J. reine angew. Math. **611** (2007), 75—108
DOI 10.1515/CRELLE.2007.074

Journal für die reine und angewandte Mathematik
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Berlin · New York 2007

On the first and second variations of a nonlocal isoperimetric problem

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Abstract. We consider a nonlocal perturbation of an isoperimetric variational problem. The problem may be viewed as a mathematical paradigm for the ubiquitous phenomenon of energy-driven pattern formation associated with competing short and long-range interactions. In particular, it arises as a Γ-limit of a model for microphase separation of diblock copolymers. In this article, we establish precise conditions for criticality and stability (i.e. we explicitly compute the first and second variations). We also present some applications.

1. Introduction

This article is devoted to the computation and application of formulas for the first and second variation of the following nonlocal variational problem: For fixed $m \in (-1, 1)$,

$$\text{(NLIP)} \qquad \text{minimize} \quad \mathscr{E}_\gamma(u) := \frac{1}{2}\int_\Omega |\nabla u| + \gamma \int_\Omega |\nabla v|^2\,dx,$$

over all

$$u \in BV(\Omega, \{\pm 1\}) \quad \text{with} \quad \frac{1}{|\Omega|}\int_\Omega u\,dx = m \quad \text{and} \quad -\triangle v = u - m \quad \text{in } \Omega.$$

We consider both the periodic case, where $\Omega = \mathbb{T}^n$, the n-dimensional flat torus of unit volume, and the homogeneous Neumann case where Ω is a general bounded domain in $\mathbb{R}^n$. In the former, the differential equation is solved over the torus $\mathbb{T}^n$, and in the latter we impose Neumann boundary conditions (i.e. $\nabla v \cdot \nu = 0$ on $\partial\Omega$). The first integral in (NLIP) represents the total variation of u, which for a function taking only values ± 1 is simply the pe-

1) Research partially supported by an NSERC (Canada) Discovery Grant.
2) Research partially supported by NSF DMS-0401328 and by the Institute for Mathematics and its Applications at the University of Minnesota.

rimeter of the set $\{x : u(x) = 1\}$. Thus, we refer to this problem as the *nonlocal isoperimetric problem* (NLIP), since it is a volume-constrained, nonlocal perturbation of the area functional. Problem (NLIP) is directly related to modeling microphase separation of diblock copolymers (cf. [2], [9], [22], see also the Appendix) and is closely related to models arising in the study of magnetic domains and walls (cf. [11], [17]). Melts of diblock copolymers represent a physical system which exhibits the following phase separation morphology (cf. [6], [34]): the phase separation is periodic on some fixed mesoscopic scale, and within a period cell, the interfaces are close to being area-minimizing.

The modeling of (nearly) *periodic* pattern morphologies via energy minimization involving long and short-range competitions is well-established and ubiquitous in science (cf. [15], [20], [30] and the references therein). Alternatively, problem (NLIP) can be viewed as a simple mathematical paradigm for this type of pattern morphology. Let us be clear on which *periodicity* we are referring to here. There are two length scales for minimizers of (NLIP). The first is set by the size of the domain Ω, so that for example in the case $\Omega = \mathbb{T}^n$ any minimizer can naturally be regarded as periodic with period one. This is *not* the periodicity we allude to, and the choice of periodic boundary conditions imposed by working on the torus is made for convenience only: Indeed, in Remark 2.8 we indicate how all of our results can be readily adapted to the homogeneous Neumann setting. For γ sufficiently large, a smaller scale is enforced as a weak constraint via interactions between the perimeter and the nonlocal term.

Heuristically, it is not hard to see that minimization of the nonlocal term favors oscillation: the constraint $\int u\,dx = m$ fixes the measure of the sets $\{x : u(x) = \pm 1\}$ and distributing these two sets in an oscillatory way leads to oscillation of the Laplacian of v through the Poisson equation, thus keeping down the value of the L^2-norm of ∇v. Accepting that oscillations are preferred, it is not unreasonable to anticipate periodic or nearly periodic structures emerging as the value of γ increases. In one space dimension, it has been proven that minimizers of (NLIP) and its diffuse interface version (cf. (4.1) in Appendix) are periodic on a scale determined by γ (see [3], [19], [35] for the case $m = 0$ and [24] for general m). In higher space dimensions, there is evidence to support the contention that minimizers are at least nearly periodic (see [1] and the references therein). Whether or not minimizers are exactly periodic and the precise nature of their geometry within a periodic cell remains an open problem. We expect that minimizers are at least nearly periodic for large γ, and this inherent mesoscopic periodicity is one of the reasons why (NLIP) is of interest.

Setting $\gamma = 0$ leads to a fixed volume, area-minimization (isoperimetric) problem (cf. [29]). Posing this problem on the torus gives the periodic isoperimetric problem which, together with its diffuse interface counterpart (the periodic Cahn-Hilliard problem), is the focus of an earlier paper of ours [10]. Note that for these local problems, the imposed periodic boundary conditions associated with working on the torus are crucial—greatly influencing the nature of minimizers.

One of the points of this article concerns the observation that in addition to favoring periodicity on a smaller scale, the long-range (nonlocal) term will also affect the geometry of minimizing structures. In fact the phase boundary of minimizers will not, in general, be of constant mean curvature (CMC) as it is for the classical isoperimetric problem where $\gamma = 0$. This observation is made precise in Theorem 2.3, where a first variation calculation reveals that along the phase boundary of a critical point u one has the condition

$$(n-1)H(x) + 4\gamma v(x) = \lambda \quad \text{for all } x \in \partial\{x : u(x) = 1\}$$

for some constant λ. Here $H(x)$ denotes the mean curvature of $\partial\{x : u(x) = 1\}$.

From the criticality condition above, one sees that the phase boundary will be nearly of constant mean curvature precisely when it is close to being a level set of v. Indeed, one of our long term goals is to rigorously address the extent to which minimizers of (NLIP) are close to having CMC interfaces. The connection between CMC surfaces and phase boundaries in diblock copolymer melts is well-established in the literature (see for example [1], [4], [34]). The problem (NLIP) is perhaps the simplest energetic model for phase separation in diblock copolymers which takes into account long-range interactions due to the connectivity of the different monomer chains. Numerical experiments on the gradient flow for (4.1) also suggest that its minimizers are close to being CMC ([33]). Since in the appropriate regime the Γ-limit of (4.1) is (NLIP) (cf. [9], [23]), one expects similar behavior for (NLIP). In [27], [26], a spectral analysis of some simple CMC structures (stripes and spots, and rings) is presented to establish instability for sufficiently large γ (so called "wiggled" stripe and spot instabilities) for (4.1). Not surprisingly, this closeness to CMC has also been observed in general reaction-diffusion-type models for Turing patterns (see [12], [14] and the references therein).

In part to address this issue further, a second and technically more involved point of emphasis for this article is the calculation of the second variation of (NLIP). While it is clearly the case that a full understanding of the complex energy landscape for such a nonlocal problem will undoubtedly require tools beyond the second variation, which is inherently local, this calculation seems a reasonable starting point for any stability analysis. Of course the second variation formula for the area functional ($\gamma = 0$) is a standard calculation leading to the well-known and useful criterion for the stability of constant mean curvature surfaces:

$$\int_{\partial A} (|\nabla_{\partial A}\zeta|^2 - \|B_{\partial A}\|^2\zeta^2)\, d\mathscr{H}^{n-1}(x) \geqq 0$$

for all smooth functions ζ satisfying $\int_{\partial A} \zeta(x)\, dx = 0$, cf. e.g. [5], [31]. Here we have let A denote the set $\{x : u(x) = 1\}$, $\nabla_{\partial A}$ denotes the gradient relative to the manifold ∂A and $\|B_{\partial A}\|^2$ denotes the square of the second fundamental form of ∂A, i.e. the sum of the squares of the principal curvatures. However, in addition to the incorporation of the nonlocal term, we must re-compute the second variation of area since we are taking it about a critical point of $\mathscr{E}_\gamma$, not about a critical point of the pure area functional $\mathscr{E}_0$. We find in our main result, Theorem 2.6, that the stability criterion of non-negative second variation for (NLIP) in the case $\Omega = \mathbb{T}^n$ takes the form of the following inequality:

$$\int_{\partial A} (|\nabla_{\partial A}\zeta|^2 - \|B_{\partial A}\|^2\zeta^2)\, d\mathscr{H}^{n-1}(x) + 8\gamma \int_{\partial A}\int_{\partial A} G(x,y)\zeta(x)\zeta(y)\, d\mathscr{H}^{n-1}(x)\, d\mathscr{H}^{n-1}(y)$$
$$+ 4\gamma \int_{\partial A} \nabla v \cdot \nu \zeta^2\, d\mathscr{H}^{n-1}(x) \geqq 0,$$

for all smooth functions $\zeta : \mathbb{T}^n \to \mathbb{R}$ satisfying $\int_{\partial A} \zeta(x)\, dx = 0$. Here G denotes the Green's function associated with the Laplacian on the torus and ν denotes the unit normal to ∂A

pointing out of A. A slightly modified version also holds on a general bounded domain with homogeneous Neumann boundary conditions; see Remark 2.8.

The paper is organized as follows. We carry out the calculation of the first and second variations in Section 2. In Section 3, we present some applications of these formulas, including an analysis of the stability of lamellar structures for small and large values of γ (Propositions 3.5 and 3.6). Our hope is that the first and second variation formulas will prove to be useful tools in the stability analysis of critical points of (NLIP) of non-constant mean curvature as well. Finally, in the Appendix we describe the diffuse version of the functional $\mathscr{E}_\gamma$.

Acknowledgments. Part of the work for this project was carried out while P.S. was visiting the Institute for Mathematics and its Applications in Minneapolis. He would like to thank the I.M.A. for its hospitality during this visit.

2. The first and second variations of (NLIP)

In this section, we calculate both the first and second variation of the nonlocal isoperimetric problem. For the majority of this section, we choose $\Omega = \mathbb{T}^n$ and work with periodic boundary conditions. This is for convenience only and in Remark 2.8, we address the necessary modifications for working with homogeneous Neumann boundary conditions in a general domain Ω. To fix notation, we will analyze the functional $\mathscr{E}_\gamma$ given by

$$\mathscr{E}_\gamma(u) := E(u) + \gamma F(u), \tag{2.1}$$

where $E : L^1(\mathbb{T}^n) \to \mathbb{R}$ is defined by

$$E(u) = \begin{cases} \dfrac{1}{2} \displaystyle\int_{\mathbb{T}^n} |\nabla u| & \text{if } |u| = 1 \text{ a.e., } u \in BV(\mathbb{T}^n), \ \displaystyle\int_{\mathbb{T}^n} u = m, \\ +\infty & \text{otherwise,} \end{cases}$$

and $F : L^1(\mathbb{T}^n) \to \mathbb{R}$ denotes the functional

$$F(u) = \begin{cases} \displaystyle\int_{\mathbb{T}^n} |\nabla v|^2 \, dx & \text{if } |u| = 1 \text{ a.e., } u \in BV(\mathbb{T}^n), \ \displaystyle\int_{\mathbb{T}^n} u = m, \\ +\infty & \text{otherwise,} \end{cases}$$

where $v : \mathbb{T}^n \to \mathbb{R}^1$ depends on u as the solution to the problem

$$-\Delta v = (u - m) \quad \text{in } \mathbb{T}^n, \quad \int_{\mathbb{T}^n} v = 0. \tag{2.2}$$

In the definition of E, we use $\int_{\mathbb{T}^n} |\nabla u|$ to denote the *total variation measure* of u evaluated on $\mathbb{T}^n$ (cf. [13]).

Note in particular that v satisfies periodic boundary conditions on the boundary of $[0, 1]^n$. We can write v in terms of the Green's function $G = G(x, y)$ associated with the periodic Poisson problem (2.2). Precisely, for each $x \in \mathbb{T}^n$, let $G(x, y)$ be the solution to

(2.3) $$-\triangle_y G(x, y) = (\delta_x - 1) \quad \text{on } \mathbb{T}^n, \quad \int_{\mathbb{T}^n} G(x, y)\, dx = 0,$$

where δ_x is a delta-mass measure supported at x. In particular, if we introduce the fundamental solution

$$\Phi(x) := \begin{cases} -\dfrac{1}{2\pi} \log|x| & \text{if } n = 2, \\ \dfrac{1}{\omega_n (n-2)|x|^{n-2}} & \text{if } n > 2, \end{cases}$$

then, for any fixed $x \in \mathbb{T}^n$,

(2.4) $G(x, y) - \Phi(x - y)$ is a C^∞ function (of y) in a neighborhood of x.

