Ground State Energy Scaling Laws During the Onset and Destruction of the Intermediate State in a Type I Superconductor

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Abstract

The intermediate state of a type I superconductor is a classical example of energy-driven pattern formation, first studied by Landau in 1937. Three of us recently derived five different rigorous upper bounds for the ground-state energy, corresponding to different microstructural patterns, but only one of them was complemented by a lower bound with the same scaling [Choksi, Kohn, and Otto, *J. Nonlinear Sci.* **14** (2004), 119–171]. This paper completes the picture by providing matching lower bounds for the remaining four regimes, thereby proving that exactly those five different regimes are traversed with an increasing magnetic field. © 2007 Wiley Periodicals, Inc.

1 Introduction

The intermediate state of a type I superconductor is characterized by penetration of the magnetic field in selected parts of the material. A microscopic mixture of normal and superconducting domains is formed [7], as was first predicted by Landau [10, 11] (see [3] for a discussion of the literature). The mathematical study of the problem via energy minimization was begun by three of us in [3], building upon mathematical studies of related pattern formation problems in materials science (see, e.g., [1, 2, 8, 9]). We examined the scaling law of the minimum energy and the qualitative properties of domain patterns achieving this law, which are expected to represent not only the ground state but also most low-energy metastable states.

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As explained in [3], the minimum energy has the form

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1,$$

where $\mathcal{E}_0$ is the value obtained by ignoring the surface energy between the normal and superconducting regions and $\mathcal{E}_1$ is the correction due to nonzero surface energy. The leading-order term $\mathcal{E}_0$, which was completely understood by Landau, corresponds to a "thermodynamic" theory of the intermediate state and determines, e.g., the volume fraction of the normal domains. Mathematically, it is associated with the *relaxation* of the underlying nonconvex variational problem.

The term of interest here, which determines the geometry of the microstructure, is the correction $\mathcal{E}_1$, and in particular its dependence on the surface tension ε of the normal-superconductor interface and on the magnitude b_a of the applied magnetic field. Different regimes exhibit different scaling laws. The aim of this paper is to provide matching lower bounds for all the regimes discussed in [3].

Our analysis is restricted to the simplest realistic geometry: a plate of thickness L, i.e., $(0, L) \times \mathbb{R}^2$, under a uniform transverse applied field. Abusing notation slightly, we denote the normalized applied field (applied field / critical field) by $b_a = (b_a, 0, 0)$ with $0 < b_a < 1$, and in what follows, we will use b_a to denote both the scalar and the vector $(b_a, 0, 0)$—the choice will be clear from its context. Note that $b_a = 1$ corresponds to the saturation magnetic field, above which the material ceases to be superconducting. We denote by $B = (B_1, B_2, B_3)$ the magnetic field, and by χ the characteristic function of the superconducting phase. To avoid edge effects and facilitate spatial averaging, we assume both B and χ are periodic in y and z, with period $Q = (0, 1)^2$, so that the only remaining geometric parameter is the slab thickness L. The choice of period is unimportant, provided it is large compared to the length scale of the microstructure. We shall be interested in the phase diagram depending on the two parameters $b_a \in (0, 1)$ and $\varepsilon \ll 1$, which represent the applied field and the surface tension, respectively.

The variational problem determining $\mathcal{E}_1$ is

$$\mathcal{E}_1 = \min_{\substack{\operatorname{div} B = 0 \\ B\chi = 0 \text{ in } \Omega}} E(B, \chi)$$

where

$$\begin{aligned} E(B, \chi) := & \int_\Omega \left[B_2^2 + B_3^2 + (1 - \chi)(B_1 - 1)^2\right] dx\, dy\, dz \\ & + \varepsilon \int_\Omega |\nabla \chi| + \int_{\Omega^c} |B - b_a|^2\, dx\, dy\, dz. \end{aligned} \tag{1.1}$$

Here $\Omega = (0, L) \times Q$, where throughout this paper $Q = (0, 1)^2$ denotes the unit square with periodic boundary conditions, identified with the two-dimensional torus $\mathbb{T}^2$; $(B - b_a) \in L^2(\mathbb{R} \times Q, \mathbb{R}^3)$; and $\chi \in \mathrm{BV}((0, L) \times Q, \{0, 1\})$. Further, the fields satisfy $B\chi = 0$ a.e., representing the Meissner effect, and $\operatorname{div} B = 0$ in the

	Reduced Applied Field b_a	Optimal Energy Scaling Law/ Example of an Optimal Structure
(c)	$b_a \ll \left(\frac{\varepsilon}{L}\right)^{2/7}$	$\mathcal{E}_1 \sim b_a \varepsilon^{4/7} L^{3/7}$ clusters of branched flux tubes
(b)	$\left(\frac{\varepsilon}{L}\right)^{2/7} \ll b_a \ll 1$	$\mathcal{E}_1 \sim b_a^{2/3} \varepsilon^{2/3} L^{1/3}$ branched flux tubes
(a)	Intermediate b_a	$\mathcal{E}_1 \sim \varepsilon^{2/3} L^{1/3}$ branched lamellar
(d)	$\left(\frac{\varepsilon}{L}\right)^{2/3} \ll \frac{(1-b_a)}{\lvert\log(1-b_a)\rvert^{1/3}} \ll 1$	$\mathcal{E}_1 \sim (1-b_a)\lvert\log(1-b_a)\rvert^{1/3} \varepsilon^{2/3} L^{1/3}$ superconducting tunnels
(e)	$\frac{1-b_a}{\lvert\log(1-b_a)\rvert^{1/3}} \ll \left(\frac{\varepsilon}{L}\right)^{2/3}$	$\mathcal{E}_1 \sim (1-b_a)^2 L$ all normal

TABLE 1.1. The five regimes traversed by a type I superconducting plate with increasing field b_a. By an optimal structure we mean a structure whose energy achieves the optimal scaling law. Pictorial illustrations of the patterns were given in [3].

sense of distributions, from Maxwell's equation. A full description of the physical significance of the various terms in the energy can be found in [3].

Minimization of (1.1) reveals a number of different regimes. In [3], these regimes were explored, primarily by presenting constructions with a certain energy scaling law that were believed to be optimal in their appropriate regime. We also obtained a matching ansatz-independent lower bound for the regime of intermediate values of b_a. It is the purpose of this article to prove matching ansatz-independent lower bounds for the other regimes associated with small and large b_a, i.e., the onset and destruction of the intermediate state.

We now list the five regimes, summarizing the results of [3] and of the present paper; see also Table 1.1. By $f \lesssim g$ (respectively, $\gtrsim$, $\sim$) we mean that $f \le Cg$ (respectively, $Cf \ge g$, $f/C \le g \le Cf$) for some universal constant C.

(0)(a) For intermediate values of b_a, bounded away from 0 and 1, $\mathcal{E}_1 \sim \varepsilon^{2/3} L^{1/3}$. This scaling corresponds to a branched microstructure, which can take the form of either layers or tubes.

(0)(b) For relatively small values of b_a, in the range $(\varepsilon/L)^{2/7} \lesssim b_a \ll 1$, the prefactor is proportional to $b_a^{2/3}$. Thus $\mathcal{E}_1 \sim b_a^{2/3}\varepsilon^{2/3}L^{1/3}$; this scaling is achieved by a uniform family of branched flux tubes.

(0)(c) For the smallest values of b_a, when $b_a \lesssim (\varepsilon/L)^{2/7} \ll 1$, we obtain the lower energy $\mathcal{E}_1 \sim b_a\varepsilon^{4/7}L^{3/7}$. This energy corresponds to well-separated families of branched flux tubes.

(0)(d) For relatively large values of b_a, i.e., for b_a near 1, the situation is similar to (b), and one obtains $\mathcal{E}_1 \sim (1-b_a)|\log(1-b_a)|^{1/3}\varepsilon^{2/3}L^{1/3}$. In this regime the magnetic flux fills most of the sample, leaving a uniformly distributed family of branched superconducting tunnels.

(0)(e) For the largest values of b_a, the sample is entirely normal and $\mathcal{E}_1 \sim (1-b_a)^2L$.

Notice that the distinction of regime (a) from either (b) or (d) is made for historical reasons; from our scaling point of view, there is really only one regime. In [3] we proved all the upper bounds implicit in Table 1.1. Also, using methods developed in [2], we proved the lower bound

$$\mathcal{E}_1 \gtrsim b_a(1-b_a)\varepsilon^{2/3}L^{1/3},$$

matching the upper bound in the intermediate regime (a) and missing the upper bound by a logarithmic factor in regime (d). In this article, we prove all remaining lower bounds. Specifically, Theorems 4.2 and 4.3 provide matching lower bounds for (c) and (b), respectively, and Theorem 5.1 provides matching lower bounds for (d) and (e). Thus we have now identified the entire phase diagram for a type I superconducting plate and shown that with increasing applied field b_a at fixed $\varepsilon/L \ll 1$, the material goes through five different regimes, as listed in Table 1.1.

We summarize the general line of our argument. Basically, it examines the balance between the three different energy terms (surface, interior, and exterior magnetic field) by focusing on various sections of the form $\{x_0\} \times Q$ (see Figure 1.1). With a slight abuse of notation we use $B_1(x_0,\cdot)$ to denote the value of B_1 on sections; for a precise interpretation of this expression (in the sense of traces), see Lemma 2.1 below.

For the cases of a small applied field (Theorems 4.2 and 4.3), our approach can be summarized as follows:

(1) We use the smallness of the surface energy $\varepsilon|\nabla\chi|$ and the constraint $B\chi = 0$ to show that for certain x_0 in the interior of the sample, the magnetic field has to concentrate into regions with small boundary. Hence B cannot be uniform on that section; i.e., it has significant dependence on y and z. More specifically, Lemma 3.1 shows that if $|\nabla\chi|$ is small, then the support of $1-\chi$ can be well approximated with a regular set on which a significant part of B_1 must concentrate.

