated subgroup of a compact Lie group admits a proper action by semisimple isometries on a complete $\mathrm{CAT}(0)$ space.

For us, a *graph manifold* is a connected closed orientable irreducible non-Seifert 3-manifold all of whose JSJ blocks are Seifert. Property (ii) of the action described in Theorem 1.1 allows us to conclude the following fact about representations of fundamental groups of graph manifolds.

Theorem 1.3. *Let* M *be a graph manifold and let* $\rho : \pi_1(M) \to \mathrm{SL}_n(\mathbb{C})$ *be any representation. If* M *does not admit a nonpositively curved Riemannian metric, then there is a JSJ torus* S *of* M *and a nontrivial element* $h \in \pi_1(S) < \pi_1(M)$ *such that* $\rho(h)$ *is unipotent.*

A manifold is said to be *nonpositively curved (NPC)* if it admits a nonpositively curved Riemannian metric. By work of Agol [AGM13], Przytycki–Wise [PW18], and Liu [Liu13], the fundamental group of any closed NPC 3-manifold virtually embeds in a finitely generated right-angled Artin group (RAAG). Moreover, Agol [Ago18] showed that any finitely generated RAAG embeds in a compact Lie group (for an elaboration on Agol's argument, see [Dou21a]). On the other hand, if M is a closed aspherical non-NPC 3-manifold, then either M is Seifert, in which case there is a nontrivial element of $\pi_1(M)$ that gets mapped to a virtually unipotent matrix under any finite-dimensional linear representation of $\pi_1(M)$ (see, for example, the discussion in the introduction of [Dou21b]), or the orientation cover of M is a non-NPC graph manifold. Thus, we obtain from Theorem 1.3 the following corollary.

Corollary 1.4. *Let* M *be a closed aspherical* 3*-manifold. Then the following are equivalent:*

(i) the manifold M *is nonpositively curved;*

(ii) the fundamental group $\pi_1(M)$ *embeds in a compact Lie group;*

(iii) there is a faithful finite-dimensional $\mathbb{C}$*-linear representation of* $\pi_1(M)$ *whose image consists entirely of diagonalizable matrices;*

(iv) there is a faithful finite-dimensional $\mathbb{C}$*-linear representation of* $\pi_1(M)$ *whose image contains no nontrivial unipotent matrices.*

We remark that a result similar to Theorem 1.1 was announced in [Mat07, Theorem 1.4]. However, the proof of [Mat07, Theorem 4.8], on which that result rests, contains an error; a $\mathrm{CAT}(0)$ action of a finitely generated linear group G with proper restrictions to certain subgroups of G is desired, but what is provided is a proper $\mathrm{CAT}(0)$ action for each such subgroup of G.

Organization

In Section 2, we define the relevant objects, discuss briefly some properties of ballistic isometries of complete $\mathrm{CAT}(0)$ spaces, and introduce the central notion of a "thick flat" in such a space. In Section 3, we prove several lemmas used in the proofs of Theorems 1.1 and 1.3. The latter proofs are contained in Section 4.

Acknowledgements

I am deeply grateful to Piotr Przytycki for his encouragement, patience, and guidance. I also thank Bruno Duchesne, Zachary Munro, and Abdul Zalloum for helpful discussions.

2. Preliminaries

2.1 Complete CAT(0) spaces

Let X be a complete CAT(0) space and ∂X its visual boundary. We will make references to the cone topology on $\overline{X} := X \cup \partial X$, described in [BH99]. Under this topology, a sequence of points $x_n \in X$ converges to $\xi \in \partial X$ if and only if for some (hence any) point $x_0 \in X$, the geodesics joining x_0 to x_n converge uniformly on compact intervals to the unique geodesic ray emanating from x_0 in the class of ξ. In addition, we will use the angular metric $\angle$ on ∂X, also described in [BH99]. Note that the topology on ∂X induced by the angular metric is in general finer than the cone topology on ∂X.

An r-dimensional *flat* in X is an isometrically embedded copy of $\mathbb{R}^r$ in X. We say X is *π-visible* if for any $\xi, \eta \in \partial X$ satisfying $\angle(\xi, \eta) = \pi$, there is a geodesic line in X whose endpoints on ∂X are ξ and η. Since Euclidean spaces are π-visible, a complete CAT(0) space X with the property that any two points on ∂X lie on the boundary of a common flat in X is also π-visible. Note that if X is a Euclidean building, a symmetric space of non-compact type, or a product of such spaces, then X possesses the latter property by the building structure on ∂X, so that X is π-visible. For more information on symmetric spaces, we refer the reader to the monograph [Ebe96].

2.2 Isometries of complete CAT(0) spaces

Let (X, d_X) be a complete CAT(0) space and let $g \in \mathrm{Isom}(X)$. The *translation length* of g is the quantity $|g|_X := \inf_{x \in X} d_X(x, gx)$. The isometry g is *semisimple* if $|g|_X = d_X(x_0, gx_0)$ for some $x_0 \in X$. We say g is *ballistic* (resp., *neutral*) if $|g|_X > 0$ (resp., if $|g|_X = 0$), and *hyperbolic* if g is both ballistic and semisimple. A subgroup $H < \mathrm{Isom}(X)$ acts *neutrally* on X if each $h \in H$ is neutral.