The functions G and v are then related by

(2.5) $$v(x) = \int_{\mathbb{T}^n} G(x, y) u(y)\, dy.$$

Calculation of the first and second variations of E alone about a critical point of E is a standard procedure (see e.g. [31]) that corresponds simply to the first and second variations of area subject to fixed volume. However, what makes this particular part of our calculation non-standard is that we are not computing this second variation about a critical point of E but rather about a critical point of E_γ; hence the phase boundary will not be of constant mean curvature and the formula will necessarily include an extra term that reflects this change. More significantly, to our knowledge, the variations of F have not been previously computed in this context. Before presenting the result, we should mention that in case the requirement $|u| = 1$ a.e. is dropped, then the calculation of the second variation of F is trivial. To this end, note that

(2.6) $$\begin{aligned} F(u) &= \int_{\mathbb{T}^n} |\nabla v|^2\, dx = -\int_{\mathbb{T}^n} v(x) \Delta v(x)\, dx = \int_{\mathbb{T}^n} v(x)\big(u(x) - m\big)\, dx \\ &= \int_{\mathbb{T}^n} v(x) u(x)\, dx = \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} G(x, y) u(x) u(y)\, dx\, dy. \end{aligned}$$

With this representation in hand, one can easily compute

$$\delta^2 F(u; \tilde{u}) := \frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0} F(u + \varepsilon \tilde{u}) = \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} G(x, y) \tilde{u}(x) \tilde{u}(y)\, dx\, dy,$$

where the variations $\tilde{u}$ run over all functions in $L^1(\mathbb{T}^n)$ satisfying $\int_{\mathbb{T}^n} \tilde{u}(x)\, dx = 0$ so as to respect the mass constraint $\int_{\mathbb{T}^n} u = m$. However, taking the constraint $|u| = 1$ a.e. into account puts a severe restriction on the variations. Indeed one should really view $\mathscr{E}_\gamma$ as a functional depending on a set, say $A \subset \mathbb{T}^n$, through the formula

(2.7) $$u(x) = \begin{cases} 1 & \text{if } x \in A, \\ -1 & \text{if } x \in A^{\mathrm{c}}. \end{cases}$$

Here A^c denotes the complement of A, i.e. $A^c = \mathbb{T}^n \backslash A$, and the n-dimensional measure of A, which we write as $|A|$, is compatible with the mass constraint on u, so that $2|A| - 1 = m$.

Our goal is to compute the first and second variations of $\mathscr{E}_\gamma$ as we allow a critical point $A \subset \mathbb{T}^n$ to vary smoothly in such a way as to preserve its volume to second order. Given a set $A \subset \mathbb{T}^n$ with C^2 boundary, we say a family of sets $\{A_t\}_{t\in(-\tau,\tau)}$ for some $\tau > 0$ represents an *admissible perturbation* (*or flow*) *of A* for the purpose of calculating a first variation if the family $\{A_t\} \subset \mathbb{T}^n$ satisfies the following three conditions:

$$\chi_{A_t} \to \chi_A \quad \text{as } t \to 0 \text{ in } L^1(\mathbb{T}^n), \tag{2.8}$$

$$\partial A_t \text{ is of class } C^2, \tag{2.9}$$

$$\frac{d}{dt}_{|_{t=0}} |A_t| = 0, \tag{2.10}$$

where χ denotes the characteristic function of a set. Note that necessarily $A_0 = A$. For the purpose of calculating a second variation, we will need to consider a family of sets $\{\tilde{A}_t\}$ that in addition to (2.8)–(2.10) satisfies the second order condition

$$\frac{d^2}{dt^2}_{|_{t=0}} |\tilde{A}_t| = 0. \tag{2.11}$$

Corresponding to each of these families, we define $U(x,t)$ and $\tilde{U}(x,t)$ respectively by

$$U(x,t) = \begin{cases} 1 & \text{if } x \in A_t, \\ -1 & \text{if } x \in A_t^c, \end{cases} \quad \text{and} \quad \tilde{U}(x,t) = \begin{cases} 1 & \text{if } x \in \tilde{A}_t, \\ -1 & \text{if } x \in \tilde{A}_t^c. \end{cases} \tag{2.12}$$

Note that $U(x,0) = \tilde{U}(x,0) = u(x)$ given by (2.7). Also, for any $t \in (-\tau,\tau)$, we define $V(\cdot,t)$ and $\tilde{V}(\cdot,t)$ to be solutions of

$$-\Delta V(\cdot,t) = U(\cdot,t) - \int_{\mathbb{T}^n} U(y,t)\,dy, \quad \int_{\mathbb{T}^n} V(x,t)\,dx = 0, \tag{2.13}$$

$$-\Delta \tilde{V}(\cdot,t) = \tilde{U}(\cdot,t) - \int_{\mathbb{T}^n} \tilde{U}(y,t)\,dy, \quad \int_{\mathbb{T}^n} \tilde{V}(x,t)\,dx = 0, \tag{2.14}$$

respectively. Note that $V(x,0) = \tilde{V}(x,0) = v(x)$ where $v : \mathbb{T}^n \to \mathbb{R}^1$ satisfies

$$-\Delta v = (u - m), \quad \int_{\mathbb{T}^n} v(x)\,dx = 0, \tag{2.15}$$

with u given by (2.7).

We now state precisely our notion of criticality and stability.

Definition 2.1. A function u given by (2.7) is a critical point of $\mathscr{E}_\gamma$ if

$$\frac{d}{dt}_{|_{t=0}} \mathscr{E}_\gamma\big(U(\cdot,t)\big) = 0 \tag{2.16}$$

for every $U = U(x,t)$ associated with an admissible family $\{A_t\}$ satisfying (2.8)–(2.10).

Definition 2.2. A critical point u of $\mathscr{E}_\gamma$ is stable if

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathscr{E}_\gamma\big(\tilde{U}(\cdot,t)\big) \geqq 0 \tag{2.17}$$

for every $\tilde{U} = \tilde{U}(x,t)$ associated with an admissible family $\{\tilde{A}_t\}$ satisfying (2.8)–(2.11).

Clearly if u is a local minimizer in L^1 of $\mathscr{E}_\gamma$ such that ∂A is C^2 then in particular u will be stable in this sense.

For the remainder of the paper, we use $\mathscr{H}^{n-1}$ to denote $(n-1)$-dimensional Hausdorff measure. Our two main results are the following:

Theorem 2.3. *Let u be a critical point of $\mathscr{E}_\gamma$ given by* (2.7) *such that ∂A is C^2 with mean curvature $H : \partial A \to \mathbb{R}$. Let ζ be any smooth function on ∂A satisfying the condition*

$$\int_{\partial A} \zeta(x)\, d\mathscr{H}^{n-1}(x) = 0.$$

Then for v solving (2.2) *one has the condition*

$$\int_{\partial A} \big((n-1)H(x) + 4\gamma v(x)\big)\zeta(x)\, d\mathscr{H}^{n-1}(x) = 0. \tag{2.18}$$

Hence there exists a constant λ such that

$$(n-1)H(x) + 4\gamma v(x) = \lambda \quad \textit{for all } x \in \partial A. \tag{2.19}$$

Remark 2.4. When $\gamma = 0$, one is simply studying critical points of area subject to a volume constraint so, as is well-known, one gets constant mean curvature as a condition of criticality. For $\gamma > 0$ but small one sees from (2.1) that the curvature is almost constant, but it will not actually be constant unless it happens that ∂A is a level set of v solving (2.2). This will be the case for a periodic lamellar structure (see Proposition 3.3) but note, for example, that a sphere (or periodic array of spheres) will never be a critical point—unless one works, as in [26], within the ansatz of radial symmetry.

Remark 2.5. A comment needs to be made regarding the assumed regularity of the critical point in Theorem 2.3, as well as that to be made in Theorem 2.6 to follow. When $\gamma = 0$, it is well-known (see for example [16]) that in dimensions $n < 8$, the phase boundary associated with any L^1-local minimizer must have constant mean curvature and be an analytic $(n-1)$-dimensional manifold, while in dimensions $n \geqq 8$, the same is true off of a (perhaps empty) singular set of Hausdorff dimension at most $n - 8$. While the phase boundary associated with a local minimizer for $\gamma > 0$ will, in general, no longer have constant mean curvature, we strongly suspect that this lower-order, compact perturbation will not destroy regularity, so we expect that the phase boundary associated with a local minimizer of $\mathscr{E}_\gamma$ is still an analytic $(n-1)$-dimensional manifold in dimensions $n < 8$, and in particular is C^2. In any event, the hypothesis of a critical point with C^2 boundary is henceforth adopted, though accomodation could be made for a low dimensional singular set as was done in [32].

Computation of a second variation about a critical point u then leads to:

Theorem 2.6. *Let u be a stable critical point of $\mathscr{E}_\gamma$ given by* (2.7) *such that ∂A is C^2. Let ζ be any smooth function on ∂A satisfying the condition*

$$\int_{\partial A} \zeta(x)\, d\mathscr{H}^{n-1}(x) = 0.$$

Then for v solving (2.2) *one has the condition*

$$J(\zeta) := \int_{\partial A} (|\nabla_{\partial A}\zeta|^2 - \|B_{\partial A}\|^2\zeta^2)\, d\mathscr{H}^{n-1}(x) + 8\gamma \int_{\partial A}\int_{\partial A} G(x,y)\zeta(x)\zeta(y)\, d\mathscr{H}^{n-1}(x)\, d\mathscr{H}^{n-1}(y) + 4\gamma \int_{\partial A} \nabla v \cdot \nu \zeta^2\, d\mathscr{H}^{n-1}(x) \geqq 0. \tag{2.20}$$

Here $\nabla_{\partial A}\zeta$ denotes the gradient of ζ relative to the manifold ∂A, $B_{\partial A}$ denotes the second fundamental form of ∂A so that $\|B_{\partial A}\|^2 = \sum_{i=1}^{n-1} \kappa_i^2$ where $\kappa_1, \ldots, \kappa_{n-1}$ are the principal curvatures and ν denotes the unit normal to ∂A pointing out of A.

A formal derivation of essentially the same formula appears in the appendix of [20].

Remark 2.7. We remark that for any ζ,

$$\int_{\partial A}\int_{\partial A} G(x,y)\zeta(x)\zeta(y)\, d\mathscr{H}^{n-1}(x)\, d\mathscr{H}^{n-1}(y) \geqq 0,$$

and hence this term is stabilizing. To see this, let μ denote the measure given by $\zeta\mathscr{H}^{n-1} \llcorner \partial A$. Since μ so defined obviously satisfies $\mu \ll \mathscr{H}^{n-1} \llcorner \partial A$, it is easy to check that the potential w given by

$$w(x) = \int_{\mathbb{T}^n} G(x,y)\, d\mu(y)$$

is bounded. Then it follows from classical potential theory that in fact w is an H^1 (weak) solution to the equation

$$-\Delta w = \mu \quad \text{on } \mathbb{T}^n, \tag{2.21}$$

and satisfies

$$\int_{\partial A}\int_{\partial A} G(x,y)\zeta(x)\zeta(y)\, d\mathscr{H}^{n-1}(x)\, d\mathscr{H}^{n-1}(y) = \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} G(x,y)\, d\mu(x)\, d\mu(y) = \int_{\mathbb{T}^n} |\nabla w|^2\, dx$$

(cf. [18], Chapter 1). Alternatively, one may view the above as $\|\mu\|^2_{H^{-1}(\mathbb{T}^n)}$, a version of the H^{-1}-norm squared of the measure μ (cf. [28]).

Before beginning the proofs of our two theorems, we indicate a way to construct admissible families (flows) $\{A_t\}$ and $\{\tilde{A}_t\}$ satisfying (2.8)–(2.10) and (2.8)–(2.11) respectively. Our approach here is rather explicit and is therefore efficient for pursuing our goals. We use an ODE to produce a flow deformation of A whose boundary instantaneously moves according to a perturbation vector field X, and which instantaneously preserves volume (in the sense of (2.10)). We then apply a correction to insure it instantaneously preserves volume to second order as well (i.e. (2.11) is satisfied). This is essentially the approach of [32] where, in fact, a flow was obtained which identically preserved volume for a finite *time* interval. However, for the purposes there and here, the instantaneous notions of (2.10) and (2.11) suffice. We should note that if one only wanted to establish the existence of a volume preserving flow, instantaneously associated with a perturbation vector field X, a simple application of the implicit function theorem, as was done in [5], would suffice. However, this construction would not make the computations necessary for the second variation explicit. In any event, our hands-on approach via the correction $\tilde{A}_t$ does not significantly expand the proofs.

Let $X : \mathbb{T}^n \to \mathbb{R}^n$ be a C^2 vector field such that

$$\int_{\partial A} X \cdot \nu \, d\mathscr{H}^{n-1}(x) = 0, \tag{2.22}$$

where ν denotes the outer unit normal to ∂A. Then let $\Psi : \mathbb{T}^n \times (-\tau, \tau) \to \mathbb{T}^n$ solve

$$\frac{\partial \Psi}{\partial t} = X(\Psi), \quad \Psi(x, 0) = x, \tag{2.23}$$

for some $\tau > 0$ and define

$$A_t := \Psi(A, t). \tag{2.24}$$

Expanding in t we find that

$$D\Psi(\cdot, t) = I + t\nabla X + \frac{1}{2} t^2 \nabla Z + o(t^2), \tag{2.25}$$

where $Z := \frac{\partial^2 \Psi}{\partial t^2}_{|_{t=0}}$ has i^{th} component given by

$$Z^{(i)} = X^{(i)}_{x_j} X^{(j)}. \tag{2.26}$$

Here and throughout, we invoke the summation convention on repeated indices.

