

Bounds on the Micromagnetic Energy of a Uniaxial Ferromagnet

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Abstract

We identify the optimal scaling law for a nonconvex, nonlocal variational problem representing the magnetic energy of a uniaxial ferromagnet. Our analysis is restricted to a certain parameter regime, in which the surface tension is sufficiently small relative to the other parameters of the problem. © 1998 John Wiley & Sons, Inc.

1 Introduction

This article addresses the following nonconvex and nonlocal variational problem: Minimize

$$(P1) \quad E(\mathbf{m}) = \alpha \int_{\Omega} m_2^2 dx dy + \varepsilon \int_{\Omega} |\nabla \mathbf{m}| dx dy + \beta \int_{\mathbb{R}^2} |\nabla u|^2 dx dy$$

over all $\mathbf{m} = (m_1, m_2) \in BV(\mathbb{R}^2, \mathbb{R}^2)$ such that $|\mathbf{m}| = 1$ in Ω and $\mathbf{m} = 0$ in Ω^c . The integrand of the last term is determined by \mathbf{m} through the differential equation $\Delta u = \operatorname{div} \mathbf{m}$ in \mathbb{R}^2 . We choose the domain to be a rectangle: $\Omega = (-L, L) \times (0, 1)$.

Our functional is a two-dimensional and sharp-interface reduction of the magnetic energy of a uniaxial ferromagnet (see, e.g., Landau and Lifshitz [13]). The unknown \mathbf{m} represents the spontaneous magnetization. The first term accounts for the effect of crystalline anisotropy, the second assigns surface energy to sharp changes in the direction of \mathbf{m} , and the third is the magnetic field energy generated by \mathbf{m} . The third term can alternatively be thought of as a penalization favoring $\operatorname{div} \mathbf{m} = 0$, since ∇u is the Helmholtz projection of \mathbf{m} (its L^2 projection onto the subspace of gradients). Our functional (P1) is not quite the usual micromagnetic energy, because the second term involves $|\nabla \mathbf{m}|$ rather than $|\nabla \mathbf{m}|^2$. However, it is (at least formally) operationally equivalent, as we shall explain later.

There is an extensive physics literature on micromagnetics. One of its goals is to explain the formation and morphology of magnetic domains—regions

in which the magnetization is smoothly varying or even essentially constant, separated by sharp walls where it is discontinuous. These domain structures are believed to be (local or global) minimizers of the magnetic energy. This idea goes back at least to Landau, Lifshitz, and Kittel, see, e.g., [14], [15], [9], and [10]. More recent work related to the uniaxial case includes that of Privorotskiĭ [17], Hubert [5], Kaczer [8], Marchenko [16], and Gabay and Garel [4].

One would like to optimize the energy among all magnetic domain patterns. In practice, however, it has been customary to do something much more limited—namely, optimize the energy within a specific class of patterns parameterized by just a few degrees of freedom. For example, it is well-known that the optimal one-dimensional pattern has energy of order $\varepsilon^{1/2}L^{1/2}$ and domain width proportional to $L^{1/2}$; we shall review this calculation below (see Figures 1.2 and 1.3). Optimizing in a more complicated class of “branched” domain structures achieves an energy of order $\varepsilon^{2/3}L^{1/3}$ and a basic domain width proportional to $L^{2/3}$: this was shown in slightly different settings by Lifshitz [15], Privorotskiĭ [17], and Hubert [5]. This calculation has been used to explain why magnetic domains branch and their widths scale as $L^{2/3}$ when ε/L is sufficiently small.

Such calculations give a lot of insight, but they fall far short of identifying the minimum energy state. They leave open the possibility that there might be other, as yet undiscovered classes of domain structures with different scaling laws and smaller energies. An entirely different type of argument is needed to rule this out. Optimizing the energy within a suitable ansatz amounts to finding an upper bound for (P1). Showing that this gives the right scaling law amounts to consideration of the corresponding lower bound.

This paper provides the missing lower bound for the functional (P1). We also give a fully rigorous treatment of the upper bound. Thus we identify the precise scaling law for the minimum energy in a certain parameter regime. Our main result, Theorem 4.1, says that

$$(1.1) \quad \min E \sim \eta^{1/3} \varepsilon^{2/3} L^{1/3}$$

with $\eta = \min\{\alpha, \beta\}$ provided that $L > 1$ and the parameters α , ε , β , and L satisfy a certain condition (4.1). That condition requires $\varepsilon^{1/3}L^{2/3}$ to be sufficiently small relative to α and β . It is essentially the hypothesis that our branched domain construction be realizable in a rectangle of unit width.