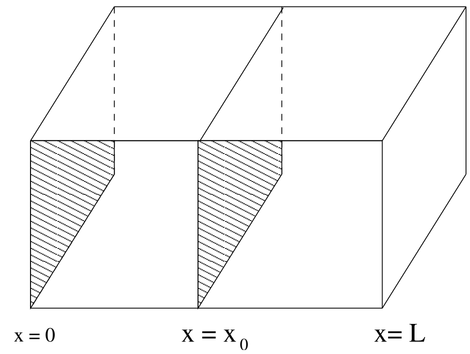

FIGURE 1.1. We consider a slab of material of thickness L. Most arguments are based on choosing a suitable $x_0 \in (0, L)$ and studying the behavior of B and χ on the section $\{x_0\} \times Q$ and on one surface $\{0\} \times Q$.

(2) Focusing on such a cross section $x = x_0$, we define a suitable test function ψ related to these concentration areas and then derive a lower bound on

$$\left| \int_Q [B_1(x_0, \cdot\,) - b_a]\psi \right|.$$

In practice, the first term ($\int_Q B_1\psi$) will dominate.

(3) A *telescoping sum* is now used to relate this lower bound to the energy. That is, one notes that

$$\int_Q [B_1(x_0, \cdot\,) - b_a]\psi = \int_Q [B_1(x_0, \cdot\,) - B_1(0, \cdot\,)]\psi + \int_Q [B_1(0, \cdot\,) - b_a]\psi.$$

The first term on the right is related to the energy by Lemma 2.2, which gives an estimate on the Monge-Kantorovich (i.e., $W^{-1,1}$) norm of the differences $B_1(x_0, \cdot\,) - B_1(x_1, \cdot\,)$ in terms of the magnetostatic energy inside the sample, exploiting again the condition div $B = 0$. The second term on the right can be related to the exterior magnetic energy via an estimate on the $H^{-1/2}$ norm of $[B_1(0, \cdot\,) - b_a]$ (Lemma 2.1).

For the case of intermediate applied field (Theorem 4.1), the approach is actually simpler. In step (2) the test function ψ is simply a mollified version of $\chi(x_0, \cdot\,)$ and the term $\int_Q b_a\psi$ provides the lower bound. We then connect back to the energy via the telescoping sum involving both $B_1(x_0, \cdot\,)$ and $B_1(0, \cdot\,)$.

The case of a large applied field (Theorem 5.1) is more involved. For step (2), the term

$$\int_Q [\chi(x_0, \cdot\,) - (1 - b_a)]\psi(\,\cdot\,)$$

now provides the lower bound, with the first term dominating. The corresponding telescoping sum involves additional terms involving $\chi(x_0,\cdot)$, $B_1(x_0,\cdot)$, and $B_1(0,\cdot)$, which are all related to parts of the energy.

In all cases, there is a length scale attached to each test function: In fact, for all but the intermediate b_a regime, there are two length scales (cf. r and ℓ in Lemma 3.1). The (larger) length scale is chosen in a certain optimal way that coincides exactly with the length scale within the center of the sample for the respective matching upper-bound construction that was presented in [3, sec. 4].

The energy of a type I superconductor is highly nonconvex. We do not suggest for a moment that the system is typically in its ground state. However, we argued in [3] that the patterns seen under increasing applied fields may be determined by the topology of energy minima for the smallest applied fields. Similarly, we argued that the patterns seen under decreasing applied fields may be determined by the topology of energy minima for near-critical applied fields. Recent experimental work on hysteresis in type I superconductors [12] lends support to this view.

The paper is organized as follows. Since the two lemmas involving negative norms are easy to prove, we present them first in Section 2. A deeper "concentration lemma" (Lemma 3.1) is presented separately in Section 3, together with Lemma 3.2, which shows that the amount of normal and superconducting phase is fixed by the external field b_a up to a constant factor. Section 3 closes with a digression about interpolation inequalities, which captures the analytical heart of our analysis in a transparent and generalizable form. Then we get down to business: Section 4 establishes the desired lower bounds for intermediate and small b_a, and Section 5 addresses the bounds for large b_a.

2 Preliminaries

In the entire paper we identify $Q = (0,1)^2$ with the torus $\mathbb{T}^2$, and assume without explicit mention that functions defined on Q are periodic. For $\omega \subset Q$ we denote by $\operatorname{dist}(p,\omega)$ the distance of p from ω on Q, i.e.,

$$\operatorname{dist}(p,\omega) = \inf\{|p-q-\delta| : q \in \omega,\ \delta \in \mathbb{Z}^2\},$$

and the perimeter $\operatorname{Per}(\omega)$ is interpreted in the Q-sense, i.e.,

$$\begin{aligned}\operatorname{Per}(\omega) &:= \sup\left\{\int_\omega \operatorname{div}\varphi : \varphi \in C^1(Q;\mathbb{R}^2),\ |\varphi| \le 1\right\}\\ &= \int_Q |\nabla\chi_\omega|,\end{aligned}$$

where χ_ω is the characteristic function of ω (notice that $\varphi \in C^1(Q)$ implies periodicity of φ). By $B_r(p)$ we mean the ball of radius r centered at p, i.e., $\{p' \in Q : \operatorname{dist}(p',p) < r\}$. We use the notation $\lesssim$, $\gtrsim$, and $\sim$ for inequalities up to some universal constant, and denote explicit constants by c_0, c_1, etc. For $v \in \mathbb{R}^3$, we denote by v' the projection on the yz-plane.

We will require two *negative* norms to capture certain energetic terms. Let $C^\infty_\sharp(Q)$ be the space of Q-periodic smooth functions with mean value 0. For any $k \in \mathbb{R}$ we define the $H^k_\sharp(Q)$ norm by

$$\|f\|^2_{H^k_\sharp(Q)} := \sum_{\substack{\xi \in 2\pi\mathbb{Z}^2 \\ \xi \neq 0}} (|\xi|^k \, |\widehat{f}(\xi)|)^2, \tag{2.1}$$

where $\widehat{f} : 2\pi\mathbb{Z}^2 \to \mathbb{R}$ denotes the Fourier coefficients of $f \in C^\infty_\sharp(Q)$ (with $\widehat{f}(0) = 0$ since f has average 0). We define the Hilbert space $H^k_\sharp(Q)$ as the completion of $C^\infty_\sharp(Q)$ with respect to this norm. It is clear that (2.1) can be used whenever the Fourier coefficients are defined. Taking the direct sum of $H^k_\sharp$ and the space of constant functions, one obtains H^k. We shall use the same symbol $\|\cdot\|_{H^k_\sharp(Q)}$ to denote the corresponding seminorm on spaces of functions where the average is not constrained to be 0.

We will primarily be interested in the cases $k = -\frac{1}{2}, \frac{1}{2}, 1$, and will often use the following elementary interpolation inequalities: For $f \in H^1(Q)$, we have

$$\|f\|^2_{H^{1/2}_\sharp} \leq \|f\|_{L^2} \, \|\nabla f\|_{L^2}, \tag{2.2}$$

and for $f \in H^{-1/2}_\sharp(Q)$ and $g \in H^{1/2}(Q)$, we have $fg \in L^1(Q)$ and

$$\int_Q fg \leq \|f\|_{H^{-1/2}_\sharp} \, \|g\|_{H^{1/2}_\sharp}. \tag{2.3}$$

Both inequalities are easily proved in Fourier space on $C^\infty_\sharp(Q)$ functions and extended by density.

We also use the Monge-Kantorovich norm of f. Formally, it is finite on the space $W^{-1,1}$; rigorously, it is defined by duality with $W^{1,\infty}$ as

$$\int_Q |\nabla^{-1}(f(y,z))| := \max_{|\nabla\psi| \leq 1} \int_Q f(y,z)\psi(y,z)\,dy\,dz,$$

where ψ is a Lipschitz function on Q. The notation is formal: we are *not* taking the L^1 norm of any well-defined function $\nabla^{-1}(f(y,z))$. Note that this norm is finite only for functions f with average 0.

The following two lemmas will be used to relate certain energetic terms to these norms evaluated on cross sectional slices. The first lemma relates the $H^{-1/2}_\sharp$ norm of the boundary cross section to the exterior magnetic energy. It will be systematically applied to $\tilde{B} := B - b_a$. Throughout this article, the equation $\operatorname{div} B = 0$ is interpreted in the sense of distributions.

LEMMA 2.1 *Let $\tilde{B} \in L^2(\mathbb{R} \times Q, \mathbb{R}^3)$ be such that* $\operatorname{div} \tilde{B} = 0$. *Then the first component of $\tilde{B}$ has a trace $\tilde{B}_1(x_0, \cdot)$ on all sections $\{x_0\} \times Q$ in the space $H_\sharp^{-1/2}$. Specifically, for any $x_0 \in \mathbb{R}$ one has*

$$\int_Q \tilde{B}_1(x_0, \cdot) = 0 \tag{2.4}$$

and

$$\|\tilde{B}_1(x_0, \cdot)\|^2_{H_\sharp^{-1/2}(Q)} \leq \int_{x \leq x_0} \int_Q |\tilde{B}|^2. \tag{2.5}$$

With a slight abuse of notation, the trace of $\tilde{B}_1$ on each section $\{x_0\} \times Q$ will be simply denoted by $\tilde{B}_1(x_0, \cdot)$. The trace is *not* a well-defined function; only expressions of the form $\int \tilde{B}_1(x_0, \cdot)\psi$ with $\psi \in H^{1/2}$ make sense.