Letting $J\Psi$ denote the Jacobian of Ψ, we then invoke the matrix identity

$$\begin{aligned} \det\left(I + tA + \frac{1}{2} t^2 B + o(t^2)\right) \\ = 1 + t \operatorname{trace} A + \frac{1}{2} t^2 [\operatorname{trace} B + (\operatorname{trace} A)^2 - \operatorname{trace}(A^2)] + o(t^2), \end{aligned} \tag{2.27}$$

which holds for any square matrices A and B. This yields

$$\frac{\partial}{\partial t}_{|_{t=0}} J\Psi = \operatorname{trace} \nabla X = \operatorname{div} X. \tag{2.28}$$

Consequently, (2.22) and the Divergence Theorem imply that

$$\frac{d}{dt}_{|_{t=0}}|A_t| = \frac{d}{dt}_{|_{t=0}}\int_A J\Psi\,dx = 0, \tag{2.29}$$

and the family of sets $\{A_t\}$ constitutes an admissible perturbation of A for the purpose of calculating a first variation, cf. (2.10). However, we note that further use of (2.25) and (2.27) yields

$$\begin{aligned}\frac{d^2}{dt^2}_{|_{t=0}}|A_t| &= \int_A \frac{\partial^2 J\Psi}{\partial t^2}_{|_{t=0}}\,dx \\ &= \int_A \operatorname{div} Z + (\operatorname{div} X)^2 - X^{(i)}_{x_j}X^{(j)}_{x_i}\,dx \\ &= \int_{\partial A}\big(\operatorname{div} X(x)\big)\big(X(x)\cdot\nu\big)\,d\mathscr{H}^{n-1}(x),\end{aligned} \tag{2.30}$$

where we obtained the last line through the two identities

$$\operatorname{div} Z = X^{(i)}_{x_i x_j}X^{(j)} + X^{(i)}_{x_j}X^{(j)}_{x_i} \tag{2.31}$$

and

$$\operatorname{div}\big((\operatorname{div} X)X\big) = (\operatorname{div} X)^2 + X^{(i)}_{x_i x_j}X^{(j)}. \tag{2.32}$$

Hence, in light of the requirement (2.11), we see through (2.30) that $\{A_t\}$ is not, in general, admissible when calculating a second variation, and we must modify it.

If it turns out that $\frac{d^2}{dt^2}_{|_{t=0}}|A_t| \neq 0$, then we define the modified perturbation $\{\tilde{A}_t\}$ as follows. We first introduce the distance function $d : \mathbb{T}^n \times (\tau,\tau) \to \mathbb{R}$ by

$$d(x,t) = \operatorname{dist}(x, \partial A_t), \tag{2.33}$$

which will be smooth on $[0,\tau)$ and $(-\tau,0]$ for small enough τ in light of the assumed regularity of ∂A. We will make frequent use of the fact that $|\nabla d(x,t)| = 1$. We next note that for A_t constructed as above, the quantity $|A_t|$ is a C^2 function of t. Hence by (2.29) and (2.30), there must be a $\tau' > 0$ such that for $|t| < \tau'$ either $|A_t| < |A|$ or $|A_t| > |A|$. If $|A_t| < |A|$, then for $|t| < \tau'$ we let

$$\tilde{A}_t := A_t \cup \{x \in A_t^c : d(x,t) < c_1 t^2\}, \tag{2.34}$$

while if $|A_t| > |A|$, we let

$$\tilde{A}_t := A_t \backslash \{x \in A_t : d(x,t) < -c_1 t^2\}. \tag{2.35}$$

Here the constant c_1 is given by

$$c_1 = -\frac{1}{2}\fint_{\partial A}\big(\operatorname{div} X(x)\big)\big(X(x)\cdot\nu\big)\,d\mathscr{H}^{n-1}(x), \tag{2.36}$$

where the average $\fint_{\partial A}$ denotes the integral divided by $\mathscr{H}^{n-1}(\partial A)$.

We will verify that condition (2.11) holds in the case $|A_t| < |A|$. The other case is handled similarly. Writing $|\tilde{A_t}| = |A_t| + (|\tilde{A_t}| - |A_t|)$ we use the co-area formula (cf. [13] or [31]) to see that

$$\frac{d^2}{dt^2}\Big|_{t=0} (|\tilde{A_t}| - |A_t|) = \frac{d^2}{dt^2}\Big|_{t=0} \int\limits_{A_t^c \cap \{0 < d(x,t) < c_1 t^2\}} |\nabla d(x,t)|\, dx$$

$$= \frac{d^2}{dt^2}\Big|_{t=0} \int\limits_0^{c_1 t^2} \mathscr{H}^{n-1}\big(\{x \in A_t^c : d(x,t) = s\}\big)\, ds = 2c_1 \mathscr{H}^{n-1}(\partial A).$$

Then (2.30) and (2.36) combine to imply (2.11), making $\{\tilde{A_t}\}$ admissible for the purpose of calculating a second variation.

Proof of Theorem 2.3. Let ζ be a smooth function on ∂A such that

$$\int\limits_{\partial A} \zeta(x)\, d\mathscr{H}^{n-1}(x) = 0. \tag{2.37}$$

Then take $X = X_\zeta \in C^2(\mathbb{T}^n, \mathbb{R}^n)$ such that

$$X_\zeta(x) = \zeta(x)\nu(x) \quad \text{on } \partial A. \tag{2.38}$$

Such a vector field will then satisfy (2.22) and through (2.23), (2.24), (2.12) and (2.13), we obtain the admissible family $\{A_t\}$ verifying (2.8)–(2.10) and corresponding functions $U(x,t)$ and $V(x,t)$. Then we have

$$\mathscr{E}_\gamma\big(U(\cdot,t)\big) = \frac{1}{2}\int\limits_{\mathbb{T}^n} |\nabla U(x,t)| + \gamma \int\limits_{\mathbb{T}^n} |\nabla V(x,t)|^2\, dx$$

$$=: E(t) + \gamma F(t).$$

Our goal is to compute the first variation $E'(t) + \gamma F'(t)$ at $t = 0$.

We begin with the calculation of $E'(0)$. Let $\Phi : \partial A \times (-\tau, \tau) \to \mathbb{T}^n$ be defined as the restriction of Ψ to the $(n-1)$-manifold ∂A. Then we have

$$E'(0) = \int\limits_{\partial A} \frac{\partial J\Phi(x)}{\partial t}\Big|_{t=0}\, d\mathscr{H}^{n-1}(x),$$

where $J\Phi$ is the Jacobian map (relative to ∂A) for Φ. Let $\{\tau_i(x)\}_{i=1,\dots,n-1}$ be an orthonormal basis for $T_x(\partial A)$, the tangent space of ∂A at x. Then following [31], Chapter 2, and making further use of (2.27), one finds

$$J\Phi = 1 + t\,\mathrm{div}_{\partial A} X + \frac{1}{2}t^2\Bigg(\mathrm{div}_{\partial A} Z + (\mathrm{div}_{\partial A} X)^2 + \sum_{i=1}^{n-1} |(D_{\tau_i} X)^\perp|^2 - \sum_{i,j=1}^{n-1} (\tau_i \cdot D_{\tau_j} X)(\tau_j \cdot D_{\tau_i} X)\Bigg) + o(t^2) \tag{2.39}$$

where $(D_{\tau_i} X)^\perp := \big((D_{\tau_i} X)\cdot \nu\big)\nu$ and $\mathrm{div}_{\partial A} X = (D_{\tau_i} X)\cdot \tau_i$. Hence, in particular we see that

$$E'(0) = \int_{\partial A} \operatorname{div}_{\partial A} X \, d\mathscr{H}^{n-1}(x) \tag{2.40}$$
$$= \int_{\partial A} (\operatorname{div}_{\partial A} X^{\perp} + \operatorname{div}_{\partial A} X^{T}) \, d\mathscr{H}^{n-1}(x),$$

where $X^{\perp} := (X \cdot v)v$ and $X^T := X - X^{\perp}$. Using (2.38), we find $X^T = 0$ on ∂A, while

$$\operatorname{div}_{\partial A} X^{\perp} = D_{\tau_i}\big((X \cdot v)v\big) \cdot \tau_i = (X \cdot v)(D_{\tau_i} v) \cdot \tau_i = (n-1)H(X \cdot v),$$

so that (2.40) gives the familiar formula for the first variation of area:

$$E'(0) = (n-1) \int_{\partial A} H(x)\zeta(x) \, d\mathscr{H}^{n-1}(x), \tag{2.41}$$

where $H(x)$ denotes the mean curvature of ∂A at x.

Now we compute $F'(t)$ at $t = 0$. We find through integration by parts that

$$F'(t) = 2 \int_{\mathbb{T}^n} \nabla V(x,t) \cdot \nabla \left(\frac{\partial}{\partial t} V(x,t) \right) dx \tag{2.42}$$
$$= -2 \int_{\mathbb{T}^n} \triangle V(x,t) \left(\frac{\partial}{\partial t} V(x,t) \right) dx$$
$$= 2 \int_{\mathbb{T}^n} \big(U(x,t) - m^t\big) \left(\frac{\partial}{\partial t} V(x,t) \right) dx,$$

where $m^t := \int_{\mathbb{T}^n} U(y,t) \, dy$. Note that

$$V(x,t) = \int_{\mathbb{T}^n} G(x,y) U(x,t) \, dy = \left(\int_{A_t} - \int_{A_t^c} \right) G(x,y) \, dy,$$

where G is defined by (2.3). Hence

$$\frac{\partial}{\partial t} V(x,t) = \frac{\partial}{\partial t} \left(\int_{A_t} G(x,y) \, dy \right) - \frac{\partial}{\partial t} \left(\int_{A_t^c} G(x,y) \, dy \right). \tag{2.43}$$

Changing variables we find:

$$\frac{\partial}{\partial t} \int_{A_t} G(x,y) \, dy \tag{2.44}$$
$$= \frac{\partial}{\partial t} \int_{\Psi(A,t)} G(x,y) \, dy$$
$$= \frac{\partial}{\partial t} \int_{A} G\big(x, \Psi(z,t)\big) J\Psi(z,t) \, dz$$
$$= \int_{A} \nabla_y G\big(x, \Psi(z,t)\big) \frac{\partial}{\partial t} \Psi(z,t) J\Psi(z,t) + G\big(x, \Psi(z,t)\big) \frac{\partial}{\partial t} \big(J\Psi(z,t)\big) \, dz.$$

Using (2.23) and (2.28), we find

$$\frac{\partial}{\partial t}\Big|_{t=0}\left(\int_{A_t} G(x,y)\,dy\right) = \int_A \nabla_y G(x,y)\cdot X(y) + G(x,y)\operatorname{div} X(y)\,dy$$

$$= \int_A \operatorname{div}_y\big(G(x,y)X(y)\big)\,dy = \int_{\partial A} G(x,y)\big(X(y)\cdot \nu_y\big)\,d\mathscr{H}^{n-1}(y).$$

Going back to (2.43), we combine this with the analogous calculation on A_t^c to obtain

$$\frac{\partial}{\partial t}\Big|_{t=0} V(x,t) = 2\int_{\partial A} G(x,y)\big(X(y)\cdot \nu_y\big)\,d\mathscr{H}^{n-1}(y). \tag{2.45}$$

Then (2.42) and (2.45) yield

$$F'(0) = 4\int_{\mathbb{T}^n}\left\{\big(u(x)-m\big)\int_{\partial A} G(x,y)\big(X(y)\cdot \nu_y\big)\,d\mathscr{H}^{n-1}(y)\right\}dx. \tag{2.46}$$

Hence by (2.41), (2.46) and the definition of a critical point (2.16), we have

$$\begin{aligned}0 &= E'(0) + \gamma F'(0) \\ &= \int_{\partial A}\left[(n-1)H(y) + 4\gamma\int_{\mathbb{T}^n}\big(u(x)-m\big)G(x,y)\,dx\right]\zeta(y)\,d\mathscr{H}^{n-1}(y) \\ &= \int_{\partial A}\{(n-1)H(y) + 4\gamma v(y)\}\zeta(y)\,d\mathscr{H}^{n-1}(y).\end{aligned} \tag{2.47}$$

Since (2.47) holds for any smooth ζ satisfying (2.37), equality (2.1) follows. □

Proof of Theorem 2.6. We again let ζ be any smooth function on ∂A such that

$$\int_{\partial A}\zeta(x)\,d\mathscr{H}^{n-1}(x) = 0.$$

This time, however, we will extend ζ to a neighborhood of ∂A such that the extension $\hat{\zeta}$ satisfies

$$\nabla\hat{\zeta}\cdot\nu = 0. \tag{2.48}$$

That is, $\hat{\zeta}$ is locally constant along normals to ∂A. There also exists a smooth vector field $\hat{\nu}$ extending the unit normal ν to ∂A such that

$$|\hat{\nu}| = 1, \tag{2.49}$$

in some neighborhood of ∂A. Consequently, there exists $X_\zeta \in C^2(\mathbb{T}^n, \mathbb{R}^n)$ such that

$$X_\zeta(x) = \hat{\zeta}(x)\hat{\nu}(x) \quad \text{in some neighborhood of } \partial A. \tag{2.50}$$

We proceed by taking $X = X_\zeta$ and then define $\{A_t\}$, $\{\tilde{A}_t\}$, $U(x,t)$, $\tilde{U}(x,t)$, $V(x,t)$ and $\tilde{V}(x,t)$ according to (2.24), (2.34), (2.35), (2.12), (2.13) and (2.14).