The restriction imposed by condition (4.1) is natural, but the restriction $L > 1$ is not. We believe that (1.1) holds even for $L < 1$, provided that $L > \varepsilon / \min\{\alpha, \beta\}$. Our constructions, which give upper bounds on the energy, work for such L . However, our techniques for proving lower bounds seem

limited to $L > 1$ (or more precisely to $L > b$ with b independent of ε , α , and β).

Our work is closely related to that of Kohn and Müller concerning branching of twins near an austenite–twinned-martensite interface [11, 12]. That work identified the scaling law of the minimum energy for a certain nonconvex but local variational problem regularized by small surface energy. Here we achieve a similar goal for the nonconvex, nonlocal problem (P1). Our methods build on those of [11]; however, there are many significant differences due to the nonlocal character of the field energy $\int |\nabla u|^2 dx dy$. See Remarks 2.4 and 3.3 for further comments.

We turn now to a discussion of the physical problem to motivate our interest in (P1). The structure of a ferromagnet is characterized by its magnetization vector field \mathbf{M} , the ionic magnetic moment per unit volume. Below the Curie temperature T_C the material experiences spontaneous magnetization; in other words, $\mathbf{M} \neq 0$ even in the absence of an applied magnetic field. For $T \ll T_C$ the degree of magnetization is constant, i.e., $|\mathbf{M}| = f(T)$ is constant in space, so the interesting variable is the normalized magnetization

$$\mathbf{m} = \frac{\mathbf{M}}{|\mathbf{M}|}.$$

In many settings \mathbf{m} is observed to be more or less “piecewise smooth.” That is, the material is divided into regions where \mathbf{m} is smooth (*magnetic domains*) separated by layers where \mathbf{m} experiences sharp changes of direction (*Bloch walls*). Micromagnetics aims to explain the formation of these domains—and their specific morphology—on the basis of energy minimization.

We specialize to the case of a uniaxial material oriented so that the x -axis is its easy axis (the preferred direction for \mathbf{m}). Let $\Omega \subset \mathbb{R}^3$ be the region occupied by the ferromagnet. We take the convention that $\mathbf{m}(x, y, z) = (m_1(x, y, z), m_2(x, y, z), m_3(x, y, z))$ is defined on all of \mathbb{R}^3 , with $|\mathbf{m}| = 1$ on Ω and $\mathbf{m} = 0$ on $\Omega^c = \mathbb{R}^3 - \Omega$. In the absence of an applied magnetic field and ignoring magnetostriction, the micromagnetic energy is the sum of these three terms:

The Anisotropy Energy.

$$E_a(\mathbf{M}) = \kappa \int_{\Omega} M_2^2 + M_3^2 dx dy dz$$

or

$$E_a(\mathbf{m}) = \alpha \int_{\Omega} m_2^2 + m_3^2 dx dy dz,$$

which favors $m_1 = \pm 1$. Here κ is the anisotropy constant and $\alpha = \kappa |\mathbf{M}|^2$.

The Exchange Energy.

$$E_{ex}(\mathbf{M}) = \delta \int_{\Omega} |\nabla \mathbf{M}|^2 dx dy dz$$

or $E_{ex}(\mathbf{m}) = \delta |\mathbf{M}|^2 \int_{\Omega} |\nabla \mathbf{m}|^2 dx dy dz,$

which penalizes changes in the direction of \mathbf{m} . Here δ is the exchange constant.