PROOF: Without loss of generality, we can assume $x_0 = 0$. As is usual in this kind of argument, we use the Fourier series in the y- and z-components but not in x. We introduce the notation

$$\tilde{B}(x, y, z) = \sum_{\xi \in 2\pi\mathbb{Z}^2} b^\xi(x) e^{i\xi \cdot (y,z)},$$

where for every $\xi \in 2\pi\mathbb{Z}^2$ and $x \in \mathbb{R}$, $b^\xi(x) \in \mathbb{C}^3$. Then

$$\int_{x \leq 0} \int_Q |\tilde{B}|^2 = \sum_{\xi \in 2\pi\mathbb{Z}^2} \int_{-\infty}^0 |b^\xi|^2(x) dx.$$

The constraint $\operatorname{div} \tilde{B} = 0$ reduces, in Fourier space, to

$$i\xi \cdot (b^\xi)' + \frac{\partial b_1^\xi}{\partial x} = 0. \tag{2.6}$$

Taking $\xi = 0$, we get that $b_1^0(x)$ does not depend on x, and since

$$\infty > \int_{\mathbb{R}} \int_Q |\tilde{B}|^2 > \int_{\mathbb{R}} |b^0|^2,$$

we have $b_1^0(x) = 0$ for all x. This implies in particular (2.4).

The components with $\xi \neq 0$ of (2.6) give $|(b^\xi)'| \geq |\partial b_1^\xi / \partial x| / |\xi|$; hence

$$\begin{aligned} \int_{x \leq 0} \int_Q |\tilde{B}|^2 &\geq \sum_{\xi \neq 0} \int_{-\infty}^0 |b_1^\xi|^2 + \frac{1}{|\xi|^2} \left| \frac{\partial b_1^\xi}{\partial x} \right|^2 dx \\ &\geq \sum_{\xi \neq 0} \int_{-\infty}^0 2 \frac{1}{|\xi|} |b_1^\xi(x)| \left| \frac{\partial b_1^\xi(x)}{\partial x} \right| dx \end{aligned}$$

$$\geq \sum_{\xi \neq 0} \int_{-\infty}^{0} \frac{1}{|\xi|} \frac{\partial(|b_1^\xi(x)|^2)}{\partial x}\,dx$$

$$\geq \sum_{\xi \neq 0} \frac{1}{|\xi|} |b_1^\xi(0)|^2$$

$$= \|\tilde{B}_1(0,\cdot)\|^2_{H_\sharp^{-1/2}(Q)}.$$

This concludes the proof. $\square$

Next we prove a simple lemma that will be used to relate variations of the magnetic field B on two interior cross sections to the magnetostatic energy inside.

LEMMA 2.2 *Let $B \in L^2((a,b)\times Q, \mathbb{R}^3)$ be such that* $\operatorname{div} B = 0$ *for some $a, b \in \mathbb{R}$, $a < b$. Then for any $x_0, x_1 \in (a,b)$, $x_0 < x_1$, one has*

$$\int\limits_Q |(\nabla')^{-1}(B_1(x_1,\cdot) - B_1(x_0,\cdot))| \leq \int\limits_{(x_0,x_1)\times Q} |B'| \tag{2.7}$$

in the sense that

$$\int\limits_Q (B_1(x_1,\cdot) - B_1(x_0,\cdot))\psi(y,z)dy\,dz \leq \|\nabla\psi\|_{L^\infty} \int\limits_{(x_0,x_1)\times Q} |B'| \tag{2.8}$$

for any $\psi \in W^{1,\infty}(Q) \subset H^{1/2}(Q)$. If additionally $B - b_a \in L^2(\mathbb{R}\times Q, \mathbb{R}^3)$, then for any $x \in \mathbb{R}$ we have

$$\int\limits_Q B_1(x,\cdot) = b_a. \tag{2.9}$$

PROOF: We claim that the condition $\operatorname{div} B = 0$ implies that, for any $\psi \in W^{1,\infty}(Q)$,

$$\int\limits_Q (B_1(x_1,\cdot) - B_1(x_0,\cdot))\psi(y,z) = \int\limits_Q \int_{x_0}^{x_1} B' \cdot \nabla'\psi\,dx\,dy\,dz. \tag{2.10}$$

By a standard mollification argument it suffices to assume B is smooth. In this case, note that

$$\begin{aligned}\int\limits_Q (B_1(x_1,\cdot) - B_1(x_0,\cdot))\psi(y,z) &= \int\limits_Q \int_{x_0}^{x_1} \frac{\partial B_1}{\partial x}(x,y,z)\psi(y,z)dx\,dy\,dz\\ &= -\int\limits_Q \int_{x_0}^{x_1} (\nabla'\cdot B')\psi\,dx\,dy\,dz\\ &= \int\limits_Q \int_{x_0}^{x_1} B'\cdot\nabla'\psi\,dx\,dy\,dz.\end{aligned}$$

Hence (2.8) follows. Finally, if $B - b_a \in L^2(\mathbb{R} \times Q, \mathbb{R}^3)$, the choice of $\psi = 1$ trivially implies (2.9) (which can also be derived from (2.4)). $\square$

3 Two Lemmas

We first focus on the trace of the BV function χ on a surface $\{x_0\} \times Q$, which for a.e. x_0 is also in BV. Lemma 3.1 shows that if $|\nabla\chi|$ is small, then the support of χ can be well approximated with a regular set on which a significant part of B_1 must concentrate. This result was already presented in [4]; moreover, a very similar lemma was proved long ago by De Giorgi [5, lemma II]. But since Lemma 3.1 lies at the heart of our analysis, we include a complete proof for the reader's convenience. (We state the lemma in space dimension 2 because this is what we use. However, a similar result holds in any space dimension with a similar proof.)

LEMMA 3.1 *Let $S \subset Q$ be a set of finite perimeter, and let $\ell > 0$ be such that*

$$\ell\, \mathrm{Per}(S) \leq \frac{1}{4}|S|. \tag{3.1}$$

Then there exists an open set $S_\ell \subset Q$ with the properties:

(i) $|S \cap S_\ell| \geq \frac{1}{2}|S|$.
(ii) *For all $r > 0$, the set $S_\ell^r := \{p \in Q : \mathrm{dist}(p, S_\ell) < r\}$ satisfies $|S_\ell^r| \leq C|S|(1 + (\frac{r}{\ell})^2)$.*

The lemma states that for a given set S of controlled perimeter there is a "regular" set S_ℓ "close by." Here "close by" is meant in the sense of (i): S_ℓ covers at least half of the volume of S. "Regular" is meant in the sense of (ii): the thickened sets S_ℓ^r have controlled volume.

PROOF: We can assume without loss of generality $0 < |S| \leq \frac{1}{2}$ (if not, it suffices to take $S_\ell = Q$). Let χ be the characteristic function of S, and χ_ℓ the convolution of χ with the normalized characteristic function of B_ℓ, i.e.,

$$\chi_\ell(p) := \frac{1}{|B_\ell|} \int\limits_{B_\ell(p)} \chi(p')dp' = \frac{|S \cap B_\ell(p)|}{|B_\ell|}.$$

Consider the set

$$S_\ell := \left\{p : \chi_\ell(p) > \frac{1}{2}\right\} = \left\{p : |S \cap B_\ell(p)| > \frac{1}{2}|B_\ell|\right\}. \tag{3.2}$$

We claim that it satisfies the claimed properties. To prove (i), observe that

$$\chi - \chi_\ell \geq 1 - \frac{1}{2} = \frac{1}{2} \quad \text{on } S \setminus S_\ell$$

so that

$$
\begin{aligned}
|S \setminus S_\ell| &\le 2 \int_Q |\chi(p) - \chi_\ell(p)| dp \\
&\le 2 \frac{1}{|B_\ell|} \int_Q \int_{\operatorname{dist}(p,p+h)<\ell} |\chi(p) - \chi(p+h)| dh\, dp \\
&\le 2 \sup_{|h|\le \ell} \int_Q |\chi(p) - \chi(p+h)| dp \\
&\le 2\ell \int_Q |\nabla \chi| \\
&= 2\ell \operatorname{Per}(S) \overset{(3.1)}{\le} \frac{1}{2}|S|. \qquad (3.3)
\end{aligned}
$$

Thus

$$
|S \cap S_\ell| = |S| - |S \setminus S_\ell| \ge \frac{1}{2}|S|,
$$

and (i) is proved.

Now let $A \subset S_\ell$ be a maximal family such that

$$
\{B_\ell(p)\}_{p\in A} \quad \text{are disjoint.} \qquad (3.4)
$$

We claim that

$$
S_\ell \subset \bigcup_{p\in A} B_{2\ell}(p). \qquad (3.5)
$$

If not, there would be $p \in S_\ell$ such that

$$
\forall p' \in A \quad B_\ell(p) \cap B_\ell(p') = \varnothing,
$$

and this would contradict the maximality of A. Furthermore, since $A \subset S_\ell$, we have

$$
\#A|B_\ell| = \sum_{p\in A} |B_\ell(p)| \overset{(3.2)}{<} 2 \sum_{p\in A} |S \cap B_\ell(p)| \overset{(3.4)}{\le} 2|S|
$$

where $\#A$ denotes the number of elements in A. We thus obtain

$$
\#A \le \frac{2|S|}{|B_\ell|}. \qquad (3.6)
$$

We are finally ready to prove (ii). Indeed, (3.5) implies

$$S_\ell^r \subset \bigcup_{p\in A} B_{2\ell+r}(p).$$

Thus

$$|S_\ell^r| \le \sum_{p\in A} |B_{2\ell+r}(p)| \overset{(3.6)}{\le} \frac{2|S|}{|B_\ell|}|B_{2\ell+r}| = 2|S|\left(\frac{2\ell+r}{\ell}\right)^2.$$

□

Our last lemma shows that the amount of normal and superconducting phase is fixed by the external field b_a up to a constant factor.