If $g \in \mathrm{Isom}(X)$ is ballistic, then there is a point $\omega_g \in \partial X$ such that for any $x \in X$, we have $g^n x \to \omega_g$ as $n \to \infty$ with respect to the cone topology on $\overline{X}$ [CM09]; we call ω_g the *canonical attracting fixed point* of g. We use repeatedly the following fact, due to Duchesne [Duc15, Prop. 6.2]. For an arbitrary group G and $g_1, \ldots, g_m \in G$, we denote by $\mathscr{Z}_G(g_1, \ldots, g_m)$ the centralizer of $g_1, \ldots, g_m$ in G.

Theorem 2.1. *Let X be a complete π-visible CAT(0) space and suppose $g \in \mathrm{Isom}(X)$ is ballistic. Then there is a closed convex subspace $Y \subset X$ and a metric decomposition $Y = Z \times \mathbb{R}$ such that*

- *$\mathscr{Z}_{\mathrm{Isom}(X)}(g)$ preserves Y and acts diagonally with respect to the decomposition $Y = Z \times \mathbb{R}$, acting by translations on the second factor;*
- *the isometry g acts neutrally on the factor Z.*

In accordance with [BH99], we define an isometric action of a group H on a metric space X to be *proper* if for any point $x \in X$, there is a neighborhood $U \subset X$ of x such that $\{h \in H : U \cap hU \neq \infty\}$ is finite. In this case, the set $\{h \in H : K \cap hK \neq \infty\}$ is finite for any compact subset $K \subset X$ (see, for example, [BH99, Remark I.8.3(1)]). Note, however, that if the metric space X is not proper, then X may contain balls B such that $\{h \in H : B \cap hB \neq \infty\}$ is infinite;

that is, the notion of properness for isometric actions used here is strictly weaker than *metric properness*.

We will make use of the following well-known theorem [BH99, Theorem II.7.1].

Theorem 2.2. *Let H be a free abelian group of rank r acting properly by semisimple isometries on a complete* CAT(0) *space X. Then H preserves and acts as a lattice of translations on an r-dimensional flat in X.*

2.3 Thick flats

A closed convex subspace $Y \subset X$ together with an isometry $\varphi : Y \to Z \times \mathbb{R}^r$, where $r \geqslant 0$ and Z is some complete CAT(0) space, is called a *thick flat* of dimension r in X. We say a group H acting isometrically on X *preserves* the thick flat (Y, φ) if H preserves Y. Such a group H *acts as a lattice of translations* on the thick flat (Y, φ) if H acts diagonally with respect to the decomposition $Z \times \mathbb{R}^r$, acting neutrally on the first factor and by translations on the second, so that the induced map $H \to \mathbb{R}^r$ embeds H as a lattice of $\mathbb{R}^r$.

3. Lemmata

Lemmas 3.1 and 3.2 are probably well known, but we include their proofs for completeness. The objective is to determine the canonical attracting fixed point of a ballistic isometry acting diagonally on a product.

Lemma 3.1. *Let Y, Z be complete* CAT(0) *spaces and $X = Y \times Z$. Suppose $g_Y \in \mathrm{Isom}(Y)$ is neutral and $g_Z \in \mathrm{Isom}(Z)$ is hyperbolic, and let $g, g' \in \mathrm{Isom}(X)$ be the isometries $g_Y \times g_Z$, $\mathrm{Id}_Y \times g_Z$ of X, respectively. Then $\omega_g = \omega_{g'}$.*

Proof. There exist a geodesic line $\gamma_Z : \mathbb{R} \to Z$ in Z and a positive number ℓ such that $g_Z(\gamma_Z(t)) = \gamma_Z(t+\ell)$ for any $t \in \mathbb{R}$. The point $\omega_{g'} \in \partial X$ is represented by a geodesic ray of the form $(y_0, \gamma_Z(t))$, $t \geqslant 0$, $y_0 \in Y$. Thus, we reduce to the case that $Z = \mathbb{R}$ and g_Z is a translation by $\ell > 0$. Setting $x_0 = (y_0, 0)$, we show that the geodesics $\gamma^{(n)}$ in X joining x_0 to $g^n x_0$ converge uniformly on compact subsets as $n \to \infty$ to the geodesic ray $\gamma : [0, \infty) \to X$ given by $t \mapsto (y_0, t)$.