The Self Field (or Nonlocal) Energy.

$$(1.2) \quad E_f(\mathbf{M}) = \int_{\mathbb{R}^3} |\mathbf{H}|^2 dx dy dz = \int_{\mathbb{R}^3} |\nabla v|^2 dx dy dz,$$

which favors $\operatorname{div} \mathbf{m} = 0$. Here $H = -\nabla v$ is the (self-) magnetic field associated with the magnetization \mathbf{M} . It is determined, along with the magnetic induction \mathbf{B} , by Maxwell's equations, which assert that

$$(1.3) \quad \operatorname{div} \mathbf{B} = 0, \quad \operatorname{curl} \mathbf{H} = 0, \quad \mathbf{B} = \mathbf{H} + \mathbf{M}.$$

Since $\operatorname{curl} \mathbf{H} = 0$, we can write $\mathbf{H} = -\nabla v$. Observing that (1.2) gives the Helmholtz decomposition of \mathbf{M} , we see that ∇v is the L^2 orthogonal projection of \mathbf{M} onto the subspace of gradients. In particular, the potential v is determined (up to an additive constant) by the equation

$$(1.4) \quad \Delta v = \operatorname{div} \mathbf{M}$$

supplemented by the condition that $\nabla v \in L^2(\mathbb{R}^3)$. The definition (1.2) of E_f expresses it in terms of \mathbf{M} , but of course it can also be expressed in terms of \mathbf{m} :

$$E_f(\mathbf{m}) = \beta \int_{\mathbb{R}^3} |\nabla u|^2 dx dy dz,$$

where $\beta = |\mathbf{M}|^2$ and ∇u satisfies

$$(1.5) \quad \Delta u = \operatorname{div} \mathbf{m}.$$

The field energy E_f depends continuously on the magnetization \mathbf{M} in the L^2 norm, because the Helmholtz projection is a bounded, linear operator from L^2 to L^2 . We will make extensive use of this elementary and well-known fact, so we record it here as a lemma:

LEMMA 1.1 *For any magnetization $\mathbf{M} \in L^2(\mathbb{R}^3, \mathbb{R}^3)$, let $\mathbf{H} = -\nabla v$ be the associated magnetic field determined by (1.2). Then $|\mathbf{H}|_{L^2} \leq |\mathbf{M}|_{L^2}$.*

To arrive at our variational problem (P1) from the standard micromagnetic energy presented above, we must make two changes. First, we want to treat the Bloch walls as sharp interfaces rather than thin layers of rapidly varying magnetization. Second, we want to consider a two-dimensional reduction.

We prefer a sharp-interface treatment of the Bloch walls because it permits us to focus on the morphology of domain structure without simultaneously having to resolve the internal structure of the walls. To get such a treatment, we replace the exchange energy E_{ex} by a term of the form

$$E_s(\mathbf{m}) = \varepsilon \int_{\Omega} |\nabla \mathbf{m}| dx dy,$$

representing *surface energy*. Since the integrand is $|\nabla \mathbf{m}|$ rather than $|\nabla \mathbf{m}|^2$, this term permits \mathbf{m} to be discontinuous, but it penalizes discontinuities by assigning them a certain energy—namely, the surface integral of $\varepsilon|[\mathbf{m}]|$, where $[\mathbf{m}]$ is the jump in \mathbf{m} .

The use of sharp-interface models is common in the physics literature. For the physical problem to remain unchanged, one must choose ε so that E_s represents the micromagnetic energy of a Bloch wall. The formal calculation goes back at least to Landau and Lifshitz [13]. Assuming that (a) \mathbf{m} jumps from $(1, 0, 0)$ to $(-1, 0, 0)$ across the wall, (b) \mathbf{m} depends only on the variable transverse to the wall, and (c) \mathbf{m} is divergence free within the wall, one arrives at a one-dimensional variable problem that is easily solved. The upshot is a formula for the “surface tension” ε in terms of the basic micromagnetic parameters, which scales as

$$(1.6) \quad \varepsilon \sim |\mathbf{M}|^2 \sqrt{\kappa \delta}.$$

One also gets a formula for the thickness w of the Bloch wall, which scales as

$$(1.7) \quad w \sim \sqrt{\frac{\delta}{\kappa}} \sim \frac{\varepsilon}{\alpha}.$$

We are interested in w as well as ε , because the sharp-interface model loses physical validity as the length scale of its magnetic domains approaches w .

It would be natural to seek a more rigorous passage from micromagnetics to a sharp-interface model. The recent work of Anzellotti, Baldo, and Visintin [1] represents important progress. However, their analysis is restricted to a special scaling in which the nonlocal energy is relatively unimportant. The cases of interest here are not restricted to that regime.

The attentive reader will have noticed that besides treating the interfaces differently, E_{ex} and E_s also behave differently inside the “magnetic domains.” Indeed, in regions where $|\nabla \mathbf{m}| \sim 1$, the exchange energy E_{ex} has order $\delta |\mathbf{M}|^2$ while E_s has order $\sqrt{\delta \kappa} |\mathbf{M}|^2$. But these terms will be negligible, in other words, much smaller than the total energy. So this difference between E_{ex} and E_s is insignificant.