LEMMA 3.2 *If* $\operatorname{div} B = 0$, $B\chi = 0$, $b_a \in (0,1)$, *and*

$$E(B,\chi) \le \frac{1}{16}\min\{b_a, (1-b_a)^2\}L,$$

then:

(i) *The function* χ *satisfies*

$$\int_0^L \int_Q \chi \sim (1-b_a)L \tag{3.7}$$

and

$$\int_0^L \int_Q 1-\chi \sim b_a L. \tag{3.8}$$

(ii) *There exists a subset* $\mathcal{I} \subset (0,L)$ *with* $|\mathcal{I}| \ge L/2$ *such that for all* $x \in \mathcal{I}$ *one has*

$$\int_{\{x\}\times Q} \chi \sim 1-b_a \tag{3.9}$$

and

$$\int_{\{x\}\times Q} 1-\chi \sim b_a. \tag{3.10}$$

PROOF: (i) Since $B\chi = 0$ and by using (2.9) to evaluate the integral of B_1, we have for all $x \in (0,L)$

$$\begin{aligned}\int_{\{x\}\times Q} (1-\chi)(1-B_1) &= \int_{\{x\}\times Q} 1-B_1-\chi \\ &= \int_{\{x\}\times Q} 1-b_a-\chi.\end{aligned} \tag{3.11}$$

Integrating (3.11) in x and exploiting the fact the $(1-\chi)=(1-\chi)^2$, we find

$$\left|\int_{(0,L)\times Q} 1-b_a-\chi\right| = \left|\int_{(0,L)\times Q}(1-\chi)(1-B_1)\right| \overset{\text{Hölder}}{\leq} L^{1/2}\Big(\int_{(0,L)\times Q}(1-\chi)(1-B_1)^2\Big)^{1/2} \leq L^{1/2}E^{1/2}.$$

Hence we have

$$\left|(1-b_a)L-\int_{(0,L)\times Q}\chi\right| \leq L^{1/2}E^{1/2} \leq \frac{1}{4}(1-b_a)L$$

and (3.7) is valid.

Consider now (3.8). We again integrate (3.11) over x, exploiting the fact that $(1-\chi)=(1-\chi)^2$, to obtain

$$\begin{aligned}\left|\int_{(0,L)\times Q} 1-b_a-\chi\right| &= \left|\int_{(0,L)\times Q}(1-\chi)^2(1-B_1)\right| \\ &\overset{\text{Hölder}}{\leq} \Big(\int_{(0,L)\times Q}(1-\chi)\Big)^{1/2}\Big(\int_{(0,L)\times Q}(1-\chi)(1-B_1)^2\Big)^{1/2} \\ &\leq \Big(\int_{(0,L)\times Q}(1-\chi)\Big)^{1/2}E^{1/2} \\ &\leq \frac{1}{4}\int_{(0,L)\times Q}(1-\chi)+E. \qquad (3.12)\end{aligned}$$

Thus we have

$$\left|b_aL-\int_{(0,L)\times Q}1-\chi\right| \leq \frac{1}{4}\int_{(0,L)\times Q}(1-\chi)+\frac{1}{16}b_aL,$$

which implies (3.8).

(ii) Consider the set $\mathcal{J}_1$ of $x\in(0,L)$ with the property that

$$\left|\int_{\{x\}\times Q}1-b_a-\chi\right| \geq \frac{2E^{1/2}}{L^{1/2}}.$$

Integrating in x and arguing as in (3.11) and the lines just after it, we find

$$
\begin{aligned}
|\mathcal{J}_1|\frac{2E^{1/2}}{L^{1/2}} &\le \int_{\mathcal{J}_1}\left|\int_Q 1-b_a-\chi\right| \\
&\le \int_{\mathcal{J}_1\times Q}\left|1-B_1-\chi\right| \\
&\overset{\text{Hölder}}{\le} |\mathcal{J}_1|^{1/2}\left(\int_{(0,L)\times Q}(1-\chi)(1-B_1)^2\right)^{1/2} \le |\mathcal{J}_1|^{1/2}E^{1/2},
\end{aligned}
$$

which gives $|\mathcal{J}_1| \le L/4$. For all $x \in \mathcal{I}_1 := (0,L)\setminus\mathcal{J}_1$ we have

$$
\left|1-b_a-\int_{\{x\}\times Q}\chi\right| \le \frac{2E^{1/2}}{L^{1/2}} \le \frac{1}{2}(1-b_a),
$$

which implies (3.9). Further, $|\mathcal{I}_1| \ge 3L/4$.

Finally, consider the subset $\mathcal{J}_2$ of $x \in (0,L)$ with the property that

$$
\left|\int_{\{x\}\times Q} b_a-(1-\chi)\right| \ge c_0 b_a
$$

for some constant c_0 to be chosen shortly. Then combining (3.12) with (3.8), we find

$$
\begin{aligned}
|\mathcal{J}_2|c_0 b_a &\le \frac{1}{4}\int_{(0,L)\times Q}(1-\chi)+E \\
&\le c_1 b_a L + \frac{1}{16}b_a L
\end{aligned}
$$

for some constant c_1. Hence

$$
|\mathcal{J}_2| \le \frac{c_1+1/16}{c_0}L.
$$

We choose c_0 such that the right-hand side is less than $L/4$. Then (3.10) holds for all $x \in \mathcal{I}_2 := (0,L)\setminus\mathcal{J}_2$ and $|\mathcal{I}_2| \ge 3L/4$. Finally, taking $\mathcal{I} = \mathcal{I}_1\cap\mathcal{I}_2$, we have that $|\mathcal{I}| \ge L/2$ and for all $x \in \mathcal{I}$ both (3.9) and (3.10) hold, and the proof is concluded.

□

The rest of this section is a digression. Its goal is to explain the mathematical heart of our lower bounds in a transparent and generalizable way. (Impatient readers can skip to Section 4 without loss of continuity.)

Let χ be a periodic characteristic function with unit cell $Q = [0, 1]^2$ and mean $\bar{\chi} = \int_Q \chi$. The interpolation inequalities

$$\int_Q (\chi - \overline{\chi})^2 \lesssim \Big(\int_Q |\nabla \chi|\Big)^{2/3} \Big(\int |\nabla \Delta^{-1}(\chi - \overline{\chi})|^2\Big)^{1/3} \tag{3.13}$$

and

$$\int_Q |\chi - \overline{\chi}| \lesssim \Big(\int_Q |\nabla \chi|\Big)^{1/2} \Big(\int_Q |\nabla^{-1}(\chi - \overline{\chi})|\Big)^{1/2} \tag{3.14}$$

are relatively easy to prove. (An efficient proof of the former can be found, for example, in [4], and the latter can be proved using the same technique.)

Now consider the low-volume-fraction regime: suppose the area fraction of the set where $\chi = 1$ is $\theta \ll 1$. Since $\overline{\chi} = \theta$, we have $\int_Q (\chi - \overline{\chi})^2 = \theta(1-\theta) \sim \theta$ and $\int_Q |\chi - \overline{\chi}| = 2\theta(1-\theta) \sim \theta$. Therefore (3.13) and (3.14) become

$$\Big(\int_Q |\nabla \chi|\Big)^{2/3} \|\chi - \theta\|_{H_\sharp^{-1}}^{2/3} \geq C\theta \tag{3.15}$$

and

$$\Big(\int_Q |\nabla \chi|\Big)^{1/2} \Big(\int_Q |\nabla^{-1}(\chi - \theta)|\Big)^{1/2} \geq C\theta \tag{3.16}$$

with C independent of θ.

It is natural to ask whether these estimates are optimal or, more precisely, whether the dependence of the right-hand side on θ is optimal. The answer turns out to be no. Indeed, (3.15) can be improved to

$$\Big(\int_Q |\nabla \chi|\Big)^{2/3} \|\chi - \theta\|_{H_\sharp^{-1}}^{2/3} \geq C\theta|\log\theta|^{1/3} \quad \text{for } \theta \ll 1, \tag{3.17}$$

and (3.16) can be improved to

$$\Big(\int_Q |\nabla \chi|\Big)^{1/2} \Big(\int_Q |\nabla^{-1}(\chi - \theta)|\Big)^{1/2} \geq C\theta^{3/4} \quad \text{for } \theta \ll 1. \tag{3.18}$$

The proof of (3.17) can be found in [4]; the argument uses Lemma 3.1 and is somewhat similar to the proof of Theorem 5.1. The proof of (3.18) is easier; we shall give it in a moment. The argument is similar to the proof of Theorem 4.2.

The preceding comments are specific to space dimension 2. Let us briefly discuss what happens in space dimension $n \geq 3$. The elementary estimates (3.13)–(3.16) are valid in any dimension. In dimension $n \geq 3$ the right-hand side of (3.15) *cannot* be improved, as we show below. The situation is different for (3.16): the analogue of (3.18) in space dimension n is

$$\left(\int_Q |\nabla\chi|\right)^{1/2}\left(\int_Q |\nabla^{-1}(\chi-\theta)|\right)^{1/2} \geq C\theta^{1-1/(2n)}. \tag{3.19}$$

This estimate also has the optimal scaling; see below.

We now give the proof of (3.18). We shall apply Lemma 3.1 with $S = \{\chi = 1\}$ and ℓ defined by

$$\ell\,\mathrm{Per}(S) = \frac{1}{4}|S| = \frac{1}{4}\theta.$$

The value of the parameter r in Lemma 3.1 will be chosen later, in (3.20); for now we leave it unspecified but assume $r > \ell$. The lemma provides a set S_ℓ such that

- $|S \cap S_\ell| \geq \frac{1}{2}|S|$.
- $|S_\ell^r| \leq c_1(r/\ell)^2|S|$ where $S_\ell^r = \{p : \mathrm{dist}(p, S_\ell) < r\}$.

Define a test function ψ on Q by

$$\psi(p) = \max\{r - \mathrm{dist}(p, S_\ell), 0\}$$

so that $\psi \geq 0$ and

$$\psi = r \ \text{ on } S_\ell, \quad |\nabla\psi| \leq 1, \ \ \psi = 0 \ \text{ off } S_\ell^r.$$

We have

$$\int_Q (\chi-\theta)\psi \leq \int_Q |\nabla^{-1}(\chi-\theta)|$$

by definition, since $|\nabla\psi| \leq 1$. Now,

$$\int_Q \chi\psi \geq r|S_\ell \cap S| \geq \frac{1}{2}r\theta,$$

while

$$\int_Q \theta\psi \leq \theta r|S_\ell^r| \leq c_1\theta^2 r\left(\frac{r}{\ell}\right)^2.$$

We are ready to choose r: it should satisfy

$$c_1\theta\left(\frac{r}{\ell}\right)^2 = \frac{1}{4}; \tag{3.20}$$

notice that this gives $r \sim \ell\theta^{-1/2}$. It follows that

$$\int_Q (\chi-\theta)\psi \geq \frac{1}{4}r\theta \sim \ell\theta^{1/2}.$$

Recalling the definition of ℓ, we conclude that

$$\int\limits_Q |\nabla^{-1}(\chi - \theta)| \geq C\bigg(\int\limits_Q |\nabla\chi|\bigg)^{-1} \theta^{3/2},$$

which is equivalent to (3.18). The same argument, using the n-dimensional version of Lemma 3.1, proves (3.19).