To that end, write $\gamma^{(n)}(t) = (\gamma_Y^{(n)}(t), \alpha_n t)$, where $\alpha_n > 0$ and $\gamma_Y^{(n)}$ is a linearly reparameterized geodesic in Y joining y_0 to $g_Y^n y_0$, and let $R > 0$. Note that the maximum value of $d_X(\gamma(t), \gamma^{(n)}(t))$ on $[0, R]$ is attained at $t = R$; indeed, for $0 \leqslant t \leqslant R$, we have

$$d_X(\gamma(t), \gamma^{(n)}(t))^2 = d_Y(y_0, \gamma_Y^{(n)}(t))^2 + t^2(1 - \alpha_n)^2.$$

Thus, it suffices to show that $d_X(\gamma(R), \gamma^{(n)}(R)) \to 0$. This will follow if we can show that $d_Y(y_0, \gamma_Y^{(n)}(R)) \to 0$ since

$$R^2 = d_X(x_0, \gamma^{(n)}(R))^2 = d_Y(y_0, \gamma_Y^{(n)}(R))^2 + \alpha_n^2 R^2.$$

To see that $d_Y(y_0, \gamma_Y^{(n)}(R)) \to 0$, note that since $\gamma_Y^{(n)}$ is a linearly reparameterized geodesic, we have

$$\frac{d_Y(y_0, \gamma_Y^{(n)}(R))}{d_Y(y_0, g_Y^n y_0)} = \frac{R}{d_X(x_0, g^n x_0)}$$

and so

$$\begin{aligned} d_Y(y_0, \gamma_Y^{(n)}(R))^2 &= R^2 \frac{d_Y(y_0, g_Y^n y_0)^2}{d_X(x_0, g^n x_0)^2} \\ &= R^2 \frac{d_Y(y_0, g_Y^n y_0)^2}{d_Y(y_0, g_Y^n y_0)^2 + n^2\ell^2} \\ &= R^2 \frac{\left(\frac{d_Y(y_0, g_Y^n y_0)}{n}\right)^2}{\left(\frac{d_Y(y_0, g_Y^n y_0)}{n}\right)^2 + \ell^2}. \end{aligned}$$

Now the latter approaches 0 as $n \to 0$ since

$$\lim_{n\to\infty} \frac{d_Y(y_0, g_Y^n y_0)}{n} \leqslant |g_Y|_Y$$

and $|g_Y|_Y = 0$ by assumption. □

Lemma 3.2. *Let X_1, X_2 be complete π-visible CAT(0) spaces, let $g_i \in \operatorname{Isom}(X_i)$ for $i = 1, 2$, and suppose g_1 is ballistic. Let $X = X_1 \times X_2$ and let $g = g_1 \times g_2 \in \operatorname{Isom}(X)$. Then g acts ballistically on X and*

$$\omega_g = (\arctan(|g_2|/|g_1|), \omega_{g_1}, \omega_{g_2})$$

in the spherical join $\partial X_1 * \partial X_2 = \partial X$.

Proof. We suppose first that g_1, g_2 are both ballistic, so that we may assume that X_i admits a decomposition $X_i = Y_i \times Z_i$ with respect to which g_i acts diagonally, where Z_i is isometric to $\mathbb{R}$, and where g_i acts neutrally on the first factor and acts by a translation of $|g_i|$ on the second factor. Let $g_i' \in \operatorname{Isom}(X_i)$ be the product of the identity on Y_i with the translation by $|g_i|$ on Z_i, and let $g' = g_1' \times g_2' \in \operatorname{Isom}(X)$. Note we have $|g_i| = |g_i'|$, and by Lemma 3.1, we have $\omega_{g_i} = \omega_{g_i'}$. Moreover, by viewing X as the product $X = (Y_1 \times Y_2) \times (Z_1 \times Z_2)$, we also have $\omega_g = \omega_{g'}$ by Lemma 3.1. Thus, to establish the lemma, it suffices to show

$$\omega_{g'} = (\arctan(|g_1'|/|g_2'|), \omega_{g_1'}, \omega_{g_2'})$$

but this follows from plane geometry since g_1', g_2' preserve and act as translations on the 2-dimensional flat $\{(y_1, y_2)\} \times (Z_1 \times Z_2) \subset X$, where y_i is any point in Y_i.

If g_2 is neutral, then we may only assume that X_1 admits a decomposition $X_1 = Y_1 \times Z_1$ as above, and now the lemma follows immediately from Lemma 3.1 by viewing X as the product $X = (Y_1 \times X_2) \times Z_1$. □

We apply Lemma 3.2 to the special case of matrices acting on symmetric spaces.

Lemma 3.3. *Let M be a symmetric space associated to $\mathrm{GL}_n(\mathbb{C})$ and let $g \in \mathrm{GL}_n(\mathbb{C})$ be of the form*

$$g = \operatorname{diag}(\lambda_1 U_1, \ldots, \lambda_m U_m)$$

where $\lambda_1, \ldots, \lambda_m \in \mathbb{C}^*$ with $|\lambda_k| \neq 1$ for at least one $k \in \{1, \ldots, m\}$, and $U_k \in \mathrm{SL}_{n_k}(\mathbb{C})$ is an upper unitriangular matrix for $k \in \{1, \ldots, m\}$. Then g acts ballistically on M and has the same canonical attracting fixed point as

$$g' := \operatorname{diag}(\lambda_1 I_{n_1}, \ldots, \lambda_m I_{n_m})$$

on ∂M. The same statement holds when $\mathrm{GL}_n(\mathbb{C})$ is replaced with $\mathrm{SL}_n(\mathbb{C})$.