Our second reduction is from three dimensions to two. This, too, is common in the physics literature. We suppose that the uniaxial ferromagnet occupies a rectangular slab infinite in the z -direction: $(-L, L) \times (0, 1) \times (-\infty, \infty)$. We suppose further that the magnetization \mathbf{m} is independent of z , and its direction remains in the xy -plane: $\mathbf{m} = (m_1(x, y), m_2(x, y), 0)$. The energy is formally infinite because of invariance in the z -direction, so the sensible goal is to minimize the two-dimensional energy density (the energy per unit length in the z direction). For the local terms—the anisotropy energy E_a and the surface energy E_s —the interpretation of this reduction is obvious. For the nonlocal term E_f it is a bit less obvious, because the potential u defined by (1.5) no longer has $\nabla u \in L^2(\mathbb{R}^3)$. The proper interpretation is that $u = u(x, y)$ has $\nabla u \in L^2(\mathbb{R}^2)$, and ∇u is the two-dimensional Helmholtz projection of (m_1, m_2) onto the subspace of gradients.

Our sharp-interface and two-dimensional reductions have rather different status. The former leaves the essential physical problem unchanged (or so we conjecture). The latter does not: In truth, the magnetization of a uniaxial slab forms interesting, fully three-dimensional patterns near the “basal plane” $x = \pm L$ (see, e.g., [8]). However, our two-dimensional reduction captures many features of the three-dimensional behavior, including the length scale and branching of magnetic domains. This fact and other three-dimensional extensions of the present analysis will appear in a subsequent article ([2]).

These reductions lead directly to our two-dimensional variational problem (P1). We now discuss it more precisely, addressing issues such as the space of admissible \mathbf{m} and the existence of minimizers. From now on, the domain Ω represents the cross section of our slab, $\Omega = (-L, L) \times (0, 1)$. Since our surface energy has the form $\int_{\Omega} |\nabla \mathbf{m}| dx dy$, the natural space for \mathbf{m} is BV, the class of functions with bounded variation (see, e.g., [3] or [18]). Our variational problem is

$$(P1) \quad \min_{\mathbf{m} \in \mathcal{A}} E(\mathbf{m}) = \int_{\Omega} \alpha m_2^2 + \varepsilon |\nabla \mathbf{m}| dx dy + \beta \int_{\mathbb{R}^2} |\nabla u|^2 dx dy,$$

where $\Delta u = \operatorname{div} \mathbf{m}$, $\Omega = (-L, L) \times (0, 1)$, and

$$(1.8) \quad \mathcal{A} := \{\mathbf{m} \in \operatorname{BV}(\mathbb{R}^2, \mathbb{R}^2) \mid |\mathbf{m}| = 1 \text{ a.e. in } \Omega, \mathbf{m} = 0 \text{ on } \Omega^c\}.$$

We note that in this language, $|\nabla \mathbf{m}|$ is a finite Borel measure, the surface energy is ε times the total variation of \mathbf{m} on Ω , and the equation for u is interpreted in the sense of distributions. Throughout this article, boundary values of $\mathbf{m} \cdot \mathbf{n}$ are to be interpreted in the sense of inner trace.

THEOREM 1.2 *Problem (P1) attains its minimum.*

PROOF: Let \mathbf{m}_n be a minimizing sequence. We have \mathbf{m}_n bounded in $BV(\Omega)$ and hence (cf. [3]) there exists a subsequence (not relabeled) such that

$$\mathbf{m}_n \rightarrow \mathbf{m} \text{ in } L^1 \quad \text{and} \quad \int_{\Omega} |\nabla \mathbf{m}| dx dy \leq \liminf_{n \rightarrow \infty} \int_{\Omega} |\nabla \mathbf{m}_n| dx dy.$$