Writing

$$E(t) := \frac{1}{2}\int_{\mathbb{T}^n} |\nabla U(x,t)|, \quad \tilde{E}(t) := \frac{1}{2}\int_{\mathbb{T}^n} |\nabla \tilde{U}(x,t)|,$$

$$F(t) := \int_{\mathbb{T}^n} V(x,t)U(x,t)\,dx \quad \text{and} \quad \tilde{F}(t) := \int_{\mathbb{T}^n} \tilde{V}(x,t)\tilde{U}(x,t)\,dx$$

(cf. (2.6)), the second variation of $\mathscr{E}_\gamma$ with respect to the admissible perturbation $\{\tilde{A}_t\}$ is expressed as $\tilde{E}''(0) + \gamma\tilde{F}''(0)$.

We will begin with the calculation of $\tilde{E}''(0)$. We wish to emphasize here that while the calculation of the second variation of area is a well-known procedure, there are two complicating factors here. The first is that we are computing it with respect to the specific volume-preserving variations $\{\tilde{A}_t\}$. The second is that we are computing a second variation of area about an $(n-1)$-manifold ∂A that is *not* in general a critical point of the area functional.

Step 1: *Calculation of the difference* $\tilde{E}''(0) - E''(0)$. Let us assume, without loss of generality, that $|A_t| < |A|$ for $|t|$ small so that $\tilde{A}_t$ is given by (2.34). The other case is handled similarly. In light of the fact that $\partial\tilde{A}_t$ is the level set $A_t^{\mathrm{c}} \cap \{x : d(x,t) = c_1 t^2\}$, and since $|\nabla d(x,t)| = 1$, we can apply the Divergence Theorem and the co-area formula to obtain

$$\begin{aligned}\tilde{E}(t) - E(t) &= \int_{\partial\tilde{A}_t} \nabla d \cdot \nu_{\partial\tilde{A}_t}\,d\mathscr{H}^{n-1}(x) + \int_{\partial A_t} \nabla d \cdot \nu_{\partial A_t}\,d\mathscr{H}^{n-1}(x) = \int_{\tilde{A}_t\setminus A_t} \Delta d(x,t)\,dx \\ &= \int_0^{c_1 t^2} \int_{\{x:d(x,t)=\tau\}} \Delta d(x,t)\,d\mathscr{H}^{n-1}(x)\,d\tau =: \int_0^{c_1 t^2} f(t,\tau)\,d\tau.\end{aligned}$$

In the first line above, $\nu_{\partial\tilde{A}_t}$ and $\nu_{\partial A_t}$ denote unit normals pointing out of the set $\tilde{A}_t\setminus A_t$. Direct calculation and (2.36) then yield

$$\begin{aligned}\tilde{E}''(0) - E''(0) &= 2c_1 f(0,0) = 2c_1(n-1)\int_{\partial A} H(x)\,d\mathscr{H}^{n-1}(x) \\ &= -(n-1)^2\overline{H}\int_{\partial A} H(x)\zeta^2(x)\,d\mathscr{H}^{n-1}(x),\end{aligned} \tag{2.51}$$

where we have used $\overline{H}$ to denote the average of mean curvature $\fint_{\partial A} H(x)\,d\mathscr{H}^{n-1}(x)$ and we have computed

$$\Delta d(x,0) = \operatorname{div}\nu = (n-1)H(x) \quad \text{for } x \in \partial A.$$

Step 2: *Calculation of* $E''(0)$. We appeal again to the expansion (2.39) to see that

$$\begin{aligned}E''(0) &= \int_{\partial A} \frac{\partial^2 J\Phi(x)}{\partial t^2}\Big|_{t=0}\,d\mathscr{H}^{n-1}(x) \\ &= \int_{\partial A}\bigg(\operatorname{div}_{\partial A} Z + (\operatorname{div}_{\partial A} X)^2 + \sum_{i=1}^{n-1}|(D_{\tau_i}X)^\perp|^2 \\ &\qquad - \sum_{i,j=1}^{n-1}(\tau_i\cdot D_{\tau_j}X)(\tau_j\cdot D_{\tau_i}X)\bigg)\,d\mathscr{H}^{n-1}(x).\end{aligned} \tag{2.52}$$

Writing $\operatorname{div}_{\partial A} Z = \operatorname{div}_A Z^T + \operatorname{div}_A Z^\perp$ where

$$Z^T := \sum_{i=1}^{n-1} (Z \cdot \tau_i)\tau_i \quad \text{and} \quad Z^\perp := (Z \cdot \nu)\nu,$$

a straightforward calculation leads to the conclusion that

$$\text{(2.53)} \qquad \int_{\partial A} \operatorname{div}_{\partial A} Z \, d\mathscr{H}^{n-1}(x) = \int_{\partial A} \operatorname{div}_{\partial A} Z^\perp \, d\mathscr{H}^{n-1}(x)$$
$$= (n-1) \int_{\partial A} H(x)(Z \cdot \nu) \, d\mathscr{H}^{n-1}(x) = 0$$

since (2.26), (2.48), (2.50), and differentiation of the expression in (2.49) imply that $Z \cdot \nu = 0$ on ∂A. Here we have also used the Divergence Theorem on ∂A ([31]) to conclude that $\int_{\partial A} \operatorname{div}_{\partial A} Z^T \, d\mathscr{H}^{n-1}(x) = 0$ since ∂A itself has no boundary.

The other three terms in expression (2.52) are easily evaluated using (2.50) to obtain

$$\int_{\partial A} (\operatorname{div}_{\partial A} X)^2 \, d\mathscr{H}^{n-1}(x) = (n-1)^2 \int_{\partial A} H^2 \zeta^2 \, d\mathscr{H}^{n-1}(x),$$

$$\int_{\partial A} \sum_{i=1}^{n-1} |(D_{\tau_i} X)^\perp|^2 \, d\mathscr{H}^{n-1}(x) = \int_{\partial A} |\nabla_{\partial A} \zeta|^2 \, d\mathscr{H}^{n-1}(x)$$

and

$$\int_{\partial A} \sum_{i,j=1}^{n-1} (\tau_i \cdot D_{\tau_j} X)(\tau_j \cdot D_{\tau_i} X) \, d\mathscr{H}^{n-1}(x) = \int_{\partial A} \|B_{\partial A}\|^2 \zeta^2 \, d\mathscr{H}^{n-1}(x).$$

Combining these identities with (2.53), we see through (2.52) that

$$E''(0) = \int_{\partial A} \left(|\nabla_{\partial A} \zeta|^2 - \|B_{\partial A}\|^2 \zeta^2 + (n-1)^2 H^2 \zeta^2 \right) d\mathscr{H}^{n-1}(x).$$

Hence, in light of (2.51), we have

$$\text{(2.54)} \qquad \tilde{E}''(0) = \int_{\partial A} (|\nabla_{\partial A} \zeta|^2 - \|B_{\partial A}\|^2 \zeta^2) \, d\mathscr{H}^{n-1}(x)$$
$$+ (n-1)^2 \int_{\partial A} (H - \overline{H}) H \zeta^2 \, d\mathscr{H}^{n-1}(x).$$

Having completed the calculation of the second variation of the perimeter term, we now turn to the evaluation of the second variation of the nonlocal term, i.e. $\tilde{F}''(0)$. To this end, we decompose $\tilde{F}$ as follows:

$$\text{(2.55)} \qquad \tilde{F}(t) = F(t) + \int_{\mathbb{T}^n} V(x,t)\big(\tilde{U}(x,t) - U(x,t)\big) \, dx$$
$$+ \int_{\mathbb{T}^n} \tilde{U}(x,t)\big(\tilde{V}(x,t) - V(x,t)\big) \, dx$$
$$=: F(t) + F_1(t) + F_2(t).$$

Step 3: *Calculation of* $F''(0)$. To calculate $F''(t)$, we note that

$$F(t) = \Big(\int_{A_t} - \int_{A_t^c}\Big) V(x,t)\,dx = \Big(\int_A - \int_{A^c}\Big)\big[V\big(\Psi(z,t),t\big)J\Psi(z,t)\big]\,dz.$$

Hence,

$$F'(t) = \Big(\int_A - \int_{A^c}\Big)\Big[\nabla_x V\big(\Psi(z,t),t\big)\cdot\Psi_t(z,t)J\Psi(z,t) + V\big(\Psi(z,t),t\big)\frac{\partial}{\partial t}J\Psi(z,t) + V_t\big(\Psi(z,t),t\big)J\Psi(z,t)\Big]\,dz.$$

Then adopting the summation convention,

$$\begin{aligned}F''(t) = \Big(\int_A - \int_{A^c}\Big)\Big[&V_{x_ix_j}\big(\Psi(z,t),t\big)\Psi_t^{(i)}(z,t)\Psi_t^{(j)}(z,t)J\Psi(z,t)\\ &+ 2\nabla_x V_t\big(\Psi(z,t),t\big)\cdot\Psi_t(z,t)J\Psi(z,t)\\ &+ 2\nabla_x V\big(\Psi(z,t),t\big)\cdot\Psi_t(z,t)\frac{\partial}{\partial t}J\Psi(z,t)\\ &+ \nabla_x V\big(\Psi(z,t),t\big)\cdot\Psi_{tt}(z,t)J\Psi(z,t) + 2V_t\big(\Psi(z,t),t\big)\frac{\partial}{\partial t}J\Psi(z,t)\\ &+ V\big(\Psi(z,t),t\big)\frac{\partial^2}{\partial t^2}J\Psi(z,t) + V_{tt}\big(\Psi(z,t),t\big)J\Psi(z,t)\Big]\,dz.\end{aligned}$$

Recalling (2.26), we can then write

$$\begin{aligned}F''(0) = \Big(\int_A - \int_{A^c}\Big)\Big[&V_{y_iy_j}(y,0)X^{(i)}(y)X^{(j)}(y) + 2\nabla_y V_t(y,0)\cdot X(y)\\ &+ 2\nabla_y V(y,0)\cdot X(y)\,\mathrm{div}\,X(y) + \nabla_y V(y,0)\cdot Z(y)\\ &+ 2V_t(y,0)\,\mathrm{div}\,X(y) + V(y,0)\frac{\partial^2}{\partial t^2}J\Psi(y,0) + V_{tt}(y,0)\Big]\,dy.\end{aligned} \tag{2.56}$$

We now proceed to compute all seven terms in (2.56). We will often exploit, without citation, the Divergence Theorem.

Recall from formula (2.45) of the proof of Theorem 2.3 that

$$V_t(x,0) = 2\int_{\partial A} G(x,y)\big(X(y)\cdot\nu_y\big)\,d\mathscr{H}^{n-1}(y).$$

Consequently, grouping together the second and fifth terms of (2.56) we find

$$\begin{aligned}&2\Big(\int_A - \int_{A^c}\Big)\{\nabla_y V_t(y,0)\cdot X(y) + V_t(y,0)\,\mathrm{div}\,X(y)\}\,dy\\ &= 2\Big(\int_A - \int_{A^c}\Big)\mathrm{div}\big(V_t(y,0)X(y)\big)\,dy\end{aligned} \tag{2.57}$$

$$= 4 \int_{\partial A} V_t(y,0) X(y) \cdot \nu_y \, d\mathscr{H}^{n-1}(y)$$

$$= 8 \int_{\partial A} \int_{\partial A} G(x,y) \big(X(x) \cdot \nu_x\big) \big(X(y) \cdot \nu_y\big) \, d\mathscr{H}^{n-1}(x) \, d\mathscr{H}^{n-1}(y).$$

Next, using (2.26), we note that

$$\operatorname{div}\big([\nabla_y V(y,0) \cdot X(y)] X(y)\big)$$
$$= \big(\nabla_y V(y,0) \cdot X(y)\big) \operatorname{div} X(y) + V_{y_i y_j}(y,0) X^{(i)}(y) X^{(j)}(y) + \nabla_y V(y,0) \cdot Z(y).$$

Hence, the first and fourth terms of (2.56), along with one factor of the third term gives

$$\text{(2.58)} \qquad \left(\int_A - \int_{A^c} \right) V_{y_i y_j}(y,0) X^{(i)}(y) X^{(j)}(y) + \nabla_y V(y,0) \cdot X(y) \operatorname{div} X(y)$$
$$+ \nabla_y V(y,0) \cdot Z(y) \, dy$$
$$= 2 \int_{\partial A} \big(\nabla v(y) \cdot X(y)\big) \big(X(y) \cdot \nu_y\big) \, d\mathscr{H}^{n-1}(y).$$

By (2.25), (2.27), (2.31) and (2.32) we have

$$\text{(2.59)} \qquad \frac{\partial^2}{\partial t^2} J\Psi(x,0) = \operatorname{div} Z(x) + \big(\operatorname{div} X(x)\big)^2 - X^{(i)}_{x_j} X^{(j)}_{x_i}$$
$$= X^{(i)}_{x_i x_j} X^{(j)} + \big(\operatorname{div} X(x)\big)^2$$
$$= \operatorname{div}\big((\operatorname{div} X) X\big).$$

Hence the sixth term in (2.56) can be computed as

$$\text{(2.60)} \qquad \left(\int_A - \int_{A^c} \right) V(x,0) \frac{\partial^2}{\partial t^2} J\Psi(x,0) \, dx$$
$$= \left(\int_A - \int_{A^c} \right) v \operatorname{div}\big((\operatorname{div} X) X\big) \, dx$$
$$= 2 \int_{\partial A} v (\operatorname{div} X)(X \cdot \nu) \, d\mathscr{H}^{n-1}(x) - \left(\int_A - \int_{A^c} \right) (\nabla v \cdot X) (\operatorname{div} X) \, dx.$$

Note that the last term in (2.60) will cancel with the remaining factor of the third term of (2.56).