Proof. For $k = 1, \ldots, m$, let X, X_k, Y_k, Z_k be the projections of the subgroups

$$\{\mathrm{diag}(h_1, \ldots, h_m) \,:\, h_k \in \mathrm{GL}_{n_k}(\mathbb{C})\}$$
$$\{\mathrm{diag}(I_{n_1}, \ldots, I_{n_{k-1}},\, h,\, I_{n_{k+1}}, \ldots, I_{n_m}) \,:\, h \in \mathrm{GL}_{n_k}(\mathbb{C})\}$$
$$\{\mathrm{diag}(I_{n_1}, \ldots, I_{n_{k-1}},\, h,\, I_{n_{k+1}}, \ldots, I_{n_m}) \,:\, h \in \mathrm{SL}_{n_k}(\mathbb{C})\}$$
$$\{\mathrm{diag}(I_{n_1}, \ldots, I_{n_{k-1}},\, e^t I_{n_k},\, I_{n_{k+1}}, \ldots, I_{n_m}) \,:\, t \in \mathbb{R}\}$$

of $\mathrm{GL}_n(\mathbb{C})$ to M under the quotient map $\mathrm{GL}_n(\mathbb{C}) \to M = \mathrm{GL}_n(\mathbb{C})/\mathrm{U}(n)$, respectively. Then X is a closed convex subspace of M admitting a decomposition $X = \prod_{k=1}^m X_k$. The subspace X_k in turn admits a decomposition $X_k = Y_k \times Z_k$, and the factor Z_k is isometric to $\mathbb{R}$. Each of the isometries g, g' preserves X and acts diagonally with respect to the decomposition $X = \prod_{k=1}^m X_k$. On each factor X_k, each of g, g' also acts diagonally with respect to the decomposition $X_k = Y_k \times Z_k$, acting neutrally on the first factor and as a translation by $\alpha_k \ln|\lambda_k|$ on the second for some $\alpha_k > 0$. Thus, the lemma follows from a repeated application of Lemma 3.2.

To see that the lemma remains true when $\mathrm{GL}_n(\mathbb{C})$ is replaced with $\mathrm{SL}_n(\mathbb{C})$, note that a symmetric space for $\mathrm{SL}_n(\mathbb{C})$ embeds as a closed convex $\mathrm{SL}_n(\mathbb{C})$-invariant subspace of a symmetric space for $\mathrm{GL}_n(\mathbb{C})$. □

We now observe that a collection of pairwise commuting matrices over $\mathbb{C}$ can be simultaneously put into the form described in Lemma 3.3.

LEMMA 3.4. *Let K be an algebraically closed field and let $h_\alpha \in \mathrm{M}_n(K)$ be a collection of pairwise commuting matrices. Then there are $s \in \mathbb{N}$ and $C \in \mathrm{SL}_n(K)$ such that*

$$C h_\alpha C^{-1} = \mathrm{diag}(h_{\alpha,1}, \ldots, h_{\alpha,s})$$

where $h_{\alpha,\ell} \in \mathrm{M}_{n_\ell}(K)$ is upper triangular and has a single eigenvalue for $\ell = 1, \ldots, s$.

Proof. Since K is algebraically closed, it suffices to find such $C \in \mathrm{GL}_n(K)$; indeed, we may ultimately replace C with μC, where μ is an n^{th} root of $1/\det(C)$. We now proceed by induction on n. The case $n = 1$ is trivial. Now let $n > 1$ and suppose the above claim has been established for matrices of smaller dimension. If each of the h_α has a single eigenvalue, then the statement follows from the fact that any collection of pairwise commuting elements of $\mathrm{M}_n(K)$ are simultaneously upper triangularizable [RR00, Theorem 1.1.5]. Now suppose a matrix $h \in \{h_\alpha\}_\alpha$ has more than one eigenvalue. By putting h into Jordan canonical form, for instance, we may assume h is of the form

$$h = \mathrm{diag}(h_1, h_2)$$

where $h_i \in \mathrm{M}_{n_i}(K)$ for $i = 1, 2$ and h_1, h_2 do not share an eigenvalue. Since the h_α commute with h, they preserve the generalized eigenspaces of h, and so h_α also has a block-diagonal structure

$$h_\alpha = \mathrm{diag}(h_{\alpha,1}, h_{\alpha,2})$$

where $h_{\alpha,i} \in \mathrm{M}_{n_i}(K)$ for $i = 1, 2$. The lemma now follows by applying the induction hypothesis to the collections $\{h_{\alpha,i}\}_\alpha$, $i = 1, 2$. □

We now prove what one might call a "thick flat torus theorem."

LEMMA 3.5. *Suppose X is a complete π-visible CAT(0) space and H is a free abelian subgroup of* $\mathrm{Isom}(X)$ *with a basis $h_1, \ldots, h_r \in H$ consisting of ballistic isometries such that for each $m \in \{1, \ldots, r\}$, there is no $(m-1)$-dimensional flat in X whose boundary contains the canonical* *attracting fixed points* $\omega_{h_1}, \dots, \omega_{h_m}$. *Then* H *preserves and acts as a lattice of translations on a thick flat of dimension* r *in* X.