Passing to a further subsequence (to obtain almost everywhere convergence of \mathbf{m}_n), we obtain $|\mathbf{m}| = 1$ a.e. in Ω and $\mathbf{m} = 0$ outside Ω . Lemma 1.1 implies that

$$\lim_{n \rightarrow \infty} E_f(\mathbf{m}_n) = E_f(\mathbf{m}),$$

and the result follows. ■

The presence of nonzero surface energy is crucial for the existence of minimizers. When $\varepsilon = 0$ the situation is different. Then the anisotropy energy and the nonconvex constraint $|\mathbf{m}| = 1$ prefer $\mathbf{m} = (\pm 1, 0)$, and the absence of surface energy permits arbitrary oscillation between these. One can easily construct a sequence whose energies tend to 0: Consider $\mathbf{m}_a(x, y) = \mathbf{m}_a(y)$ which alternates between $(\pm 1, 0)$ in intervals of length a (see Figure 1.2). The associated energy, which in this case is purely field energy, is of order a (see below) and tends to 0 with a . Thus the minimum energy is 0, and it is not attained, since no element of \mathcal{A} has energy 0. Analogous questions of attainment and nonattainment for micromagnetics without the exchange energy have been discussed by several authors; see, e.g., James and Kinderlehrer [6, 7].

We end this introduction by discussing three basic “one-dimensional” domain structures, and the scaling laws giving their energies E in terms of the parameters α , ε , β , and L . These constructions are *not* optimal for the parameter regimes considered in this paper; the branched structures introduced in Sections 2 and 3 do better. It is nevertheless useful to discuss them, partly to review some important ideas such as “closure domains” and partly to emphasize that these familiar, unbranched structures are suboptimal.

Our first example is that of uniform magnetization; see Figure 1.1. This structure has field energy of order βL , attributable to the left and right non-divergence-free boundaries. There is no anisotropy or surface energy. The total energy is therefore of order βL .

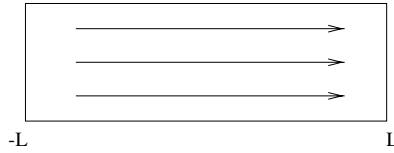


Figure 1.1. Homogeneous magnetization.

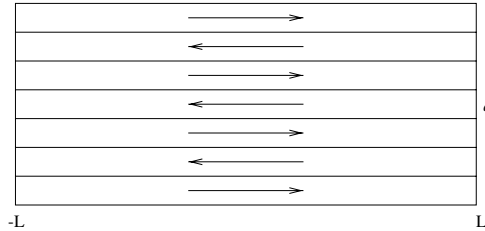


Figure 1.2. Kittel structure.

Our second example is often referred to as the Kittel structure; see Figure 1.2. It achieves a smaller value of the field energy by having the magnetization oscillate between $(\pm 1, 0)$ in layers. Of course, this costs surface energy. When the layers have width a , the extra surface energy is of order $\varepsilon \frac{L}{a}$. The field energy becomes essentially independent of L , of order βa . (An upper bound with this scaling law can be given, for example, through a calculation based on Fourier series.) Thus for given a the energy $E(a)$ scales as

$$E(a) \sim \beta a + \varepsilon \frac{L}{a}.$$

Optimization over a gives

$$a_{opt} \sim \frac{\varepsilon^{1/2} L^{1/2}}{\beta^{1/2}}, \quad E_{opt} \sim \varepsilon^{1/2} \beta^{1/2} L^{1/2}.$$

Our third example is often referred to as the Landau-Lifshitz structure; see Figure 1.3. It differs from the second example by the introduction of “closure domains” at the right- and left-hand boundaries so that $\operatorname{div} \mathbf{m} = 0$ in Ω and $\mathbf{m} \cdot \mathbf{n} = 0$ at $\partial\Omega$. This eliminates the field energy entirely, trading it for anisotropy energy and a little extra surface energy. When the layers have width a , the sum of surface and anisotropy energy scales as

$$E(a) \sim \alpha a + \varepsilon \frac{L}{a}.$$

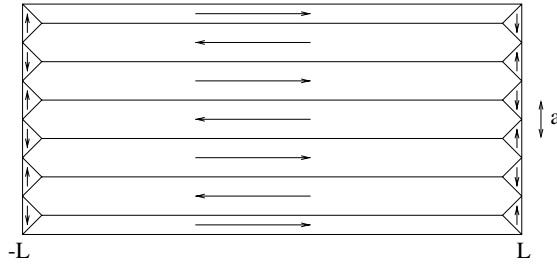


Figure 1.3. Landau-Lifshitz structure.