It remains to compute the last term of (2.56). Recall from (2.43) and (2.44) that

$$V_t(x,t) = \left(\int_A - \int_{A^c} \right) \left[\nabla_y G\big(x, \Psi(z,t)\big) \frac{\partial}{\partial t} \Psi(z,t) J\Psi(z,t) + G\big(x, \Psi(z,t)\big) \frac{\partial}{\partial t} J\Psi(z,t) \right] dz.$$

We must take some care in differentiating again with respect to t: since the second derivatives of G are not integrable close to the singularity, differentiation under the integral sign is illegal. In the end, we care only about the integral of $V_{tt}(x,0)$, and hence we circumvent differentiating G twice by first integrating over x, applying the divergence theorem, and then performing the t-differentiation. We wish to compute

$$
\begin{aligned}
(2.61)\qquad & \Big(\int_A - \int_{A^c}\Big) V_{tt}(x,0)\,dx \\
&= \frac{d}{dt}\Big|_{t=0} \Big(\int_A - \int_{A^c}\Big) V_t(x,t)\,dx \\
&= \frac{d}{dt}\Big|_{t=0} \Big(\int_A - \int_{A^c}\Big)\Big(\int_A - \int_{A^c}\Big)\Big\{\nabla_y G\big(x,\Psi(z,t)\big)\frac{\partial \Psi}{\partial t}(z,t) J\Psi(z,t) \\
&\qquad + G\big(x,\Psi(z,t)\big)\frac{\partial}{\partial t} J\Psi(z,t)\Big\}\,dz\,dx.
\end{aligned}
$$

Focusing on the first term, note that

$$\nabla_z G\big(x,\Psi(z,t)\big) = \nabla_y G\big(x,\Psi(z,t)\big) D\Psi(z,t)$$

so that

$$\nabla_y G\big(x,\Psi(z,t)\big) = \nabla_z G\big(x,\Psi(z,t)\big)[D\Psi(z,t)]^{-1}.$$

Hence,

$$
\begin{aligned}
(2.62)\qquad & \Big(\int_A - \int_{A^c}\Big)\Big(\int_A - \int_{A^c}\Big)\Big\{\nabla_y G\big(x,\Psi(z,t)\big)\frac{\partial \Psi}{\partial t}(z,t) J\Psi(z,t)\Big\}\,dz\,dx \\
&= \Big(\int_A - \int_{A^c}\Big)\Big(\int_A - \int_{A^c}\Big)\Big\{\nabla_z G\big(x,\Psi(z,t)\big)[D\Psi(z,t)]^{-1}\frac{\partial \Psi}{\partial t}(z,t) J\Psi(z,t)\Big\}\,dz\,dx \\
&= -\Big(\int_A - \int_{A^c}\Big)\Big(\int_A - \int_{A^c}\Big) \\
&\quad \times \Big\{G\big(x,\Psi(z,t)\big)\,\mathrm{div}\Big([D\Psi(z,t)]^{-1}\frac{\partial \Psi}{\partial t}(z,t) J\Psi(z,t)\Big)\Big\}\,dz\,dx \\
&\quad + 2\Big(\int_A - \int_{A^c}\Big)\int_{\partial A}\Big(G\big(x,\Psi(z,t)\big)[D\Psi(z,t)]^{-1}\frac{\partial \Psi}{\partial t}(z,t) J\Psi(z,t)\Big) \\
&\quad \cdot \nu_z\, d\mathscr{H}^{n-1}(z)\,dx.
\end{aligned}
$$

Having performed the integration by parts above, we are now free to take the t-derivative inside the integrals in (2.61). We first note that

$$D\Psi(x,t) = I + DX(x)t + \mathcal{O}(t^2) \quad \text{so that} \quad [D\Psi]^{-1} = I - DX(x)t + \mathcal{O}(t^2)$$

and hence,

$$\frac{d}{dt}([D\Psi]^{-1})_{|_{t=0}} = -DX.$$

Using this, along with (2.23), (2.28) and (2.26), one has

$$\begin{aligned} (2.63) \qquad & \frac{d}{dt}_{|_{t=0}} \left\{ G\big(x, \Psi(z,t)\big) \operatorname{div}\left([D\Psi(z,t)]^{-1} \frac{\partial \Psi}{\partial t}(z,t) J\Psi(z,t) \right) \right\} \\ & = G(x,y) \operatorname{div}\big(X(y) \operatorname{div} X(y)\big) + \nabla_y G(x,y) \cdot X(y) \operatorname{div} X(y) \\ & = \operatorname{div}_y\big(G(x,y) X(y) \operatorname{div} X(y)\big). \end{aligned}$$

We also note that

$$\begin{aligned} (2.64) \qquad & \frac{d}{dt}_{|_{t=0}} \left(G\big(x, \Psi(y,t)\big) [D\Psi(y,t)]^{-1} \frac{\partial \Psi}{\partial t}(y,t) J\Psi(y,t) \right) \\ & = \big(\nabla_y G(x,y) \cdot X(y)\big) X(y) + G(x,y) X(y) \operatorname{div} X(y) \\ & = X(y) \operatorname{div}_y\big(G(x,y) X(y)\big) \end{aligned}$$

and, through an appeal to (2.59), that

$$\begin{aligned} (2.65) \qquad & \frac{d}{dt}_{|_{t=0}} \left(G\big(x, \Psi(y,t)\big) \frac{\partial}{\partial t} J(y,t) \right) \\ & = \nabla_y G(x,y) \cdot X(y) \operatorname{div} X(y) + G(x,y) \operatorname{div}(X \operatorname{div} X) \\ & = \operatorname{div}_y\big(G(x,y) X(y) \operatorname{div} X(y)\big). \end{aligned}$$

Working with identities (2.63)–(2.65) to carry out the differentiation of the integrals in (2.61) with respect to t, we can return to (2.61) to find

$$\begin{aligned} (2.66) \qquad & \left(\int_A - \int_{A^c} \right) V_{tt}(x,0)\, dx \\ & = 2\left(\int_A - \int_{A^c} \right) \int_{\partial A} \operatorname{div}_y\big(G(x,y) X(y)\big) \big(X(y) \cdot \nu_y\big)\, d\mathscr{H}^{n-1}(y)\, dx \\ & = 2 \int_{\partial A} \operatorname{div}_y \left\{ \left(\int_A G(x,y)\, dx - \int_{A^c} G(x,y)\, dx \right) \cdot X(y) \right\} X(y) \\ & \quad \cdot \nu_y\, d\mathscr{H}^{n-1}(y) \\ & = 2 \int_{\partial A} \operatorname{div}\big(v(x) X(x)\big) X(x) \cdot \nu_x\, d\mathscr{H}^{n-1}(x), \end{aligned}$$

thus obtaining an expression for the seventh term of (2.56).

Combining (2.57), (2.58), (2.60) and (2.66), along with an appeal to (2.50), we then see that (2.56) implies

$$F''(0) = 8\int_{\partial A}\int_{\partial A} G(x,y)\zeta(x)\zeta(y)\,d\mathscr{H}^{n-1}(x)\,d\mathscr{H}^{n-1}(y) + 4\int_{\partial A}\operatorname{div}[v(x)X(x)]\zeta(x)\,d\mathscr{H}^{n-1}(x). \tag{2.67}$$

Step 4: *Calculation of* $F_1''(0)$ *and* $F_2''(0)$. Recall from (2.55) that

$$F_1(t) = \int_{\mathbb{T}^n} V(x,t)\big(\tilde{U}(x,t) - U(x,t)\big)\,dx = 2\int_{\tilde{A}_t\setminus A_t} V(x,t)\,dx - 2\int_{A_t\setminus\tilde{A}_t} V(x,t)\,dx.$$

From the construction of $\tilde{A}_t$, recall that either $A_t \subset \tilde{A}_t$ or $\tilde{A}_t \subset A_t$. We will again address the case where $A_t \subset \tilde{A}_t$. The other case is handled in a similar manner. Using the co-area formula and (2.33), we have

$$F_1(t) = 2\int_{\tilde{A}_t\setminus A_t} V(x,t)\,dx = 2\int_0^{c_1t^2}\int_{A_t^c\cap\{x:d(x,t)=\tau\}} V(x,t)\,d\mathscr{H}^{n-1}(x)\,d\tau.$$

$$= 2\int_0^{c_1t^2}\Lambda(\tau,t)\,d\tau,$$

where

$$\Lambda(\tau,t) := \int_{A_t^c\cap\{x:d(x,t)=\tau\}} V(x,t)\,d\mathscr{H}^{n-1}(x).$$

Hence differentiating with respect to t, one finds

$$F_1'(0) = 0 \quad \text{while} \quad F_1''(0) = 4c_1\Lambda(0,0) = 4c_1\int_{\partial A} v(x)\,d\mathscr{H}^{n-1}(x).$$

Thus, by (2.36), we have

$$F_1''(0) = -2\left(\int_{\partial A} v(x)\,d\mathscr{H}^{n-1}(x)\right)\left(⨍_{\partial A}\big(\operatorname{div}X(x)\big)\zeta(x)\,d\mathscr{H}^{n-1}(x)\right). \tag{2.68}$$

Now recall that

$$F_2(t) = \int_{\mathbb{T}^n}\tilde{U}(x,t)\big(\tilde{V}(x,t) - V(x,t)\big) = \int_{\tilde{A}_t}\big(\tilde{V}(x,t) - V(x,t)\big)\,dx - \int_{\tilde{A}_t^c}\big(\tilde{V}(x,t) - V(x,t)\big)\,dx.$$

As before, we will only pursue the case where $A_t \subset \tilde{A}_t$. Hence, we have

$$F_2(t) = 2 \int_{\tilde{A}_t \setminus A_t} \big(\tilde{V}(x,t) - V(x,t)\big)\, dx + \left(\int_{A_t} - \int_{A_t^c} \right) \big(\tilde{V}(x,t) - V(x,t)\big)\, dx$$

$$= 2 \int_0^{c_1 t^2} \int_{A_t^c \cap \{x : d(x,t) = \tau\}} \big(\tilde{V}(x,t) - V(x,t)\big)\, d\mathscr{H}^{n-1}(x)\, d\tau$$

$$+ \left(\int_A - \int_{A^c} \right) \big(\tilde{V}\big(\Psi(y,t),t\big) - V\big(\Psi(y,t),t\big)\big) J\Psi(y,t)\, dy$$

$$=: G_1(t) + G_2(t).$$

Since $\tilde{V}(x,0) = V(x,0) = v(x)$, we find as in the calculation of $F_1''(0)$ above that

$$G_1''(0) = 4c_1 \int_{\partial A} \big(\tilde{V}(x,0) - V(x,0)\big)\, d\mathscr{H}^{n-1}(x) = 0.$$

On the other hand, setting $w(x,t) := \tilde{V}(x,t) - V(x,t)$, we have

$$G_2'(t) = \left(\int_A - \int_{A^c} \right) \Big[\nabla_x w\big(\Psi(y,t),t\big) \cdot \Psi_t(y,t) J\Psi(y,t) + w_t\big(\Psi(y,t),t\big) J\Psi(y,t) + w\big(\Psi(y,t),t\big) \frac{\partial}{\partial t} J\Psi(y,t) \Big]\, dy. \tag{2.69}$$

To differentiate (2.69) again, first note that $w(x,0) = 0$ and hence $\nabla_x w(x,0) = 0 = w_{x_i x_j}(x,0)$ for $i, j = 1, \ldots, n$. Moreover,

$$w_t(x,t) = 2 \frac{\partial}{\partial t} \int_{\tilde{A}_t \setminus A_t} G(x,y)\, dy = 2 \frac{\partial}{\partial t} \int_0^{c_1 t^2} \int_{A_t^c \cap \{x : d(x,t) = \tau\}} G(x,y)\, d\mathscr{H}^{n-1}(y)\, d\tau.$$

It follows that $w_t(x,0) = 0$, and

$$w_{tt}(x,0) = 4c_1 \int_{\partial A} G(x,y)\, dH^{n-1}(y)$$

$$= -2 \left(\fint_{\partial A} (\operatorname{div} X(x)) \zeta(x)\, d\mathscr{H}^{n-1}(x) \right) \left(\int_{\partial A} G(x,y)\, d\mathscr{H}^{n-1}(y) \right).$$

As in (2.56), we differentiate (2.69) with respect to t and evaluate at $t = 0$. However, this time, only one term survives:

$$G_2''(0) = \left(\int_A - \int_{A^c} \right) w_{tt}(x,0)\, dx \tag{2.70}$$

$$= -2 \left(\fint_{\partial A} (\operatorname{div} X(x)) \zeta(x)\, d\mathscr{H}^{n-1}(x) \right) \int_{\partial A} \left(\int_A - \int_{A^c} \right) G(x,y)\, dx\, d\mathscr{H}^{n-1}(y)$$

$$= -2 \left(\fint_{\partial A} (\operatorname{div} X(x)) \zeta(x)\, d\mathscr{H}^{n-1}(x) \right) \int_{\partial A} v(y)\, d\mathscr{H}^{n-1}(y).$$