Proof. We prove by induction the following statement: for $m \in \{1, \dots, r\}$, there is a closed convex subspace Y_m of X and a decomposition $Y_m = Z_m \times \mathbb{R}^m$ such that

- $\mathscr{Z}_{\mathrm{Isom}(X)}(h_1, \dots, h_m)$ preserves Y_m and acts diagonally with respect to the decomposition $Y_m = Z_m \times \mathbb{R}^m$, acting by translations on the second factor;
- the subgroup $\langle h_1, \dots, h_m \rangle$ acts neutrally on the first factor and as a lattice of translations on the second.

The base case $m = 1$ is given by Theorem 2.1. Now suppose the above holds for $m - 1$, where $m \in \{2, \dots, r\}$. Then h_m must act ballistically on the factor Z_{m-1}, since otherwise $\omega_{h_1}, \dots, \omega_{h_m}$ would be contained in the boundary of $\{z\} \times \mathbb{R}^{m-1}$ by Lemma 3.1, where z is any point in Z_{m-1}. Now Z_{m-1} is a complete π-visible CAT(0) space, so that by Theorem 2.1 there is a closed convex subspace Y of Z_{m-1} and a decomposition $Y = Z \times \mathbb{R}$ satisfying

- $\mathscr{Z}_{\mathrm{Isom}(Z_{m-1})}(h_m)$ preserves Y and acts diagonally with respect to the decomposition $Y = Z \times \mathbb{R}$, acting by translations on the second factor;
- the action of h_m on the first factor Z is neutral.

Then the subspace $Y_m := Y \times \mathbb{R}^{m-1} \subset Z_{m-1} \times \mathbb{R}^{m-1}$ has the desired properties. □

The following observation is used in the proof of Lemma 3.7.

Lemma 3.6. *Let* X *be a complete* CAT(0) *space and suppose* $H < \mathrm{Isom}(X)$ *is a free abelian subgroup with a basis* $h_1, \dots, h_r \in H$. *Suppose* H *preserves and acts as a lattice of translations on thick flats* Y, Y' *in* X, *and let* ϕ, ϕ' *be the maps* $H \to \mathbb{R}^r$ *induced by the actions of* H *by translations on the Euclidean factors of* Y, Y', *respectively. Then the unique linear map* $T : \mathbb{R}^r \to \mathbb{R}^r$ *satisfying* $T(\phi(h_i)) = \phi'(h_i)$ *for* $i = 1, \dots, r$ *is orthogonal.*

Proof. We wish to show that T preserves the standard inner product on $\mathbb{R}^r$. Since the $\phi(h_i)$ constitute a basis for $\mathbb{R}^r$, it suffices to show that $\langle \phi'(h_i), \phi'(h_j) \rangle = \langle \phi(h_i), \phi(h_j) \rangle$ for $i, j \in \{1, \dots, r\}$. This is equivalent to saying that for $i, j \in \{1, \dots, r\}$, we have $\|\phi(h_i)\| = \|\phi'(h_i)\|$ and $\angle(\phi(h_i), \phi(h_j)) = \angle(\phi'(h_i), \phi'(h_j))$. The former is true since

$$\|\phi(h_i)\| = |h_i|_X = \|\phi'(h_i)\|$$

and the latter is true since $\angle(\phi(h_i), \phi(h_j))$ and $\angle(\phi'(h_i), \phi'(h_j))$ are both equal to the Tits distance between ω_{h_i} and ω_{h_j} on ∂X by Lemma 3.1. □

The proof of the following lemma borrows heavily from an argument of Leeb; see the proof of Theorem 2.4 in [KL96]. Note that we work with the JSJ decomposition of a graph manifold as opposed to its geometric decomposition, so that, for example, the twisted circle bundle over the Möbius band may appear as a JSJ block of a graph manifold.

Lemma 3.7. *Let* M *be a graph manifold and suppose* $\pi_1(M)$ *acts by isometries on a complete* CAT(0) *space* X *such that for each JSJ torus* S *of* M, *the subgroup* $\pi_1(S) < \pi_1(M)$ *preserves and acts as a lattice of translations on a thick flat in* X. *Then* M *admits a nonpositively curved Riemannian metric.*

Proof. Let B be a JSJ block of M, and let $f \in \pi_1(B)$ be an element representing a generic fiber of B. The element f acts ballistically on X since f is a nontrivial element of $\pi_1(S)$, where S is a torus boundary component of B, and $\pi_1(S)$ preserves and acts as a lattice of translations on a thick flat in X by assumption. By Theorem 2.1, there is a closed convex subspace $Y \subset X$ with a metric decomposition $Y = Z \times \mathbb{R}$ such that

- any element of $\pi_1(B)$ preserves Y and acts diagonally with respect to the decomposition $Y = Z \times \mathbb{R}$, acting as a translation on the second factor;
- the action of f on the first factor Z is neutral.