Optimization over a gives

$$a_{opt} \sim \frac{\varepsilon^{1/2} L^{1/2}}{\alpha^{1/2}}, \quad E_{opt} \sim \varepsilon^{1/2} \alpha^{1/2} L^{1/2}.$$

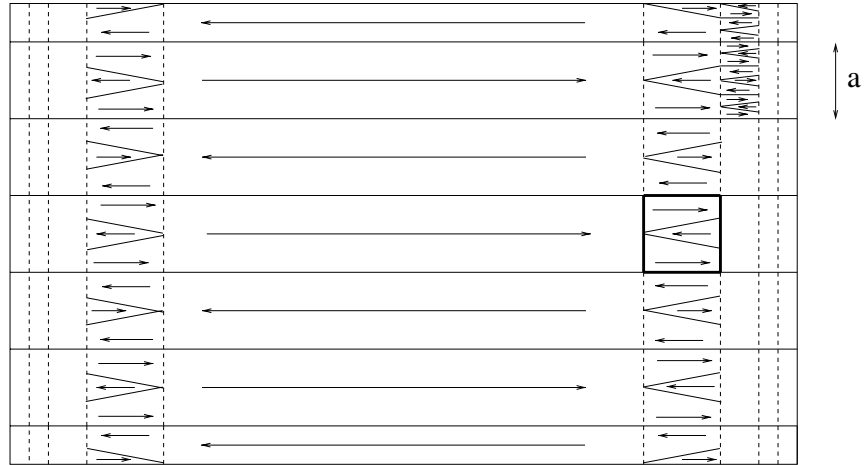
The structure of Figure 1.3 is energetically preferable over that of Figure 1.2 when anisotropy is less expensive than field energy, in other words, when $\alpha < \beta$ or (equivalently) $\kappa < 1$.

For the parameter regimes considered in this article, these one-dimensional structures are not optimal. In the subsequent sections we will give alternative structures achieving energies of order $\varepsilon^{2/3} \alpha^{1/3} L^{1/3}$ and $\varepsilon^{2/3} \beta^{1/3} L^{1/3}$. These alternative scaling laws are better whenever $L > \max(\varepsilon/\alpha, \varepsilon/\beta)$. The inequality $L > \frac{\varepsilon}{\alpha}$ holds whenever a sharp-interface model is physically justified, i.e., whenever the sample size L is larger than the Bloch wall width w (see (1.7)).

Finally, a word about the organization of the paper. Sections 2 and 3 are devoted to a pair subproblems (P2) and (P3). They are derived from (P1) by constraining either the field energy or the anisotropy energy to be zero. These subproblems represent, at least formally, the asymptotic behavior of (P1) in the limits of small and large anisotropy. They require rather different arguments, both for the upper bound (obtained in each case by a suitable branched construction) and for the lower bound (which must consider an arbitrary divergence-free magnetization in one case, and an arbitrary anisotropy-free magnetization in the other). The branched construction we use for (P2) is essentially due to Privorotskiĭ [17].

Then we turn, in Section 4, to the full problem (P1). The ideas required to handle the general case are essentially the union of the ideas required for the two subproblems. In fact, there are basically two regimes. When $\alpha < \beta$ the behavior of (P1) is very much like the small-anisotropy limit, and when $\alpha > \beta$ it is very much like the large-anisotropy limit.

Our goal throughout is to get the scaling laws, not sharp estimates for the constants implicit in them. Therefore, in discussing the constructions we often

Figure 2.1. Rough sketch of the Privorotskiĭ construction \mathbf{m}_0 .

let C be a generic constant whose value varies from line to line (but does not depend on α , β , ε , κ , or L). In discussing the lower bounds, we will make the constants explicit since it is easy to do so; however, we have not tried to optimize them.

2 The Case of Small Anisotropy: $\beta \rightarrow \infty$

This section addresses a constrained version of (P1) in which we impose the additional conditions $\operatorname{div} \mathbf{m} = 0$ in Ω and $\mathbf{m} \cdot \mathbf{n} = 0$ at $\partial\Omega$ —in other words $\operatorname{div} \mathbf{m} = 0$ everywhere, in the sense of distributions:

$$(P2) \quad \min_{\substack{\mathbf{m} \in \mathcal{A} \\ \operatorname{div} \mathbf{m} = 0}} \int_{\Omega} \alpha m_2^2 + \varepsilon |\nabla \mathbf{m}| dx dy.$$

Our goal is to identify the optimal scaling law for (P2). We will show, roughly speaking, that when $L > 1$ and $\varepsilon^{1/3} L^{2/3} \alpha^{-1/3} < 1$ the minimum energy scales like $\alpha^{1/3} \varepsilon^{2/3} L^{1/3}$.