Step 5: *Calculation of* $\tilde{F}''(0)$. Returning to (2.55), we combine (2.67), (2.68), and (2.70) to find

$$\begin{aligned}(2.71)\qquad \tilde{F}''(0) = {} & 8\int_{\partial A}\int_{\partial A} G(x,y)\zeta(x)\zeta(y)\,d\mathscr{H}^{n-1}(x)\,d\mathscr{H}^{n-1}(y)\\ & + 4\int_{\partial A}\operatorname{div}[v(x)X(x)]\zeta(x)\,d\mathscr{H}^{n-1}(y)\\ & - 4\left(\fint_{\partial A}(\operatorname{div}X(x))\zeta(x)\,d\mathscr{H}^{n-1}(x)\right)\int_{\partial A} v(y)\,d\mathscr{H}^{n-1}(y).\end{aligned}$$

Again invoking (2.48)–(2.50), we have

$$\operatorname{div}X(x) = (n-1)\zeta(x)H(x) \quad \text{for } x \in \partial A$$

and so

$$\begin{aligned}(2.72)\qquad \tilde{F}''(0) = {} & 8\int_{\partial A}\int_{\partial A} G(x,y)\zeta(x)\zeta(y)\,d\mathscr{H}^{n-1}(x)\,d\mathscr{H}^{n-1}(y)\\ & + 4\int_{\partial A}\nabla v(x)\cdot \nu(x)\zeta^2(x)\,d\mathscr{H}^{n-1}(x)\\ & + 4(n-1)\int_{\partial A} v(x)H(x)\zeta^2(x)\,d\mathscr{H}^{n-1}(x)\\ & - 4(n-1)\left(\int_{\partial A} v(y)\,d\mathscr{H}^{n-1}(y)\right)\left(\fint_{\partial A} H(x)\zeta^2(x)\,d\mathscr{H}^{n-1}(x)\right)\\ = {} & 8\int_{\partial A}\int_{\partial A} G(x,y)\zeta(x)\zeta(y)\,d\mathscr{H}^{n-1}(x)\,d\mathscr{H}^{n-1}(y)\\ & + 4\int_{\partial A}\nabla v(x)\cdot \nu(x)\zeta^2(x)\,d\mathscr{H}^{n-1}(x)\\ & + 4(n-1)\gamma\int_{\partial A}\left[v(x) - \fint_{\partial A} v(y)\,d\mathscr{H}^{n-1}(y)\right]H(x)\zeta^2(x)\,d\mathscr{H}^{n-1}(x).\end{aligned}$$

Lastly, we invoke the hypothesis that u is a critical point of $\mathscr{E}_\gamma$. By Theorem 2.3 this means for some constant λ and every $x \in \partial A$,

$$(n-1)H(x) + 4\gamma v(x) = \lambda.$$

Hence the last term in (2.72) reduces to simply

$$-(n-1)^2\int_{\partial A}\big(H(x) - \overline{H}\big)H(x)\zeta^2(x)\,d\mathscr{H}^{n-1}(x).$$

Taking this observation into account, we see that (2.54) and (2.72) combine to yield

$$\frac{d^2\mathscr{E}_\gamma(\tilde{U}(\cdot,t))}{dt^2}\Big|_{t=0} = \tilde{E}''(0) + \gamma\tilde{F}''(0)$$
$$= \int_{\partial A} (|\nabla_{\partial A}\zeta|^2 - \|B_{\partial A}\|^2\zeta^2)\, d\mathscr{H}^{n-1}(x)$$
$$+ 8\gamma \int_{\partial A}\int_{\partial A} G(x,y)\zeta(x)\zeta(y)\, d\mathscr{H}^{n-1}(x)\, d\mathscr{H}^{n-1}(y)$$
$$+ 4\gamma \int_{\partial A} \nabla v\cdot \nu\zeta^2\, d\mathscr{H}^{n-1}(x). \tag{2.73}$$

By the definition of stability (2.17), we obtain the desired inequality (2.20). □

Remark 2.8 (Natural boundary conditions). Throughout this section we have posed the nonlocal problem on a torus; that is, we have taken periodic boundary conditions. The motivation for this choice is two-fold. First, it represents perhaps the easiest setting in which to carry out the somewhat involved calculation of the second variation. Second, the periodic boundary conditions represent what is perhaps the most relevant choice when attempting to model the behavior of phase boundaries in diblock copolymers, one of our goals in an on-going investigation.

We wish to emphasize here, however, that there is nothing essential about this choice. For example, having carried out the calculations of this section, we can now readily adopt the results to the setting of the natural homogeneous Neumann boundary conditions on an arbitrary bounded, smooth domain $\Omega \subset \mathbb{R}^n$. For such an Ω, we may consider:

(NLIP) $$\text{Minimize } \frac{1}{2}\int_\Omega |\nabla u| + \gamma \int_\Omega |\nabla v|^2\, dx,$$

over all

$$u \in BV(\Omega, \{\pm 1\}), \quad \frac{1}{|\Omega|}\int_\Omega u\, dx = m$$

with

$$-\triangle v = u - m \quad \text{in } \Omega, \quad \nabla v\cdot \nu = 0 \quad \text{on } \partial\Omega \quad \text{and} \quad \int_\Omega v = 0. \tag{2.74}$$

Then one simply replaces the condition of periodicity on the deforming vector field X by the condition that $X(x)$ is tangent to $\partial\Omega$ for all $x \in \partial\Omega$, as was done for the local problem in [32]. Note that in light of this tangency, every application of the Divergence Theorem in the calculation of the first and second variations of the nonlocal term will not produce any new boundary terms. Indeed, the only deviation from the results of Theorems 2.3 and 2.6 when adapting them to the Neumann setting comes from consideration of the local term E. From the first variation of E, one will find that if A is critical, then ∂A must meet $\partial\Omega$ orthogonally (if at all), but condition (2.1) will still hold inside Ω. From the second variation, one finds that the quantity $J(\zeta)$ defined by (2.20) will additionally include the term

$$-\int_{\partial A\cap\partial\Omega} B_{\partial\Omega}(\nu,\nu)\zeta^2\, d\mathscr{H}^{n-2}(x),$$

where as before, $B_{\partial\Omega}$ represents the second fundamental form associated with $\partial\Omega$, and ν represents the outer unit normal to ∂A. As was just mentioned, criticality of A implies that ν lies in the tangent plane to the boundary of Ω so that the quantity $B_{\partial\Omega}(\nu,\nu)$ is sensible. This additional term can be traced back to (2.53) where now there will be a non-zero contribution from $\int_{\partial A} \mathrm{div}_{\partial A}\, Z^T\, d\mathscr{H}^{n-1}(x)$ in the form of an integral over $\partial A \cap \partial\Omega$. See [32] for more details.

Thus, in a manner analogous to that used for the (NLIP) on $\mathbb{T}^n$, we may define critical points and stable critical points of the Neumann boundary (NLIP) as formulated above on a general domain Ω. Let $u \in BV(\Omega, \pm 1)$ with $A := \{u(x) = 1\}$ having C^2 boundary ∂A with mean curvature H, and let v solve (2.74). Then if u is a critical point of (NLIP), we have for all $x \in \partial A \cap \Omega$ and some constant λ,

$$(n-1)H(x) + 4\gamma v(x) = \lambda, \tag{2.75}$$

and for each $x \in \partial A \cap \partial\Omega$,

$$\partial A \text{ is orthogonal to } \partial\Omega. \tag{2.76}$$

Moreover, if u is stable then for every smooth ζ defined on ∂A with

$$\int_{\partial A} \zeta(x)\, d\mathscr{H}^{n-1}(x) = 0,$$

one has the condition

$$\begin{aligned} J(\zeta) := {} & \int_{\partial A} (|\nabla_{\partial A}\zeta|^2 - \|B_{\partial A}\|^2\zeta^2)\, d\mathscr{H}^{n-1}(x) - \int_{\partial A\cap\partial\Omega} B_{\partial\Omega}(\nu,\nu)\zeta^2\, d\mathscr{H}^{n-2}(x) \\ & + 8\gamma \int_{\partial A}\int_{\partial A} \mathscr{N}(x,y)\zeta(x)\zeta(y)\, d\mathscr{H}^{n-1}(x)\, d\mathscr{H}^{n-1}(y) \\ & + 4\gamma \int_{\partial A} \nabla v \cdot \nu\zeta^2\, d\mathscr{H}^{n-1}(x) \geqq 0, \end{aligned} \tag{2.77}$$

where $\mathscr{N}(x,y)$ is the Green's function satisfying $-\triangle\mathscr{N} = \delta - \frac{1}{|\Omega|}$ on Ω with Neumann boundary conditions.

3. Applications of the first and second variation formulas

In this section we present a few applications of Theorems 2.3 and 2.6 posed on $\mathbb{T}^n$ and in the Neumann setting of Remark 2.8.

We begin with an identity coming from the translation invariance of $\mathscr{E}_\gamma$.

Proposition 3.1. *Let $u : \mathbb{T}^n \to \mathbb{R}$ be a critical point of $\mathscr{E}_\gamma$ with $A := \{x : u(x) = 1\}$ and assume ∂A is C^2. Then the following identity holds*:

$$(3.1) \qquad \sum_{i,j=1}^{n} \int_{\partial A} \int_{\partial A} G(x,y)\nu^{(i)}(x)\nu^{(j)}(y)\mathscr{H}^{n-1}(x)\mathscr{H}^{n-1}(y) = (1-|A|)|A|,$$

where $\nu = (\nu^{(1)}, \ldots, \nu^{(n)})$ denotes the outer unit normal to ∂A.

Proof. In light of the Divergence Theorem, we begin by observing that

$$\int_{\partial A} \nu^{(i)}\, d\mathscr{H}^{n-1}(x) = \int_{A} \operatorname{div} e_i\, dx = 0,$$

where e_i is the unit vector pointing in the positive i^{th} coordinate direction. Hence, the choice $\zeta = \nu^{(i)}$ is allowable in the second variation formula (2.20). This choice corresponds to pure translation in that arises by taking the deforming vector field X introduced above (2.22) to be simply e_i. In light of the invariance of the energy $\mathscr{E}_\gamma$ under translation of A, we note that necessarily one has the condition

$$(3.2) \qquad J(\nu^{(i)}) = 0 \quad \text{for } i = 1, \ldots, n.$$

Now we invoke the identity

$$\Delta_{\partial A}\nu^{(i)} = -\|B_{\partial A}\|^2\nu^{(i)} + (n-1)e_i \cdot \nabla_{\partial A} H,$$

(where $\Delta_{\partial A}$ is the surface Laplacian), which holds along any C^2 hypersurface (cf. [13]). Multiplying both sides by $\nu^{(i)}$ and integrating then yields

$$\int_{\partial A} \left(|\nabla_{\partial A}\nu^{(i)}|^2 - \|B_{\partial A}\|^2(\nu^{(i)})^2\right) d\mathscr{H}^{n-1}(x) = -(n-1)\int_{\partial A} \nabla_{\partial A} H \cdot (\nu^{(i)}e_i)\, d\mathscr{H}^{(n-1)}(x).$$

Then since $\nabla_{\partial A} H \cdot \nu = 0$, we can sum over i in this identity to see that

$$\int_{\partial A} \left(|\nabla_{\partial A}\nu^{(i)}|^2 - \|B_{\partial A}\|^2(\nu^{(i)})^2\right) d\mathscr{H}^{n-1}(x) = 0.$$

Thus, we find from (2.20) that

$$\begin{aligned} 0 &= \sum_{i=1}^{n} J(\nu^{(i)}) \\ &= 8\gamma \sum_{i=1}^{n} \int_{\partial A}\int_{\partial A} G(x,y)\nu^{(i)}(x)\nu^{(i)}(y)\, d\mathscr{H}^{n-1}(x)\, d\mathscr{H}^{n-1}(y) + 4\gamma \int_{\partial A} \nabla v \cdot \nu\, d\mathscr{H}^{n-1}(x). \end{aligned}$$

Finally, we apply the divergence theorem to see that

$$(3.3) \qquad \begin{aligned} \int_{\partial A} \nabla v \cdot \nu\, d\mathscr{H}^{n-1}(x) &= \int_{A} \Delta v\, dx \\ &= \int_{A} (m-u)\, dx = (m-1)|A| = 2(|A|-1)|A|, \end{aligned}$$

since $u \equiv 1$ on A and (3.1) follows. $\square$

Remark 3.2. Applied to the case of a single vertical strip $A := [\beta, \alpha] \times [0, 1]$, Proposition 3.1 yields the following identity for the two-dimensional periodic Green's function:

$$\int_0^1\int_0^1 G((\beta, x_2), (\beta, y_2)) + G((\alpha, x_2), (\alpha, y_2))\, dx_2\, dy_2 + 2\int_0^1\int_0^1 G((\beta, x_2), (\alpha, y_2))\, dx_2\, dy_2 = (\alpha - \beta) - (\alpha - \beta)^2. \tag{3.4}$$

An analogous identity holds in for strips in dimension $n > 2$.