Moreover, for each element $z \in \pi_1(B)$ representing a boundary component of the base orbifold O of B, the action of z on Z is ballistic since the subgroup $\langle f, z\rangle < \pi_1(B)$ preserves and acts as a lattice of translations on a thick flat in X.

We now realize B as a nonpositively curved Riemannian manifold with totally geodesic flat boundary as follows. Endow the orbifold O with a nonpositively curved Riemannian metric that is flat near the boundary so that the length of each boundary component c of O is equal to the translation length on Z of an element in $\pi_1(B)$ representing c. We let $\pi_1(B)$ act on the universal cover $\tilde{O}$ of O via the projection $\pi_1(B) \to \pi_1(O)$, where $\pi_1(O)$ acts on $\tilde{O}$ by deck transformations. The product of this action with the action of $\pi_1(B)$ on $\mathbb{R}$ coming from the decomposition $Y = Z \times \mathbb{R}$ yields a covering space action of $\pi_1(B)$ on $\tilde{O} \times \mathbb{R}$. The quotient of $\tilde{O} \times \mathbb{R}$ by this action is the desired geometric realization of B. We may do this for each Seifert component of M; the flat metrics on any pair of boundary tori that are matched in M will coincide by Lemma 3.6, so that we may glue the metrics on the Seifert components to obtain a smooth nonpositively curved metric on M. □

The following lemma will not be used in the proofs of Theorems 1.1 or 1.3, but will be applied to derive Corollary 4.1 from Theorem 1.1.

Lemma 3.8. *Let Γ be a finitely generated group and H_0 a free abelian subgroup of Γ of rank $r \geqslant 0$. Suppose Γ acts on a complete* CAT(0) *space X such that H_0 preserves and acts as a lattice of translations on a thick flat in X. Then H_0 is undistorted in Γ.*

Proof. Let $\mathcal{B} = \{h_1, \dots, h_r\} \subset H_0$ be a basis for H_0, and let $|\cdot|_{\mathcal{B}}$ be the word metric on H_0 with respect to $\mathcal{B}$. Let $\mathcal{S} \subset \Gamma$ be a finite generating set for Γ and let $|\cdot|_{\mathcal{S}}$ be the word metric on Γ with respect to $\mathcal{S}$. Let $\phi : H_0 \to \mathbb{R}^r$ be the homomorphism to $\mathbb{R}^r$ induced by the action of H_0 on a thick flat in X, and let $y_0 \in Y$, $K = \max_{s \in \mathcal{S} \cup \mathcal{S}^{-1}} d_X(y_0, sy_0)$. Since any two norms on $\mathbb{R}^r$ are equivalent, there is some $C > 0$ such that $\|\phi(h)\| \geqslant C|h|_{\mathcal{B}}$ for any $h \in H_0$. Thus, for $h \in H_0$, we have

$$K|h|_{\mathcal{S}} \geqslant d_X(y_0, hy_0) \geqslant \|\phi(h)\| \geqslant C|h|_{\mathcal{B}}$$

where the first inequality follows from the triangle inequality. □

4. Proof of Theorems 1.1 and 1.3

Proof of Theorem 1.1. (i) Since Γ is finitely generated, we have that $\Gamma \subset \mathrm{SL}_n(A)$ for some finitely generated subdomain $A \subset \mathbb{C}$. Let $E = \mathbb{Q}(A) \subset \mathbb{C}$, so that E is a finitely generated field extension of $\mathbb{Q}$. The extension $E/\mathbb{Q}$ has the structure $\mathbb{Q} \subset F \subset F(T) \subset E$, where F is the algebraic closure of $\mathbb{Q}$ in E, and T is a (possibly empty) transcendence basis for E over F. Since the extension $E/\mathbb{Q}$ is finitely generated, the set T is finite and the extensions $F/\mathbb{Q}$ and $E/F(T)$ are of finite degree.

Let $d = \deg(F/\mathbb{Q})$, and let $\sigma_1, \dots, \sigma_d$ be the embeddings of F in $\mathbb{C}$. Since $\sigma_j(F)$ is countable but $\mathbb{C}$ is not, the extension $\mathbb{C}/\sigma_j(F)$ has infinite transcendence degree, and hence we may extend σ_j to an embedding $\sigma_j : F(T) \to \mathbb{C}$. The latter may in turn be extended to an embedding $\sigma_j : E \to \mathbb{C}$ since $E/F(T)$ is algebraic and $\mathbb{C}$ is algebraically closed. The embedding $\sigma_j : E \to \mathbb{C}$ induces an embedding $\sigma_j : \mathrm{SL}_n(E) \to \mathrm{SL}_n(\mathbb{C})$. Let

$$\sigma : \mathrm{SL}_n(E) \to G_1 := \prod_{j=1}^{d} \mathrm{SL}_n(\mathbb{C})$$

be the diagonal embedding induced by the maps $\sigma_j : \mathrm{SL}_n(E) \to \mathrm{SL}_n(\mathbb{C})$. Then $\mathrm{SL}_n(E)$ acts by isometries on the Hadamard manifold $X_1 := \prod_{j=1}^{d} M_j$ via the embedding σ, where each M_j is a copy of the symmetric space (unique up to scaling of the Riemannian metric) associated to the simple Lie group $\mathrm{SL}_n(\mathbb{C})$.