We finally show the optimality of the mentioned scalings. We start with (3.15). Let $n > 2$, and for any $\theta > 0$ sufficiently small, let $B^\theta \subset Q$ be a ball such that $|B^\theta| = \theta$. We set χ to be the characteristic function of B^θ. Clearly

$$\int\limits_Q |\nabla\chi| = \mathcal{H}^{n-1}(\partial B^\theta) = c\theta^{(n-1)/n}.$$

At the same time, by duality

$$\|\chi - \theta\|_{H^{-1}_\#} = \sup\bigg\{\int\limits_Q (\chi - \theta)\phi : \phi \in H^1_\#(Q),\ \|\phi\|_{H^1_\#} \leq 1\bigg\}.$$

Since $\int_Q \phi = 0$, the integral can be estimated by

$$\int\limits_Q (\chi - \theta)\phi = \int\limits_Q \chi\phi \leq \|\chi\|_{L^{p'}}\|\phi\|_{L^p} \leq c\theta^{1/p'}\|\phi\|_{H^1_\#}.$$

Here $p = 2n/(n-2)$ is the Sobolev-conjugate exponent to 2 in dimension n and $p' = p/(p-1) = 2n/(n+2)$. We also used the continuous embedding of $H^1_\#$ into L^p and Hölder's inequality between L^p and $L^{p'}$. We conclude that, for the present choice of χ,

$$\bigg(\int\limits_Q |\nabla\chi|\bigg)^{2/3} \|\chi - \theta\|^{2/3}_{H^{-1}_\sharp} \leq c\theta^{\frac{2}{3}(\frac{n-1}{n})}\theta^{\frac{2}{3}(\frac{n+2}{2n})} = c\theta.$$

This proves optimality of (3.15) for $n > 2$. In the case $n = 2$ this construction fails because of the failure of the critical embedding of H^1 into L^∞.

We now turn to (3.19). We take the same test function and write

$$\int\limits_Q |\nabla^{-1}(\chi - \theta)| = \sup\bigg\{\int\limits_Q (\chi - \theta)\phi : \phi \in W^{1,\infty}(Q),\ \|\nabla\phi\|_{L^\infty} \leq 1,\ \int\limits_Q \phi = 0\bigg\}.$$

Proceeding as above and using the duality L^1-L^∞ to estimate the product and the continuous embedding of $W^{1,\infty}$ into L^∞, we obtain

$$\int_Q (\chi - \theta)\phi = \int_Q \chi\phi \le \|\phi\|_{L^\infty(Q)} \int_Q \chi \le c\theta.$$

Therefore

$$\left(\int_Q |\nabla\chi|\right)^{1/2}\left(\int_Q |\nabla^{-1}(\chi - \theta)|\right)^{1/2} \le c\theta^{(n-1)/2n}\theta^{1/2} = c\theta^{1-1/(2n)}.$$

All constants depend on dimension, but not on θ.

4 Geometry-Independent Lower Bounds for Small and Intermediate Applied Fields

4.1 Lower Bound at Intermediate Fields

As a warm-up, we present a short proof of the bound at intermediate fields, which illustrates in a simpler setting the strategy followed for large and small fields. This bound was previously proved in [3], in which a less direct argument was used. In this intermediate regime one does not need the concentration estimate of Lemma 3.1; it suffices to take, as a test function, a suitable mollification of χ.

THEOREM 4.1 *For any $\gamma > 0$ there exists $C(\gamma)$ (a constant depending on γ) such that the following holds: For any $\varepsilon < L$, any $b_a \in (\gamma, 1-\gamma)$, and any $\chi \in \mathrm{BV}((0,L)\times Q; \{0,1\})$ and B such that $B - b_a \in L^2(\mathbb{R}\times Q; \mathbb{R}^3)$, both Q-periodic and obeying the compatibility conditions* $\operatorname{div} B = 0$ *and* $B\chi = 0$ *a.e., we have*

$$E(B,\chi) \ge C(\gamma)\varepsilon^{2/3}L^{1/3}.$$

PROOF: Let (B,χ) be an admissible pair. We may assume $E = E(B,\chi) \lesssim L$, with a constant that permits us to apply Lemma 3.2. Otherwise, there is nothing to prove. We choose $x_0 \in (0,L)$ such that (3.9) and (3.10) hold, and

$$\int_{\{x_0\}\times Q} \varepsilon|\nabla\chi| + (1-\chi)(1-B_1)^2 \lesssim \frac{E}{L}. \tag{4.1}$$

Fix a small parameter $\rho > 0$ to be chosen below, and let χ_ρ be a mollification of $\chi(x_0,\cdot)$ on a scale ρ. This means $\chi_\rho = \chi(x_0,\cdot) * \varphi_\rho$ where $\varphi_\rho(y,z) = \rho^{-2}\varphi_1(y/\rho, z/\rho)$ and $\varphi_1 \in C_c^\infty(\mathbb{R}^2)$ with $\int \varphi_1 = 1$. Then standard estimates give

$$\|\nabla\chi_\rho\|_{L^\infty} \lesssim \frac{1}{\rho}, \qquad \|\chi - \chi_\rho\|_{L^1} \lesssim \rho \int_{\{x_0\}\times Q} |\nabla\chi| \overset{(4.1)}{\lesssim} \frac{E\rho}{\varepsilon L} \tag{4.2}$$

(for the second inequality, see the proof of (3.3) above; the constants may depend on the choice of φ_1). Hence we have (remembering that $|Q| = 1$)

$$\begin{aligned} \|\chi_\rho\|_{H_\sharp^{1/2}} &\overset{(2.2)}{\leq} \|\chi_\rho\|_{L^2}^{1/2} \|\nabla \chi_\rho\|_{L^2}^{1/2} \\ &\leq \|\chi_\rho\|_{L^\infty}^{1/2} \|\nabla \chi_\rho\|_{L^\infty}^{1/2} \\ &\overset{(4.2)}{\lesssim} \frac{1}{\rho^{1/2}}. \end{aligned} \tag{4.3}$$

We intend to use χ_ρ as a test function in the estimates of Lemma 2.1 and 2.2. First note that by (3.9) in Lemma 3.2, we have the lower bound

$$\int_Q b_a \chi_\rho = b_a \int_{\{x_0\}\times Q} \chi \gtrsim C(\gamma), \tag{4.4}$$

where $C(\gamma)$ is a positive “constant” (which can depend on γ). We now use the telescoping sum

$$\begin{aligned} \int_Q b_a \chi_\rho = &\int_Q [b_a - B_1(0,\cdot)]\chi_\rho \\ &+ \int_Q [B_1(0,\cdot) - B_1(x_0,\cdot)]\chi_\rho + \int_Q B_1(x_0,\cdot)\chi_\rho \end{aligned}$$

to relate back to the energy. To this end, we note that by Lemma 2.1, the energy bounds the fluctuation of B_1 at the boundary of the sample in the sense that

$$\begin{aligned} \int_Q [b_a - B_1(0,\cdot)]\chi_\rho &\overset{(2.3)}{\leq} \|B_1(0,\cdot) - b_a\|_{H_\sharp^{-1/2}} \|\chi_\rho\|_{H_\sharp^{1/2}} \\ &\overset{\substack{(4.3)\\ \text{Lemma 2.1}}}{\lesssim} \frac{E^{1/2}}{\rho^{1/2}}. \end{aligned} \tag{4.5}$$

Now, from Lemma 2.2 we find that the relevant projection of B_1 does not change much from the boundary to the section $\{x_0\} \times Q$, namely,

$$\begin{aligned} \int_Q [B_1(0,\cdot) - B_1(x_0,\cdot)]\chi_\rho &\overset{\text{Lemma 2.2}}{\leq} \|\nabla \chi_\rho\|_{L^\infty} \int_{(0,L)\times Q} |B'| \\ &\overset{\substack{(4.2)\\ \text{Hölder}}}{\lesssim} \frac{(EL)^{1/2}}{\rho}. \end{aligned}$$

Lastly, since $B\chi = 0$, we obtain

$$\begin{aligned}\int_Q B_1(x_0,\cdot)\chi_\rho &= \int_Q B_1(x_0,\cdot)[\chi_\rho - \chi(x_0,\cdot)]\\ &\leq \int_{\{x_0\}\times Q} |\chi_\rho - \chi| + \int_{\{x_0\}\times Q} B_1^2|\chi_\rho - \chi|\\ &\lesssim \int_{\{x_0\}\times Q} |\chi_\rho - \chi| + \int_{\{x_0\}\times Q\cap\{|B_1|>2\}} [B_1 - 1]^2[1-\chi]\\ &\overset{(4.1)(4.2)}{\lesssim} \frac{E\rho}{\varepsilon L} + \frac{E}{L}. \end{aligned} \tag{4.6}$$

In the second line above we used that $B_1 \leq 1 + B_1^2$; in the third line we used that if $|B_1| > 2$ then $\chi = 0$, so $|\chi_\rho - \chi| \leq 1 = |1-\chi|$.