We now turn our attention to investigating the criticality and stability of horizontal or vertical strips. Following Remark 2.4, one readily finds that lamellar structures are critical points if and only if they are periodic in the following sense:

Proposition 3.3. *Let N be a positive integer, $m \in (-1, 1)$, and*

$$0 = a_0 < a_1 < a_2 < \cdots < a_{2N-1} < a_{2N} = 1,$$

such that

$$\sum_{i=0}^{N-1} (a_{2i+1} - a_{2i}) = \frac{m+1}{2}, \quad \sum_{i=0}^{N-1} (a_{2i+2} - a_{2i+1}) = \frac{1-m}{2}.$$

Suppose $u(x) = u(x_1, \dots, x_n) = f(x_1)$ where

$$f(x_1) := \begin{cases} 1 & \text{if } x_1 \in [a_{2i}, a_{2i+1}), \quad \text{for } i = 0, 1, \dots, N-1, \\ -1 & \text{if } x_1 \in [a_{2i+1}, a_{2i+2}), \quad \text{for } i = 0, 1, \dots, N-1. \end{cases}$$

Then for any $\gamma > 0$, u is a critical point of $\mathscr{E}_\gamma$ if and only if for $i = 0, 1, \dots, N-1$,

$$(a_{2i+1} - a_{2i}) = \frac{m+1}{2N}, \quad (a_{2i+2} - a_{2i+1}) = \frac{1-m}{2N}.$$

Proof. Let $S := \{x \in \mathbb{T}^n \mid u(x) = 1\}$. Since $H(x) = 0$ for all $x \in \partial S$, Theorem 2.3 implies that if u is a critical point, then ∂S must be a level set of the associated v solving (2.2). Clearly, $v(x) = v(x_1)$ and without loss of generality, we may assume $v(0) = 0$. Hence we require $v(a_i) = 0$ for all $i = 0, \dots, 2N$. Since

$$-v''(x_1) = f(x_1) - m \quad \text{on } \mathbb{T}^1,$$

v consists of piecewise parabolas defined on the subinterval (a_i, a_{i+1}) which alternate pointing upwards (with convexity $1 - m$) and pointing downwards (with concavity $-1 - m$). The conditions that $v(a_i) = 0$ for each i, and the fact that v must be C^1 at each a_i directly imply that the parabolas must repeat themselves every second step. This is illustrated in Figure 1. □

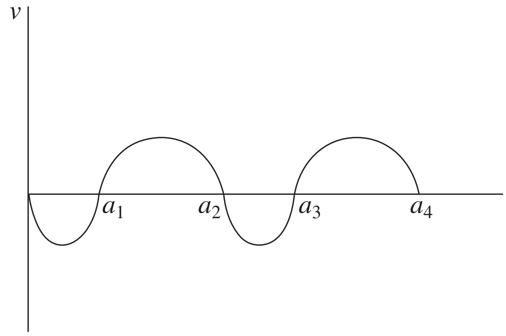

Figure 1. A function v corresponding to a lamellar critical point.

Remark 3.4. We point out that the conditions for criticality of strips described in Proposition 3.3 are independent of γ. Thus, a lamellar critical point for one value of γ is automatically a critical point for all values of γ. We presume that this is the only example of a subset of $\mathbb{T}^n$ that is critical for two different γ values. Observe from the criticality condition (2.19) that any such subset would necessarily have to possess a boundary of constant mean curvature with that boundary being a level set of the associated v solving Poisson's equation.

We now examine the stability of lamellar critical points of $\mathscr{E}_\gamma$. In particular, we study how the value of γ affects stability. One certainly expects that as γ increases, stability will require a smaller length scale for periodic structures. We will focus on the case of a single strip S. We will argue that S is stable for small γ and unstable for γ sufficiently large. For simplicity of presentation only, we will work in two space dimensions. Both the stability and the instability result below readily generalize to strips in higher dimensions with only minor changes. For $0 < \beta < \alpha < 1$, let $S \subset \mathbb{T}^2$ denote the set $[\beta, \alpha] \times [0, 1]$. Then

$$\partial S = (\{\beta\} \times [0,1]) \cup (\{\alpha\} \times [0,1]).$$

Also note that the function u associated with S via (2.7) is given by

$$u(x_1, x_2) = \begin{cases} 1 & \text{for } x_1 \in (\beta, \alpha), \\ -1 & \text{for } x_1 \notin (\beta, \alpha). \end{cases}$$

Proposition 3.5. *Let $S \subset \mathbb{T}^2$ be a horizontal or vertical strip. Then there exists a value γ_0 such that S is a stable critical point of $\mathscr{E}_\gamma$ for all positive $\gamma < \gamma_0$.*

Proposition 3.6. *For γ sufficiently large, S is unstable.*

We first prove Proposition 3.5.

Proof. We first note that since $u = u(x_1)$, one has $v = v(x_1)$ satisfying $-v'' = u - m \in L^p$ for all $p > 2$. It then follows from standard regularity theory that $v \in W^{2,p}$ for all $p > 2$, and in particular that $v \in C^1$. If we write $\partial S = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 = \{\beta\} \times [0, 1]$ and $\Gamma_2 = \{\alpha\} \times [0, 1]$, then the first component of the outer unit normal to S, which we denote here by ζ_0, is given by

$$\zeta_0(x) = \begin{cases} -1 & \text{for } x \in \Gamma_1, \\ +1 & \text{for } x \in \Gamma_2. \end{cases}$$

We can then invoke Proposition 3.1 and (3.3) to see that

$$0 = J(\zeta_0) = 8\gamma \int_{\partial S} \int_{\partial S} G(x, y)\zeta_0(x)\zeta_0(y)\, d\mathscr{H}^1(x)\, d\mathscr{H}^1(y) + 4\gamma \int_{\partial S} \nabla v \cdot \nu\, d\mathscr{H}^1(x), \tag{3.5}$$

since of course, $B_{\partial A} \equiv 0$ for the strip. Now let $\zeta : \partial S \to \mathbb{R}$ be any smooth function satisfying

$$\int_{\partial S} \zeta\, d\mathscr{H}^1(x) = 0. \tag{3.6}$$

We will write ζ as

$$\zeta = f - c\zeta_0$$

where $c := \int_{\Gamma_1} \zeta(x)\, d\mathscr{H}^1(x)$ and f is given by

$$f(x) = \begin{cases} \zeta - c & \text{for } x \in \Gamma_1, \\ \zeta + c & \text{for } x \in \Gamma_2. \end{cases} \tag{3.7}$$

Denoting $f_1 := f \llcorner \Gamma_1$ and $f_2 := f \llcorner \Gamma_2$ we observe that $\int_0^1 f_i(x_2)\, dx_2 = 0$ for $i = 1, 2$. We will argue that $J(\zeta) > 0$ for γ sufficiently small, provided $f \not\equiv 0$, that is, provided ζ does not correspond to a translation. To this end, we compute

$$\begin{aligned} J(\zeta) = {} & \int_0^1 (f_1')^2\, dx_2 + \int_0^1 (f_2')^2\, dx_2 \\ & + 8\gamma \int_{\partial S} \int_{\partial S} G(x, y) f(x) f(y)\, d\mathscr{H}^1(x)\, d\mathscr{H}^1(y) + 4\gamma \int_{\partial S} \nabla v \cdot \nu f^2\, d\mathscr{H}^1(x) \\ & + c^2 \left(8\gamma \int_{\partial S} \int_{\partial S} G(x, y)\zeta_0(x)\zeta_0(y)\, dx\, dy + 4\gamma \int_{\partial S} \nabla v \cdot \nu\, d\mathscr{H}^1(x) \right) \\ & + 8\gamma c \left(\int_{\partial S} \int_{\partial S} G(x, y) f(x)\zeta_0(y)\, d\mathscr{H}^1(x)\, d\mathscr{H}^1(y) \right. \\ & \qquad \left. + \int_{\partial S} \int_{\partial S} G(x, y)\zeta_0(x) f(y)\, d\mathscr{H}^1(x)\, d\mathscr{H}^1(y) \right) \\ & + 8\gamma c \int_{\partial S} \nabla v \cdot \nu f \zeta_0\, d\mathscr{H}^1(x) \end{aligned} \tag{3.8}$$

$$\begin{aligned} \geqq {} & \pi^2 \int_{\partial S} f^2\, d\mathscr{H}^1(x) + 8\gamma \int_{\partial S} \int_{\partial S} G(x, y) f(x) f(y)\, d\mathscr{H}^1(x)\, d\mathscr{H}^1(y) \\ & + 4\gamma \int_{\partial S} \nabla v \cdot \nu f^2\, d\mathscr{H}^1(x) \end{aligned} \tag{3.9}$$

$$+ 8\gamma c\left(\int_{\partial S}\int_{\partial S} G(x,y)f(x)\zeta_0(y)\,d\mathscr{H}^1(x)\,d\mathscr{H}^1(y) + \int_{\partial S}\int_{\partial S} G(x,y)\zeta_0(x)f(y)\,d\mathscr{H}^1(x)\,d\mathscr{H}^1(y)\right) + 8\gamma c\int_{\partial S}\nabla v\cdot \nu f\zeta_0\,d\mathscr{H}^1(x),$$

through (3.5) and the Poincaré inequality.

Now

$$\int_{\partial S}\nabla v\cdot \nu f\zeta_0\,dx = v_{x_1}(\alpha)\int_0^1 f_2(x_2)\,dx_2 + v_{x_1}(\beta)\int_0^1 f_1(x_2)\,dx_2 = 0$$

since $\int_0^1 f_i(x_2)\,dx_2 = 0$, and so the last term in (3.9) vanishes. Also, note that the function

$$w(y) := \int_{\partial S} G(x,y)\zeta_0(x)\,d\mathscr{H}^1(x)$$

is a (weak) solution to the problem

$$-\Delta w(x_1,x_2) = \mu_{\zeta_0}$$

where $\mu_{\zeta_0} = \mu_{\zeta_0}(x_1)$ is the measure given by

$$\mu_{\zeta_0} = 1\mathscr{H}^1 \llcorner \{(x_1,x_2) : x_1 = \alpha\} - 1\mathscr{H}^1 \llcorner \{(x_1,x_2) : x_1 = \beta\}.$$

Hence, in particular, $w = w(x_1)$ which means that

$$\int_{\partial S}\int_{\partial S} G(x,y)\zeta_0(x)f(y)\,d\mathscr{H}^1(x)\,d\mathscr{H}^1(y) = \int_{\partial S} w(y)f(y)\,d\mathscr{H}^1(y) = w(\alpha)\int_0^1 f_2(x_2)\,dx_2 + w(\beta)\int_0^1 f_1(x_2)\,dx_2 = 0.$$

Therefore, the second to last line of (3.9) vanishes as well. Since

$$\int_{\partial S}\int_{\partial S} G(x,y)f(x)f(y)\,d\mathscr{H}^1(x)\,d\mathscr{H}^1(y) \geqq 0,$$

we arrive at the inequality

$$J(\zeta) \geqq (\pi^2 - 4\gamma\|v'\|_{L^\infty})\int_{\partial S} f^2(x)\,d\mathscr{H}^1(x).$$

Thus, choosing $\gamma_0 = \dfrac{\pi^2}{4\|v'\|_{L^\infty}}$, we have the desired result. □

We now prove Proposition 3.6.

Proof. Note that by the divergence theorem,

$$\int_{\{\beta\}\times[0,1]\cup\{\alpha\}\times[0,1]} \nabla v(y_1,y_2)\cdot \nu\, dy_2 = \int_S \triangle v(y)\, dy \tag{3.10}$$

$$= \int_S m - u(y)\, dy$$

$$= (m-1)|S| < 0.$$

Hence, setting

$$g(y_2) := \nabla v(\alpha, y_2)\cdot \nu,$$

we may assume without loss of generality that

$$\int_0^1 g(y_2)\, dy_2 =: c_1 < 0. \tag{3.11}$$

Let $k > 1$ be a positive integer to be chosen shortly, and consider the function ζ_k set to be identically zero on $\{\beta\}\times[0,1]$ and set to equal $\sin(2\pi k y_2)$ on $\{\alpha\}\times[0,1]$. There are three non-zero terms in $J(\zeta_k)$ of (2.20) since $B_{\partial S}\equiv 0$. The first involving $\nabla_{\partial A}\zeta_k$ is readily calculated to equal $2\pi^2k^2$. The next non-zero term can be written as γ times

$$8\int_0^1\left[\int_0^1 G\big((\alpha,x_2),(\alpha,y_2)\big)\sin(2\pi k x_2)\, dx_2\right]\sin(2\pi k y_2)\, dy_2. \tag{3.12}$$

We claim that (3.12) can be made arbitrarily small by choosing k sufficiently large. To this end, denote the function in the square parenthesis of (3.12) by $f_k(y_2)$. Note that (cf. (2.4))

$$\int_0^1 \big|G\big((\alpha,x_2),(\alpha,y_2)\big)\big|\, dx_2 < \infty \quad \text{for every } y_2\in[0,1), \tag{3.13}$$

and further

$$\int_0^1\int_0^1 \big|G\big((\alpha,x_2),(\alpha,y_2)\big)\big|\, dx_2\, dy_2 < \infty. \tag{3.14}$$

By (3.13) and the Riemann-Lebesgue Lemma, for each y_2 we have

$$f_k(y_2)\to 0 \quad \text{as } k\to\infty.$$

Hence by (3.14) and the Lebesgue Dominated Convergence Theorem,

$$\int_0^1 f_k(y_2)\sin(2\pi k y_2)\, dy_2 \to 0 \quad \text{as } k\to\infty. \tag{3.15}$$

The third and last remaining non-zero term in (2.20) takes the form

$$4\gamma \int_0^1 g(y_2) \sin^2(2\pi k y_2)\, dy_2 = 4\gamma \int_0^1 g(y_2) \frac{1 - \cos(4\pi k y_2)}{2}\, dy_2,$$

and hence by (3.11) and the Riemann-Lebesgue Lemma, we have

$$\lim_{k\to\infty} \int_0^1 g(y_2) \sin^2(2\pi k y_2)\, dy_2 = \frac{c_1}{2} < 0. \tag{3.16}$$

Therefore, we can choose k sufficiently large such that for any $\gamma > 0$ the sum of the last two terms of (2.20) is, say, less than $\gamma c_1/4$. Fixing such a k, we take γ_0 to be the smallest value of γ such that $\gamma|c_1|/4 > 2\pi^2 k^2$. Hence $J(\zeta_k) < 0$ and the stability criterion (2.20) is violated. □

Remark 3.7. Proposition 3.6 can easily be extended to the case of N strips. In this case, consider S and the associated u as defined in Proposition 3.3. Then following (3.10), one finds that for some $y_1 = a_j$,

$$\int_{\{y_1 = a_j\}} \nabla v(a_j, y_2) \cdot \nu < \frac{(m-1)|S|}{N},$$

and hence the c_1 in (3.11) scales with $1/N$. This argument then implies that $\gamma_0 \sim N$, supporting our intuition that as the number of strips increase, the cut off for instability γ_0 also increases.