By [AS82, Prop. 1.2], there are finitely many discrete valuations $\nu_1, \ldots, \nu_m$ on E such that $A \cap \bigcap_{i=1}^{m} \mathcal{O}_i \subset \mathcal{O}$, where $\mathcal{O}$ is the ring of integers of F and $\mathcal{O}_i$ is the valuation ring of ν_i. Let B_i be the Bruhat–Tits building associated to $\mathrm{SL}_n(E_{\nu_i})$, where E_{ν_i} is the completion of E with respect to ν_i; let $X_2 = \prod_{i=1}^{m} B_i$; and let $\tau : \mathrm{SL}_n(E) \to G_2 := \prod_{i=1}^{m} \mathrm{SL}_n(E_{\nu_i})$ be the diagonal embedding. Then $\mathrm{SL}_n(E)$ acts by automorphisms on X_2 via the embedding τ. We claim that the diagonal action of Γ on $X := X_1 \times X_2$ via $\sigma \times \tau : \mathrm{SL}_n(E) \to G_1 \times G_2$ has the desired properties.

To that end, let H be a subgroup of Γ containing no nontrivial unipotent elements. We first claim that for any vertex v of X_2, the subgroup $\sigma(H_v) < G_1$ is discrete, where H_v is the stabilizer of v in H. Indeed, let $h \in H_v$. Then for $i = 1, \ldots, m$, the element h fixes a vertex of B_i and (since $\mathrm{GL}_n(E)$ acts transitively on the vertices of B_i) is thus conjugate within $\mathrm{GL}_n(E)$ into $\mathrm{SL}_n(\mathcal{O}_i)$; in particular, the coefficients of the characteristic polynomial χ_h of h lie in $\mathcal{O}_i$. Since this is true for each $i \in \{1, \ldots, m\}$ and since $h \in \mathrm{SL}_n(A)$, we have that the coefficients of χ_h lie in $A \cap \bigcap_{i=1}^{m} \mathcal{O}_i$ and hence in $\mathcal{O}$. We thus have a commutative diagram

$$\begin{tikzcd} G_1 = \prod_{j=1}^{d} \mathrm{SL}_n(\mathbb{C}) \arrow[r, "P"] & \prod_{j=1}^{d} \mathbb{C}^n \\ H_v \arrow[u, "\sigma"] \arrow[r, "p"] & \mathcal{O}^n \arrow[u, "\hat{\sigma}"'] \end{tikzcd} \tag{4.1}$$

where the function p maps an element $h \in H_v$ to the n-tuple whose entries are the non-leading coefficients of χ_h, the function P is the d-fold product of the analogous map $\mathrm{SL}_n(\mathbb{C}) \to \mathbb{C}^n$, and the function $\hat{\sigma}$ is given by

$$\hat{\sigma}(\alpha_1, \ldots, \alpha_n) = (\sigma_1(\alpha_1), \ldots, \sigma_1(\alpha_n), \ldots, \sigma_d(\alpha_1), \ldots, \sigma_d(\alpha_n))$$

for $\alpha_1, \ldots, \alpha_n \in \mathcal{O}$. Since $\hat{\sigma}$ has discrete image (see, for example, Lemma 25.1.10 in [KM79]) and the diagram (4.1) is commutative, it follows that $P(\sigma(H_v))$ is discrete in $\prod_{j=1}^{d} \mathbb{C}^n$. Now suppose we have a sequence $(h_k)_{k \in \mathbb{N}}$ in H_v such that $\sigma(h_k) \to 1$ in G_1. Then, by continuity of the function P, we have $P(\sigma(h_k)) \to P(1)$. By discreteness of $P(\sigma(H_v))$, this implies that $P(\sigma(h_k)) = P(1)$ for k sufficiently large. It follows that for such k the matrix h_k is unipotent and hence trivial by our assumption that H contains no nontrivial unipotent elements. We conclude that $\sigma(H_v)$ is indeed discrete in G_1.

We now argue that for any $x \in X_2$, there is a neighborhood V of x in X_2 such that $H_V \subset H_v$ for some vertex v of X_2, where

$$H_V = \{h \in H \,:\, V \cap hV \neq \emptyset\}.$$

Let c be the cell of X_2 containing x and let ℓ be the dimension of c. Let $\epsilon > 0$ be such that the intersection of the ball $B_{X_2}(x, \epsilon)$ with the ℓ-skeleton X_2^{ℓ} of X_2 is contained in c. Then we may take $V = B_{X_2}(x, \epsilon/2)$. Indeed, if $h \in H_V$, then $hx \in X_2^\ell \cap B_{X_2}(x, \epsilon) \subset c$, and so $hc = c$. Since $\mathrm{SL}_n(E)$ acts on B_i without permutations, it follows that $h \in H_v$ for any vertex v of c.