Remembering that the field energy $E_f = \beta \int_{\Omega} |\nabla u|^2 dx dy$ is a penalization for $\operatorname{div} \mathbf{m} = 0$, we see that (P2) is the formal limit of (P1) as $\beta \rightarrow \infty$. The ratio α/β is the magnetic anisotropy κ ; hence in physical terms we are considering the limit of small anisotropy (and correspondingly small exchange energy).

2.1 An Upper Bound: The Privorotskiĭ Construction

We shall prove an upper bound with the desired scaling law by displaying a suitable divergence-free test field \mathbf{m}_0 . Our construction is a formalization of one given by Privorotskiĭ [17]. It bears some resemblance to the Landau-Lifshitz structure (Figure 1.3), for example, in the use of closure domains. But it has much less surface energy, because the basic domain width is $\varepsilon^{1/3}L^{2/3}\alpha^{-1/3}$ rather than $\varepsilon^{1/2}L^{1/2}\alpha^{-1/2}$. To keep the energy of the closure domains small, the magnetic domains must refine by branching near $|x| = L$. This costs anisotropy energy, and it is the interplay of anisotropy versus surface energy that drives the construction.

We assume throughout this subsection that

$$(2.1) \quad \frac{\varepsilon^{1/3}L^{2/3}}{\alpha^{1/3}} < 1 \quad \text{and} \quad L > \frac{\varepsilon}{\alpha}.$$

We note that both L and ε/α have dimensions of length. The first inequality has a natural interpretation: It is precisely the condition that the basic width of the branched construction (see (2.4)) be small enough for the construction to make sense in our unit-width rectangular domain Ω . The second inequality implies that the scaling law achieved in this section is better than that of the Landau-Lifshitz construction, i.e., $\varepsilon^{2/3}L^{1/3}\alpha^{1/3} < \varepsilon^{1/2}L^{1/2}\alpha^{1/2}$.

A sketch of Privorotskiĭ's \mathbf{m}_0 is given in the slightly misleading Figure 2.1. The central region consists of parallel domains of width a in which $\mathbf{m}_0 = (\pm 1, 0)$. Close to the vertical boundaries, each domain branches repeatedly into three domains. The branching continues N times until the final domain width $a/3^N$ is sufficiently small so that closure domains (as in Figure 1.3) have total energy no greater than that of the rest of the structure. Note that in Figure 2.1 we have not drawn the closure domains at the right and left edges of the rectangle. Figure 2.1 is not quite accurate in the following respects: The domain walls created by the branching are not really straight, and the magnetization in the branched region is not really parallel to the easy axis. Rather, the domain walls and the magnetization must be chosen such that \mathbf{m}_0 is divergence free.

To give the details, we describe the structure of the basic cell enclosed in bold on Figure 2.1 and illustrated in Figure 2.2. For convenience, we have rotated the cell counterclockwise by 90 degrees, hence the easy axis is vertical. The magnetization is parallel to the easy axis in the central domain. At a point (x, y) outside this region, it is the unit tangent vector to the circle passing through (x, y) with center either O or O' depending on whether the point (x, y) is below or above the line $\overline{OO'}$, respectively. We let R_n be the length