The telescoping sum together with (4.5)–(4.6) and (4.4) gives

$$\frac{E^{1/2}}{\rho^{1/2}} + \frac{(EL)^{1/2}}{\rho} + \frac{E\rho}{\varepsilon L} + \frac{E}{L} \gtrsim C(\gamma)$$

or

$$E \gtrsim C(\gamma)\min\left\{\rho, \frac{\rho^2}{L}, \frac{\varepsilon L}{\rho}, L\right\}.$$

Balancing the second and third terms gives the optimal choice $\rho = \varepsilon^{1/3}L^{2/3}$ and the result follows. □

4.2 Lower Bound for the Smallest Applied Fields

THEOREM 4.2 *There exists a constant* (*implicit in the notation below*) *such that if* b_a, ε, *and* L *satisfy*

$$b_a^{7/2} \leq \frac{\varepsilon}{L} \leq \frac{1}{2}, \tag{4.7}$$

then for any $\chi \in \mathrm{BV}((0,L)\times Q;\{0,1\})$ *and any* B *such that* $B - b_a \in L^2(\mathbb{R}\times Q;\mathbb{R}^3)$, *both* Q*-periodic and obeying the compatibility conditions* div $B = 0$ *and* $B\chi = 0$ *a.e., we have*

$$E(B,\chi) \gtrsim b_a\varepsilon^{4/7}L^{3/7}.$$

PROOF: It suffices to prove the theorem under the additional assumption

$$b_a^{7/2} \leq \bar{c}\,\frac{\varepsilon}{L} \tag{4.8}$$

for some $\bar{c} \leq 1$ to be chosen below. Indeed, if (4.8) does not hold, it suffices to replace L with $L' = \bar{c}L$ and restrict all functions to $(-\infty, L')\times Q$. In doing this it is important that the following proof use only the last term in (1.1) on the restricted set $\Omega_c' = (-\infty,0)\times Q$.

Let (B, χ) be an admissible pair. Fix a constant $c_* > 0$ (chosen below). Without loss of generality we may assume

$$E := E(B, \chi) \le c_* b_a \varepsilon^{4/7} L^{3/7}. \tag{4.9}$$

Otherwise there is nothing to prove. We note that by assumption, the hypothesis of Lemma 3.2 is valid, with perhaps a suitable restriction on c_* and $\bar{c}$. Hence Lemma 3.2, (4.8), and (4.9) imply that there exists $x_0 \in (0, L)$ such that

$$\int_{\{x_0\}\times Q} 1 - \chi \sim b_a \tag{4.10}$$

and

$$\int_{\{x_0\}\times Q} \varepsilon|\nabla'\chi| + (1-\chi)(B_1 - 1)^2 \lesssim c_* b_a \left(\frac{\varepsilon}{L}\right)^{4/7} \tag{4.11}$$

hold. We now apply Lemma 3.1 on Q. We choose $\ell = \varepsilon^{3/7} L^{4/7}$ and define $S \subset Q$ to be the support of $(1-\chi)(x_0, \cdot)$. By (4.10) and (4.11),

$$\ell \,\mathrm{Per}(S) \lesssim c_* b_a \lesssim c_* |S|;$$

hence if c_* is sufficiently small, we can apply Lemma 3.1 and obtain a set $S_\ell \subset Q$ such that

$$|S \cap S_\ell| \ge \frac{|S|}{2} \gtrsim b_a, \tag{4.12}$$

and, for $r \ge \ell$ (chosen below),

$$|S_\ell^r| \lesssim |S| \frac{r^2}{\ell^2} \sim b_a \frac{r^2}{\ell^2}. \tag{4.13}$$

We shall use the set S_ℓ^r to construct a test function on Q that will permit us to estimate the magnetic energy through integrals on good sections. Specifically, for $p \in Q$ we set

$$\psi(p) := \max\{r - \mathrm{dist}(p, S_\ell); 0\}, \tag{4.14}$$

where the distance above is computed in Q (see Figure 4.1). Clearly ψ is Lipschitz-continuous on Q with $|\nabla\psi| \le 1$. Since $\psi = r$ on S_ℓ, $\psi \le r$ on Q, and $\psi = 0$ on $Q \setminus S_\ell^r$, the inequalities (4.12) and (4.13) imply

$$\int_Q [1 - \chi(x_0, \cdot)]\psi \ge r|S \cap S_\ell| \gtrsim r b_a, \tag{4.15}$$

$$\int_Q \psi^2 \le r^2 |S_\ell^r| \lesssim r^2 b_a \frac{r^2}{\ell^2}, \qquad \int_Q |\nabla\psi|^2 \le |S_\ell^r| \lesssim b_a \frac{r^2}{\ell^2}. \tag{4.16}$$

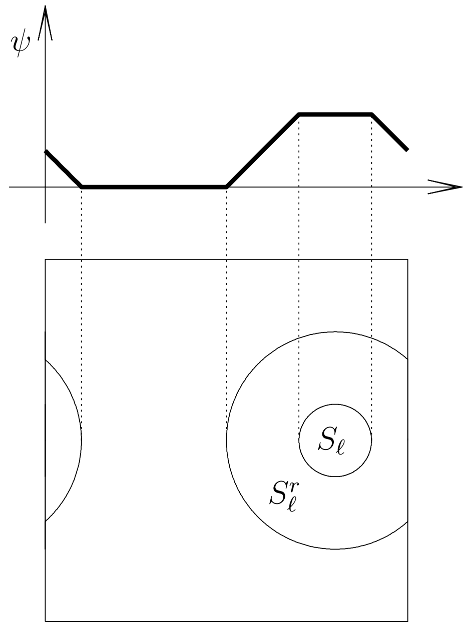

FIGURE 4.1. Sketch of the construction of ψ for the case where S_ℓ is a circle.

Next we derive a lower bound for $\int B_1(x_0, \cdot)\psi$. Using $B\chi = 0$, we write

$$\int_{\{x_0\}\times Q} B_1\psi = \int_{\{x_0\}\times Q} [1-\chi]\psi - \int_{\{x_0\}\times Q} [1-\chi][1-B_1]\psi. \tag{4.17}$$

The first term is bounded below by (4.15). The second one can be controlled, assuming

$$r \le \ell\left(\frac{L}{\varepsilon}\right)^{2/7} = \varepsilon^{1/7}L^{6/7}, \tag{4.18}$$

by

$$\begin{aligned}\int_{\{x_0\}\times Q} [1-\chi][1-B_1]\psi &\overset{\text{Hölder}}{\le} \left(\int_{\{x_0\}\times Q} [1-\chi][B_1-1]^2\right)^{1/2}\left(\int_Q \psi^2\right)^{1/2}\\ &\overset{(4.11)\,(4.16)}{\lesssim} c_*^{1/2} b_a^{1/2}\left(\frac{\varepsilon}{L}\right)^{2/7} r b_a^{1/2}\,\frac{r}{\ell}\end{aligned}$$

$$\stackrel{(4.18)}{\leq} c_*^{1/2} r b_a$$

$$\stackrel{(4.15)}{\lesssim} c_*^{1/2} \int\limits_{\{x_0\}\times Q} [1-\chi]\psi.$$

Comparing with (4.15) and (4.17), we see that, if c_* is sufficiently small, one has

$$\int\limits_{\{x_0\}\times Q} B_1\psi \geq \frac{1}{2} \int\limits_{\{x_0\}\times Q} [1-\chi]\psi \gtrsim r b_a. \tag{4.19}$$

We now relate this lower bound back to the energy via the telescoping sum

$$\int\limits_Q [B_1(x_0,\cdot)-b_a]\psi = \int\limits_Q [B_1(x_0,\cdot)-B_1(0,\cdot)]\psi + \int\limits_Q [B_1(0,\cdot)-b_a]\psi. \tag{4.20}$$

To this end, we relate the right-hand side of (4.20) to the magnetic energy. The first term is estimated by Lemma 2.2. The constraint $B\chi = 0$ implies

$$\begin{aligned}\int\limits_Q \int_0^L |B'| &= \int\limits_Q \int_0^L [1-\chi]|B'| \\ &\stackrel{\text{Hölder}}{\leq} \left(\int\limits_Q \int_0^L [1-\chi]\right)^{1/2} \left(\int\limits_Q \int_0^L |B'|^2\right)^{1/2} \stackrel{(3.8)}{\lesssim} (ELb_a)^{1/2}.\end{aligned}$$

Therefore (2.8) gives

$$\int\limits_Q [B_1(x_0,\cdot)-B_1(0,\cdot)]\psi \lesssim (ELb_a)^{1/2}. \tag{4.21}$$

For the second term, we use Lemma 2.1 to estimate the $H_\sharp^{-1/2}$ norm of $B_1(0,\cdot)-b_a$ in terms of the energy:

$$\begin{aligned}\left|\int\limits_Q [B_1(0,\cdot)-b_a]\psi\right| &\stackrel{(2.3)}{\leq} \|B_1(0,\cdot)-b_a\|_{H_\sharp^{-1/2}} \|\psi\|_{H_\sharp^{1/2}} \\ &\stackrel{\substack{\text{Lemma 2.1}\\(2.2)\\(4.16)}}{\lesssim} E^{1/2}\frac{r^{3/2}b_a^{1/2}}{\ell}.\end{aligned} \tag{4.22}$$

We now choose the value of r:

$$r = \ell^{2/3}L^{1/3} = \varepsilon^{2/7}L^{5/7}.$$

This choice is admissible since $r/\ell = (L/\varepsilon)^{1/7} \geq 1$ and $r \leq \varepsilon^{1/7}L^{6/7}$ by (4.7). The choice of r implies that the right-hand sides of (4.21) and (4.22) have the same scaling in ε, L, and b_a.