Now, to see that H acts properly on X, we observe that for any point $x \in X_2$ and any ball $B \subset X_1$, the set $U := B \times V \subset X$ has the property that $\{h \in H : U \cap hU \neq \emptyset\}$ is finite, where $V \subset X_2$ is as in the preceding paragraph. Indeed, we have $H_V \subset H_v$ for some vertex v of X_2, and H_v acts properly on X_1 since σ embeds H_v discretely in G_1.

(ii) Suppose H is free abelian with a basis $h_1, \ldots, h_r \in H$. We show that this basis is as in the statement of Lemma 3.5, so that H preserves and acts as a lattice of translations on a thick flat in X. Indeed, by Lemma 3.4, we may assume that for $j \in \{1, \ldots, d\}$, $k \in \{1, \ldots, r\}$, we have

$$\sigma_j(h_k) = \mathrm{diag}(h_{j,k,1}, \ldots, h_{j,k,s})$$

where $h_{j,k,\ell} \in \mathrm{GL}_{n_\ell}(\mathbb{C})$ is upper triangular with a single eigenvalue for $\ell \in \{1, \ldots s\}$. We now have a homomorphism $\Delta_j : H \to \mathrm{SL}_n(\mathbb{C})$ that maps $h \in H$ to the diagonal part of $\sigma_j(h)$; note that Δ_j is injective since H contains no nontrivial unipotent matrices. The embeddings Δ_j produce a diagonal embedding $\Delta : H \to G_1$. Now let $\Delta' : H \to G_1 \times G_2$ be the product of Δ with $\tau\big|_H : H \to G_2$. Then, since $\Delta_j(h)$ has the same characteristic polynomial as $\sigma_j(h)$ for each $h \in H$, and since $\Delta_j(H)$ contains no nontrivial unipotent matrices, the action of $\Delta'(H)$ on X is proper by the above arguments. Since the latter action is by semisimple isometries, by Theorem 2.2 there is a genuine r-dimensional flat in X preserved by $\Delta'(H)$ on which $\Delta'(H)$ acts as a lattice of translations. Thus, by Lemmas 3.2 and 3.3, each nontrivial $h \in H$ acts ballistically on X and the canonical attracting fixed point of h on ∂X is equal to that of $\Delta'(h)$; in particular, $\omega_{h_1}, \ldots, \omega_{h_r}$ must be of the desired form.

(iii) Suppose $g \in \Gamma$ is diagonalizable (over $\mathbb{C}$). Since any isometry of X_2 is semisimple, to show that g acts as a semisimple isometry of X, it suffices to show that $\sigma_j(g)$ is a semisimple isometry of M_j for $j = 1, \ldots, d$. To that end, we show that $\sigma_j(g)$ is diagonalizable. Indeed, since a diagonalization of g has entries in the splitting field $\tilde{E} \subset \mathbb{C}$ of χ_g over E, we in fact have $g = CDC^{-1}$ for some $C, D \in \mathrm{SL}_n(\tilde{E})$ with D diagonal (see, for example, [Rom13, Theorem 8.11]). Since $\mathbb{C}$ is algebraically closed, we may extend σ_j to an embedding $\tilde{\sigma}_j : \tilde{E} \to \mathbb{C}$. Now

$$\sigma_j(g) = \tilde{\sigma}_j(g) = \tilde{\sigma}_j(C)\, \tilde{\sigma}_j(D)\, \tilde{\sigma}_j(C)^{-1}$$

and $\tilde{\sigma}_j(D)$ is diagonal.

□

We recover the following result, due to Button [But17, Theorem 5.2].

Corollary 4.1. *Let Γ be a finitely generated group and H a distorted finitely generated abelian subgroup of Γ. Then for any representation $\rho : \Gamma \to \mathrm{SL}_n(\mathbb{C})$, there is an infinite-order element $h \in H$ such that $\rho(h)$ is unipotent.*

Proof. Let $H_0 < H$ be a free abelian subgroup of finite-index, and suppose there is a representation $\rho_0 : \Gamma \to \mathrm{SL}_n(\mathbb{C})$ that does not map any nontrivial element of H_0 to a unipotent matrix (in particular, ρ is faithful on H_0). Then, by Theorem 1.1, there is an action of Γ via ρ on a complete CAT(0) space X such that H_0 preserves and acts by translations on a thick flat in X. By Lemma 3.8, it follows that H_0 is undistorted in Γ, and hence the same is true of H. □

Proof of Theorem 1.3. Suppose otherwise, so that for each JSJ torus S of M, the representation ρ is faithful on $\pi_1(S) < \pi_1(M)$ and the image $\rho(\pi_1(S))$ contains no nontrivial unipotent matrices. Then, by Theorem 1.1, there is an action of $\pi_1(M)$ via ρ on a complete CAT(0) space X such that for each JSJ torus S of M, the subgroup $\pi_1(S)$ preserves and acts as a lattice of translations on a thick flat in X. Thus, M admits a nonpositively curved metric by Lemma 3.7. □

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