Remark 3.8 (spots and annuli). As we mentioned in Remark 2.4, there are no radial critical points on the torus. However, if one studies the Neumann setting of (NLIP) in the case where Ω is a ball, then one can look for critical points u of $\mathscr{E}_\gamma$ that are radial. In [24], [26], these structures are considered in two dimensions for the Neumann boundary (NLIP) and its diffuse interface version (cf. Appendix). In particular, critical points of both problems are produced corresponding to $\{x : u(x) = 1\}$ being a ball (so-called "spot solutions") as well as concentric annuli ("k-ring solutions"). For the latter, 2-annuli (2-ring) critical points were shown to exist for certain values of γ only. They further carried out an involved spectral calculation, showing among other things that a threshold exists (in terms of γ) for the (spectral) stability of spot and 2-ring solutions.

Here we briefly comment that the second variation inequality yields some analogous results.

(a) First let us consider single spot solutions. The instability results for large γ easily follow from the same reasoning as in Proposition 3.6. To this end, in what follows $\{x : u(x) = 1\}$ will consist of a fixed ball. For example, take the case where Ω is the unit ball in $\mathbb{R}^2$ and consider the function

$$u(r, \theta) = \begin{cases} 1 & \text{for } r < r_1, \\ -1 & \text{for } r_1 < r < 1, \end{cases}$$

for any positive value $r_1 < 1$. We observe through (2.75) that such a function will be a critical point of $\mathscr{E}_\gamma$ regardless of the value of γ since necessarily, $v = v(r)$ for such a function u, forcing the set $\{r = r_1\}$ to be a level set. Then since the second term in the first integral of (2.77) is negative while the second integral is absent since $\{x : u(x) = 1\} \cap \partial\Omega = \emptyset$, the argument presented above using the choice $\zeta = \sin(k\theta)$ with k large again applies to establish instability for big enough values of γ. The same argument can be applied to single spot solutions (i.e. balls) in any dimension.

(b) For 2-ring annuli critical points corresponding to u of the form

$$u(r,\theta) = \begin{cases} 1 & \text{for } r < r_1, \\ -1 & \text{for } r_1 < r < r_2, \\ 1 & \text{for } r_2 < r < 1, \end{cases}$$

for some $r_1 < r_2$, the arguments of Proposition 3.3 are more involved since the curvature now varies from boundary to boundary, and one must solve the ode corresponding to the radially symmetric Laplacian. Note that a particular 2-ring solution, for example, will only be a critical point for at most one value of γ. The existence of such critical points was established in [24] where it was shown that for γ sufficiently large, 2-ring critical points exist. In [26], they considered the stability of such 2-ring structures. We note that from the second variation, one obtains the following analogous instability result for large γ:

There exists γ^* such that if $\gamma > \gamma^*$ and u_γ is a 2-ring critical point for this value of γ, then u_γ is unstable.

The proof follows the same line of reasoning as Proposition 3.6. Fixing an $m \in (-1,1)$ and considering *any* 2-ring structure with relative volume fraction m, we can apply the same argument as in (3.10) to find one circular boundary (with bounded length) upon which (3.11) holds. Then again choosing ζ_k with support on this one boundary, we find a cutoff γ^* beyond which the second variation becomes negative. The key here is that this γ^* is independent of the precise 2-ring structure—it depends only on the volume fraction m.

4. Appendix. Diffuse interface version of (NLIP)

Our problem (NLIP) can be regarded as the sharp interface version of the minimization problem involving the following functional: for $\epsilon > 0$, consider

$$(4.1)\qquad \mathscr{E}_{\epsilon,\gamma} := \begin{cases} \int_\Omega \frac{\epsilon}{2}|\nabla u|^2 + \frac{1}{\epsilon}\frac{(1-u^2)^2}{4} + \gamma|\nabla v|^2\,dx & \text{if } u \in H^1(\Omega) \text{ and } \frac{1}{\Omega}\int_\Omega u = m, \\ +\infty & \text{otherwise.} \end{cases}$$

Here Ω is again either the torus $\mathbb{T}^n$ or a general domain with smooth boundary, and v is related to u and m via

$$-\triangle v = u - m \quad \text{on } \Omega, \qquad \int_\Omega v(x)\,dx = 0,$$

with the differential equation solved on $\mathbb{T}^n$ in the former case, and with homogeneous Neumann boundary conditions in the latter. This functional is essentially the energy first derived by Ohta and Kawasaki [22] to model microphase separation of diblock copolymers. As written in its non-dimensional, appropriately re-scaled form, it may be viewed as a nonlocal Cahn-Hilliard type functional (cf. [7], [8], [9], [21], [23], [25]). A full derivation can be found in [9].

Note that the third term in (4.1) represents a compact perturbation with respect to the basic L^2 (or L^1) topology. Hence it easily follows (cf. [23]) from the definition of Γ-convergence that the Γ-limit problem (as $\epsilon \to 0$ and in the L^1-topology) is equivalent to (NLIP).

References

[1] *Aksimentiev*, *A*., *Fialkowski*, *M*., and *Holyst*, *R*., Morphology of Surfaces in Mesoscopic Polymers, Surfactants, Electrons, or Reaction-Diffusion Systems: Methods, Simulations, and Measurements, Adv. Chem. Phys. **121** (2002), 141–239.

[2] *Alberti*, *G*., *Choksi*, *R*., and *Otto*, *F*., Uniform Energy Distribution for Minimizers of a Nonlocal Isoperimetric Problem, preprint 2007.

[3] *Alberti*, *G*. and *Müller*, *S*., A New Approach to Variational Problems with Multiple Scales, Comm. Pure Appl. Math. **54** (2001), 761–825.

[4] *Andersson*, *S*., *Hyde*, *S. T*., *Larsson*, *K*., and *Lidin*, *S*., Minimal Surfaces and Structures: From Inorganic and Metal Crystals to Cell Membranes and Biopolymers, Chem. Rev. **88** (1988), 221–242.

[5] *Barbosa*, *J. L*. and *do Carmo*, *M*., Stability of Hypersurfaces with Constant Mean Curvature, Math. Z. **185** (1984), 339–353.

[6] *Bates*, *F. S*. and *Fredrickson*, *G. H*., Block Copolymers-Designer Soft Materials, Physics Today **52-2** (1999), 32–38.

[7] *Cahn*, *J. W*. and *Hilliard*, *J. E*., Free Energy of a Nonuniform System I, Interfacial Free Energy, J. Chem. Phys. **28**(2) (1958), 258–267.

[8] *Choksi*, *R*., Scaling Laws in Microphase Separation of Diblock Copolymers, J. Nonlin. Sci. **11** (2001), 223–236.

[9] *Choksi*, *R*. and *Ren*, *X*., On a Derivation of a Density Functional Theory for Microphase Separation of Diblock Copolymers, J. Statist. Phys. **113** (2003), 151–176.

[10] *Choksi*, *R*. and *Sternberg*, *P*., Periodic Phase Separation: the Periodic Cahn-Hilliard and Isoperimetric Problems, Interf. and Free Bound. **8** (2006), 371–392.

[11] *Desimone*, *A*., *Kohn*, *R. V*., *Müller*, *S*., and *Otto*, *F*., Recent Analytical Developments in Micromagnetics, in: The Science of Hysteresis, G. Bertotti and I. Mayergoyz, eds., Elsevier, to appear.

[12] *De Wit*, *A*., *Borckmans*, *P*., and *Dewel*, *G*., Twist Grain Boundaries in Three-dimensional Lamellar Turing Systems, Proc. Natl. Acd. Sci. USA **94** (1997), 12765–12768.

[13] *Giusti*, *E*., Minimal Surfaces and Functions of Bounded Variation, Birkhäuser, 1984.

[14] *Glimm*, *T*. and *Hentsche*, *H*., On Isoconcentration Surfaces of Three-Dimensional Turing Patterns, preprint 2006.

[15] *Goldstein*, *R. E*., *Muraki*, *D. J*., and *Petrich*, *D. M*., Interface Proliferation and the Growths of Labyrinths in a Reaction-Diffusion System, Phys. Rev. E **53-4** (1996), 3933–3957.

[16] *Gonzalez*, *E*., *Massari*, *U*., and *Tamanini*, *I*., On the Regularity of Boundaries of Sets Minimizing Perimeter with a Volume Constraint, Indiana Univ. Math. J. **32-1** (1983), 25–37.

[17] *Kohn*, *R. V*., Energy-driven Pattern Formation, Proc. ICM2006, to appear.

[18] *Landkof*, *N. S*., Foundations of Modern Potential Theory, Springer-Verlag, 1972.

[19] *Müller*, *S*., Singular Perturbations as a Selection Criterion for Periodic Minimizing Sequences, Calc. Var. **1** (1993), 169–204.

[20] *Muratov*, *C. B*., Theory of Domain Patterns in Systems with Long-Range Interactions of Coulomb Type, Phys. Rev. E. **66** (2002), 066108.

[21] *Nishiura*, *Y*. and *Ohnishi*, *I*., Some Mathematical Aspects of the Micro-phase Separation in Diblock Copolymers, Physica D **84** (1995), 31–39.

[22] *Ohta, T.* and *Kawasaki, K.*, Equilibrium Morphology of Block Copolymer Melts, Macromolecules **19** (1986), 2621–2632.
[23] *Ren, X.* and *Wei, J.*, On the Multiplicity of Two Nonlocal Variational Problems, SIAM J. Math. Anal. **31-4** (2000), 909–924.
[24] *Ren, X.* and *Wei, J.*, Concentrically Layered Energy Equilibria of the Diblock Copolymer Problem, European J. Appl. Math. **13-5** (2002), 479–496.
[25] *Ren, X.* and *Wei, J.*, On Energy Minimizers of the Diblock Copolymer Problem, Interf. Free Bound. **5** (2003), 193–238.
[26] *Ren, X.* and *Wei, J.*, Stability of Spot and Ring Solutions of the Diblock Copolymer Equation, J. Math. Phys. **45-11** (2004), 4106–4133.
[27] *Ren, X.* and *Wei, J.*, Wiggled Lamellar Solutions and their Stability in the Diblock Copolymer Problem, SIAM J. Math. Anal. **37-2** (2005), 455–489.
[28] *Renardy, M.* and *Rogers, R. C.*, An Introduction to Partial Differential Equations, Springer-Verlag, 1993.
[29] *Ros, A.*, The Isoperimetric Problem, Proceedings from the Clay Mathematics Institute Summer School, MSRI, Berkeley, Ca., www.ugr.es/~aros.
[30] *Seul, M.* and *Andelman, D.*, Domain Shapes and Patterns: The Phenomenology of Modulated Phases, Science **267** (1995), 476.
[31] *Simon, L.*, Lectures on Geometric Measure Theory, Proc. Centre for Math. Anal., Australian Nat. Univ. **3** (1983).
[32] *Sternberg, P.* and *Zumbrun, K.*, A Poincaré inequality with applications to volume-constrained area-minimizing surfaces, J. reine angew. Math. **503** (1998), 63–85.
[33] *Teramoto, T.* and *Nishiura, Y.*, Double Gyroid Morphology in a Gradient System with Nonlocal Effects, J. Phys. Soc. Japan **71-7** (2002), 1611–1614.
[34] *Thomas, E.*, *Anderson, D. M.*, *Henkee, C. S.*, and *Hoffman, D.*, Periodic Area-Minimizing Surfaces in Block Copolymers, Nature **334** (1988), 598–601.
[35] *Yip, N. K.*, Structure of Stable Solutions of a One-dimensional Variational Problem, ESAIM: Control, Optimisation and Calculus of Variations, in press (2006).

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Eingegangen 27. Februar 2006, in revidierter Fassung 11. Juli 2006