By (4.13) and (4.19) we have

$$\int_Q b_a\psi \lesssim b_a^2 \frac{r^3}{\ell^2} \quad \text{and} \quad \int_Q B_1(x_0,\cdot)\psi \gtrsim r b_a. \tag{4.23}$$

We now choose the constant entering (4.8) to be such that

$$b_a \frac{r^2}{\ell^2} = b_a\left(\frac{L}{\varepsilon}\right)^{2/7}$$

is sufficiently small compared with (an appropriate function of) the two implicit constants entering in (4.23). This will insure that, when we include the implicit constants, the right-hand side of the second inequality of (4.23) dominates that of the first. Thus

$$\int_Q [B_1(x_0,\cdot) - b_a]\psi \gtrsim r b_a.$$

This now combines with (4.21) and (4.22), via the telescoping sum (4.20), to yield

$$\begin{aligned} r b_a &\lesssim \int_Q [B_1(x_0,\cdot) - b_a]\psi \\ &\lesssim E^{1/2}\frac{r^{3/2}b_a^{1/2}}{\ell} + E^{1/2}L^{1/2}b_a^{1/2} \\ &= 2E^{1/2}L^{1/2}b_a^{1/2} \end{aligned}$$

where in the last step we used the definition of r. Finally, inserting the definition of ℓ, we obtain

$$E^{1/2} \gtrsim \frac{r b_a}{L^{1/2}b_a^{1/2}} \quad \text{or} \quad E \gtrsim \frac{1}{Lb_a} r^2 b_a^2 = b_a \varepsilon^{4/3} L^{3/7}.$$

This concludes the proof. □

4.3 Lower Bound for Relatively Small Applied Fields

THEOREM 4.3 *There exists a constant* (*implicit in the notation below*) *such that if* b_a*,* ε*, and* L *satisfy*

$$\frac{\varepsilon}{L} \leq b_a^{7/2} \leq \frac{1}{2}, \tag{4.24}$$

then for any $\chi \in \mathrm{BV}((0,L)\times Q;\{0,1\})$ *and* B *such that* $B - b_a \in L^2(\mathbb{R}\times Q;\mathbb{R}^3)$*, both* Q*-periodic and obeying the compatibility conditions* $\operatorname{div} B = 0$ *and* $B\chi = 0$*, we have*

$$E(B,\chi) \gtrsim b_a^{2/3}\varepsilon^{2/3}L^{1/3}.$$

PROOF: The proof is analogous to that of Theorem 4.2. We mention the differences. Let (B, χ) be an admissible pair. Assumption (4.9) is naturally replaced with

$$E := E(B, \chi) \le c_* b_a^{2/3} \varepsilon^{2/3} L^{1/3}$$

(again, c_* is chosen below). Further, by Theorem 4.1, it suffices to prove the result under the assumption that

$$b_a \le \bar{c} \tag{4.25}$$

for some $\bar{c}$ chosen below. These assumptions and (4.24) imply that the hypothesis of Lemma 3.2 is valid (with suitable restrictions on c_* and $\bar{c}$). We may therefore choose $x_0 \in (0, L)$ such that

$$\int_{\{x_0\}\times Q} \varepsilon|\nabla'\chi| + (1-\chi)(B_1 - 1)^2 \lesssim c_* b_a^{2/3} \left(\frac{\varepsilon}{L}\right)^{2/3} \tag{4.26}$$

and (3.10) hold. We apply Lemma 3.1 with $\ell = b_a^{1/3}\varepsilon^{1/3}L^{2/3}$, $S \subset Q$ being the support of $(1-\chi)(x_0, \cdot)$. As above, this choice of ℓ is admissible provided c_* is sufficiently small. We obtain a set $S_\ell \subset Q$ such that (4.12) and (4.13) hold for any $r \ge \ell$ (chosen below). Next consider the test function ψ defined as in (4.14). Recall that ψ is Lipschitz-continuous with $|\nabla\psi| \le 1$ and (4.15) holds. For the estimate (4.19), we find, arguing as above and assuming

$$r \le \ell b_a^{1/6} \left(\frac{L}{\varepsilon}\right)^{1/3} = b_a^{1/2} L, \tag{4.27}$$

that

$$\begin{aligned}
\int_{\{x_0\}\times Q} [1-\chi][1-B_1]\psi \ &\le \ \left(\int_{\{x_0\}\times Q} [1-\chi][B_1-1]^2\right)^{1/2} \left(\int_Q \psi^2\right)^{1/2} \\
&\overset{\substack{(4.26)\\(4.16)}}{\lesssim} \ c_*^{1/2} b_a^{1/3} \left(\frac{\varepsilon}{L}\right)^{1/3} r b_a^{1/2} \frac{r}{\ell} \\
&\overset{(4.27)}{\le} \ c_*^{1/2} r b_a \\
&\overset{(4.15)}{\lesssim} \ c_*^{1/2} \int_{\{x_0\}\times Q} [1-\chi]\psi.
\end{aligned}$$

Therefore, if c_* is small enough, (4.15) and (4.17) imply that

$$\int_{\{x_0\}\times Q} B_1\psi \gtrsim r b_a.$$

Again, we have to ensure that this term is larger than $\int b_a \psi$, which by (4.13) can be bounded by

$$\int_Q b_a \psi \lesssim b_a^2 \frac{r^3}{\ell^2}. \tag{4.28}$$

We choose

$$r = \bar{c}^{1/2} \frac{\varepsilon^{1/3} L^{2/3}}{b_a^{1/6}} = \ell \left(\frac{\bar{c}}{b_a} \right)^{1/2},$$

where $\bar{c} \leq 1$ is the constant entering (4.25). This choice is admissible since $r/\ell = (\bar{c}/b_a)^{1/2} \geq 1$ and, by (4.24), $r \leq b_a^{1/2} L$. From (4.28) we obtain

$$\int_Q b_a \psi \lesssim b_a r \left(b_a \frac{r^2}{\ell^2} \right) = \bar{c} b_a r \lesssim \bar{c} \int_{\{x_0\} \times Q} B_1 \psi. \tag{4.29}$$

If the constant $\bar{c}$ entering (4.25) is small enough, we obtain

$$\int_Q [B_1(x_0, \cdot) - b_a] \psi \gtrsim r b_a.$$

Proceeding as in (4.21) and (4.22) of Theorem 4.2, we find both

$$\int_Q [B_1(x_0, \cdot) - B_1(0, \cdot)] \psi \lesssim (E L b_a)^{1/2}$$

and

$$\int_Q [B_1(0, \cdot) - b_a] \psi \lesssim E^{1/2} \frac{r^{3/2} b_a^{1/2}}{\ell}.$$

The telescoping sum (4.20) now implies that

$$r b_a \lesssim E^{1/2} L^{1/2} b_a^{1/2} + E^{1/2} \frac{r^{3/2} b_a^{1/2}}{\ell}$$

or

$$E^{1/2} \gtrsim \min \left\{ \frac{r b_a^{1/2}}{L^{1/2}}, \frac{\ell b_a^{1/2}}{r^{1/2}} \right\}.$$

The choice of r combined with assumption (4.24) implies that the first term is smaller, and hence

$$E \gtrsim \frac{r^2 b_a}{L} = \bar{c} b_a^{2/3} \varepsilon^{2/3} L^{1/3}.$$

$\square$

5 Geometry-Independent Lower Bounds for Large Applied Fields

THEOREM 5.1 *For any $\chi \in \mathrm{BV}((0, L) \times Q; \{0, 1\})$ and any B such that $B - b_a \in L^2(\mathbb{R} \times Q; \mathbb{R}^3)$, both Q-periodic and obeying the compatibility conditions* $\operatorname{div} B = 0$ *and $B\chi = 0$, and any $b_a \in (\frac{1}{2}, 1)$, we have*

$$E(B, \chi) \gtrsim \min\bigl\{(1 - b_a)\varepsilon^{2/3} L^{1/3} |\log(1 - b_a)|^{1/3},\ (1 - b_a)^2 L\bigr\}.$$

The first regime corresponds to families of superconducting tunnels in a normal matrix (see [3, sec. 4.4]). The tunnels have diameter of order ℓ and are separated by a distance of order τ (both length scales are defined in (5.2) below). The second regime does not have a microstructure and arises from the fact that if b_a is very close to 1, eventually the cost of field penetration in the whole sample is smaller than the cost of building the interfaces. This gives $\chi = 0$, $B_1 = b_a$ everywhere, and the total energy

$$E = (1 - b_a)^2 L.$$

PROOF: It is convenient to replace b_a by the small quantity

$$\alpha = (1 - b_a)^{1/2}.$$

By Theorem 4.1 it suffices to prove the thesis for sufficiently small α. Let (B, χ) be an admissible pair. We can assume without loss of generality that

$$E = E(B, \chi) \le \min\left\{\frac{1}{16}\alpha^4 L, c_* \alpha^2 \varepsilon^{2/3} L^{1/3} |\log \alpha|^{1/3}\right\}.$$

As in the previous cases, we shall choose c_* below. Note that the hypothesis of Lemma 3.2 is satisfied. Hence we may choose $\{x_0\} \times Q$ such that (3.9) holds,

$$\int_Q |\nabla \chi|(x_0, \cdot\,) \lesssim \frac{E}{\varepsilon L} \le c_* \frac{\alpha}{\tau} = c_* \frac{\alpha^2}{\ell}$$

and

$$\int_Q [B']^2 + [B_1 - 1]^2 [1 - \chi](x_0, \cdot\,) \lesssim \frac{E}{L}. \tag{5.1}$$

Here we denote by

$$\tau := \frac{\varepsilon^{1/3} L^{2/3}}{\alpha |\log \alpha|^{1/3}} \quad \text{and} \quad \ell := \alpha\tau = \frac{\varepsilon^{1/3} L^{2/3}}{|\log \alpha|^{1/3}}, \tag{5.2}$$

the natural length scales of the problem. If c_* is sufficiently small, we can apply Lemma 3.1 to the support of χ and obtain S_ℓ such that

$$\int_{S_\ell} \chi(x_0, \cdot\,) \gtrsim \alpha^2. \tag{5.3}$$

The test function will equal 1 on S_ℓ and then decrease logarithmically. To be more specific: suppose $\beta \in (2\alpha^{1/2}, 10)$ (in the end we will fix β, taking it equal to either 3 or 1). Define $\varphi : [0, \infty) \to \mathbb{R}$ by

$$\varphi(r) := \begin{cases} 1 & \text{if } r < \alpha\tau = \ell, \\ \frac{\log(\beta\tau/r)}{\log(\beta/\alpha)} & \text{if } \alpha\tau < r < \beta\tau, \\ 0 & \text{if } r > \beta\tau. \end{cases}$$

The test function is constructed by setting

$$\psi(p) := \varphi(\operatorname{dist}(p, S_\ell)).$$

We notice that $\psi = 1$ on S_ℓ, and that $|\nabla \operatorname{dist}(\,\cdot\,, S_\ell)| = 1$ a.e. on $Q \setminus S_\ell$. Further, by Lemma 3.1 we have, for all $r \geq \ell$,

$$|S_\ell^r| \lesssim |S| \frac{r^2}{\ell^2}. \tag{5.4}$$

We estimate, using the coarea formula [6, theorem 2 in sec. 3.4.3 and prop. 2 in sec. 3.4.4] and integrating by parts,

$$\begin{aligned} \int_Q |\psi|^2 \, dp \;&=\; |S_\ell| + \int_0^\infty \varphi^2(r)\mathcal{H}^1(\{p \in Q : \operatorname{dist}(p, S_\ell) = r\}) dr \\ &=\; -\int_0^\infty \left[\frac{d}{dr}\,\varphi^2(r)\right] |S_\ell^r| dr \\ &\overset{(5.4)}{\lesssim} \int_{\alpha\tau}^{\beta\tau} \frac{\log(\beta\tau/r)}{r \log^2(\beta/\alpha)} |S| \frac{r^2}{\ell^2}\, dr \\ &\lesssim\; \frac{|S|(\beta\tau)^2}{\ell^2 \log^2(\beta/\alpha)} \\ &\lesssim\; \frac{\beta^2}{\log^2(\beta/\alpha)}. \end{aligned}$$

Analogously we find

$$\int_Q |\psi| dp \lesssim \int_{\alpha\tau}^{\beta\tau} \frac{1}{r \log(\beta/\alpha)} |S| \frac{r^2}{\ell^2}\, dr \lesssim \frac{\beta^2}{\log(\beta/\alpha)}$$

and

$$\begin{aligned} \int_Q |\nabla\psi|^2 \, dp &\leq \int_0^\infty |\varphi'|^2(r)\, \mathcal{H}^1(\{p \in Q : \operatorname{dist}(p, S_\ell) = r\}) dr \\ &= -\int_{\alpha\tau}^{\beta\tau} \left[\frac{d}{dr} |\varphi'|^2(r)\right] |S_\ell^r| dr \\ &\leq \frac{1}{\tau^2 \log \beta/\alpha}. \end{aligned}$$

We summarize these bounds as follows:

$$\begin{aligned} &\|\psi\|_{L^2}^2 \lesssim \frac{\beta^2}{\log^2 \beta/\alpha}, && \|\psi\|_{L^1} \lesssim \frac{\beta^2}{\log \beta/\alpha}, \\ &\|\nabla\psi\|_{L^2}^2 \lesssim \frac{1}{\tau^2 \log \beta/\alpha}, && \|\psi\|_{H_\sharp^{1/2}}^2 \overset{(2.2)}{\lesssim} \frac{\beta}{\tau \log^{3/2} \beta/\alpha}. \end{aligned} \tag{5.5}$$

With the test function in hand, our remaining tasks are

- to prove the lower bound

$$\int\limits_{\{x_0\}\times Q} [\chi - (1 - b_a)]\psi \gtrsim \alpha^2, \tag{5.6}$$

- to relate this back to the energy by using the usual telescoping sum for the left-hand side.

Focusing on the first task, note that by (5.3),

$$\int\limits_{\{x_0\}\times Q} \chi\psi \geq \int\limits_{\{x_0\}\times S_\ell} \chi \gtrsim \alpha^2.$$

Also, we have

$$\int\limits_{\{x_0\}\times Q} [1 - b_a]\psi = \int\limits_{\{x_0\}\times Q} \alpha^2\psi \lesssim \frac{\alpha^2\beta^2}{\log \beta/\alpha} \lesssim \frac{\alpha^2}{|\log \alpha|}.$$

In the last step we used that $\beta \in (2\alpha^{1/2}, 10)$ and that $\alpha < 1$. Since we are working in the small-α regime, we can assume that

$$\int\limits_Q [1 - b_a]\psi \leq \frac{1}{2}\int\limits_Q \chi\psi.$$

Thus (5.6) holds.

We now use the telescoping sum

$$\begin{aligned} \int\limits_Q [\chi(x_0, \cdot) - (1 - b_a)]\psi = &\int\limits_Q [\chi(x_0, \cdot) + B_1(x_0, \cdot) - 1]\psi \\ &+ \int\limits_Q [B_1(0, \cdot) - B_1(x_0, \cdot)]\psi \\ &+ \int\limits_Q [b_a - B_1(0, \cdot)]\psi \end{aligned}$$

to relate (5.6) to the energy. To this end, from (5.1) we have

$$\begin{aligned}
\int_Q [\chi(x_0,\cdot)+B_1(x_0,\cdot)-1]\psi &= \int_Q [B_1-1][1-\chi]\psi \\
&\overset{\text{Hölder}}{\le} \left(\int_Q [B_1-1]^2[1-\chi]\right)^{1/2}\left(\int_Q \psi^2\right)^{1/2} \\
&\lesssim \left(\frac{E}{L}\right)^{1/2}\|\psi\|_{L^2} \\
&\overset{(5.5)}{\lesssim} \left(\frac{E}{L}\right)^{1/2}\frac{\beta}{\log\beta/\alpha}. \qquad (5.7)
\end{aligned}$$

Turning to the next term, we have

$$\begin{aligned}
\int_Q [B_1(0,\cdot)-B_1(x_0,\cdot)]\psi(\cdot) &\overset{(2.10)}{=} -\int_Q \int_0^{x_0} B'(x,\cdot)\nabla\psi(\cdot) \\
&\overset{\text{Hölder}}{\le} (EL)^{1/2}\|\nabla\psi\|_{L^2} \\
&\overset{(5.5)}{\lesssim} \frac{(EL)^{1/2}}{\tau\log^{1/2}\beta/\alpha}. \qquad (5.8)
\end{aligned}$$

The last term satisfies

$$\begin{aligned}
\int_Q [b_a-B_1(0,\cdot)]\psi(\cdot) &\overset{(2.3)}{\le} \|b-B_1(0,\cdot)\|_{H_\sharp^{-1/2}}\|\psi\|_{H_\sharp^{1/2}} \\
&\overset{\text{Lemma 2.1}}{\le} E^{1/2}\|\psi\|_{H_\sharp^{1/2}} \\
&\overset{(5.5)}{\lesssim} \frac{E^{1/2}\beta^{1/2}}{\tau^{1/2}\log^{3/4}\beta/\alpha}. \qquad (5.9)
\end{aligned}$$

Combining (5.7)–(5.9) with (5.6), we obtain

$$E^{1/2} \gtrsim \min\left\{\frac{\alpha^2 L^{1/2}\log\beta/\alpha}{\beta},\ \frac{\alpha^2\tau\log^{1/2}\beta/\alpha}{L^{1/2}},\ \frac{\alpha^2\tau^{1/2}\log^{3/4}\beta/\alpha}{\beta^{1/2}}\right\},$$

or

$$E \gtrsim \min\left\{\frac{\alpha^4 L\log^2\beta/\alpha}{\beta^2},\ \frac{\alpha^4\tau^2\log\beta/\alpha}{L},\ \frac{\alpha^4\tau\log^{3/2}\beta/\alpha}{\beta}\right\}. \qquad (5.10)$$

We distinguish two cases. If

$$\left(\frac{\varepsilon}{L}\right)^{2/3} \le \frac{\alpha^2}{|\log\alpha|^{1/3}}, \tag{5.11}$$

we shall show that the right-hand side of (5.10) is greater than or equal to a constant times

$$E_0 := \alpha^2 |\log\alpha|^{1/3} \varepsilon^{2/3} L^{1/3}.$$

To see this, we factor out E_0 from the right-hand side of (5.10), obtaining

$$E \gtrsim E_0 \min\left\{\left(\frac{L}{\varepsilon}\right)^{2/3}\left(\frac{\alpha}{\beta}\right)^2 \frac{\log^2\beta/\alpha}{|\log\alpha|^{1/3}},\ \frac{\log\beta/\alpha}{|\log\alpha|},\ \left(\frac{L}{\varepsilon}\right)^{1/3}\frac{\alpha}{\beta}\frac{\log^{3/2}\beta/\alpha}{|\log\alpha|^{2/3}}\right\}.$$

We claim that for some β the minimum on the right is larger than $c > 0$. To see this, note that the third term in the brackets is the geometric mean of the first and the second; hence we can neglect it. We therefore need to find $\beta \in (2\alpha^{1/2}, 10)$ such that

$$\left(\frac{L}{\varepsilon}\right)^{2/3}\left(\frac{\alpha}{\beta}\right)^2 \frac{\log^2\beta/\alpha}{|\log\alpha|^{1/3}} \gtrsim 1, \qquad \frac{\log\beta/\alpha}{|\log\alpha|} \gtrsim 1. \tag{5.12}$$

We choose $\beta = 3$, so that $\log\beta/\alpha \ge 1 + \log 1/\alpha$. The second inequality in (5.12) is obvious, and the first one follows from (5.11). This concludes the proof if (5.11) holds.

On the other hand, if

$$\left(\frac{\varepsilon}{L}\right)^{2/3} \ge \frac{\alpha^2}{|\log\alpha|^{1/3}}, \tag{5.13}$$

we need to show that the energy is larger than a constant times

$$E_1 := \alpha^4 L.$$

Arguing as before, we factor out E_1 to get

$$E \gtrsim E_1 \min\left\{\frac{\log^2\beta/\alpha}{\beta^2},\ \left(\frac{\varepsilon}{L}\right)^{2/3}\frac{\log\beta/\alpha}{\alpha^2|\log\alpha|^{2/3}},\ \left(\frac{\varepsilon}{L}\right)^{1/3}\frac{\log^{3/2}\beta/\alpha}{\alpha\beta|\log\alpha|^{1/3}}\right\}.$$

By (5.13), the result now follows with the choice of $\beta = 1$. □

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