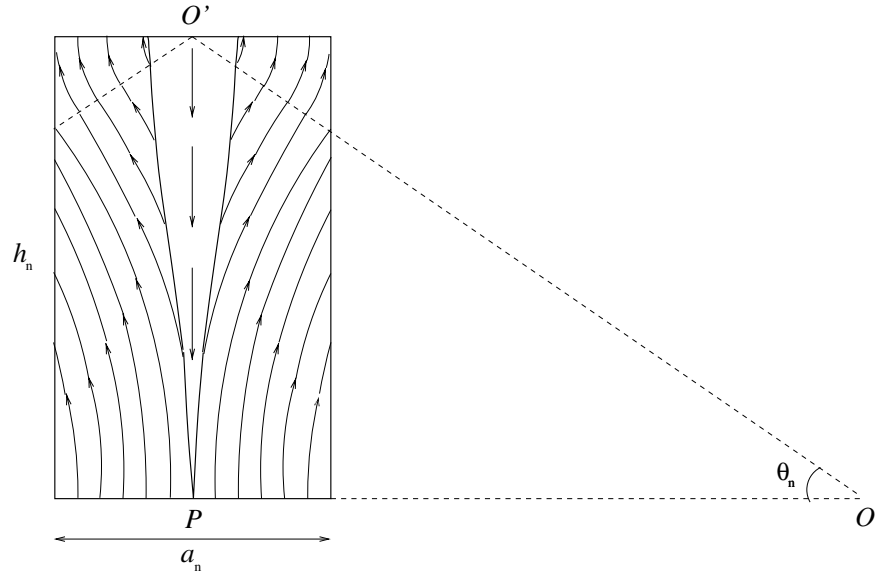


Figure 2.2. The Privorotskiĭ cell.

of the line segment \overline{PO} . Each central wall curve on which \mathbf{m} experiences a jump discontinuity is such that the tangent vector to the curve bisects the angle between \mathbf{m} immediately to either side. One can check that the part of the right wall curve below $\overline{OO'}$ has equation (in polar coordinates with origin O)

$$r \cos^2 \left(\frac{\theta}{2} \right) = R_n,$$

and the part above $\overline{OO'}$ has equation (with origin O')

$$r \cos^2 \left(\frac{\theta}{2} \right) = \frac{a_n}{6}.$$

The requirement that the two curves meet at $\theta = \theta_n$ implies that

$$\frac{a_n}{3} = R_n \left(\frac{1}{\cos \theta_n} - 1 \right).$$

Under the assumption that $\theta_n < 1$, we have $\frac{1}{\cos \theta_n} - 1 > \frac{1}{3}\theta_n^2$ and $\tan \theta_n < 2\theta_n$, so

$$R_n < \frac{a_n}{\theta_n^2} \quad \text{and} \quad h_n = R_n \tan \theta_n < 2 \frac{a_n}{\theta_n}.$$

The geometry dictates that $a_n = a/3^{n-1}$. The parameter θ_n (hence R_n and h_n) will be determined in terms of a_n and the material parameters by minimization of the energy associated with splitting in the basic cell. This amounts to an equipartition between the anisotropy and surface energy of the cell. With this in hand, we will sum a geometric series to obtain the energy $E(a)$ of the structure in terms of a . Then we will optimize in a . As a result of this procedure, our construction has its energy roughly equipartitioned between the unbranched, branched, and closure regions of the ferromagnet.

To compute the energy of the n^{th} cell, i.e., the structure shown in Figure 2.2, we note that the length of the two domain walls is bounded by a constant times the height of the cell h_n . The contribution of the surface energy due to the domain walls is bounded by twice the perimeter of the interface. Thus the surface energy of the domain wall is bounded above by $C\varepsilon h_n$. An elementary argument (given in detail by Privorotskiĭ in [17]) shows that anisotropy energy is bounded above by $C\alpha a_n^2 \theta_n$. Thus the sum of anisotropy and wall energy in the cell is roughly minimized when

$$(2.2) \quad \theta_n = c_1 \frac{\sqrt{\varepsilon}}{\sqrt{\alpha a_n}}$$

for some constant c_1 . Note that we did not include the contribution of the “surface energy” term away from the walls. Strictly speaking, this term will of course have a positive contribution in the smooth regions. However, one can check that this contribution is negligible, that is, for the chosen values of R_n and θ_n , this energy is of lower order than the surface energy concentrated on the domain walls. The energy of the unit cell now satisfies

$$\begin{aligned} E_n &\leq C\varepsilon h_n + C\alpha a_n^2 \theta_n \\ &\leq C\alpha^{1/2} \varepsilon^{1/2} a_n^{3/2}. \end{aligned}$$

The branching continues N times, at which point closure domains have an energy bounded above by αa_{N+1} . We write $h = \sum_1^N h_n$ for the length of the branched region. Then the energy in the entire domain Ω satisfies

$$\begin{aligned} E(a) &\leq \frac{1}{a} \left(\varepsilon 2(L - h) + C \sum_{n=1}^N 3^{n-1} \alpha^{1/2} \varepsilon^{1/2} a_n^{3/2} + C\alpha a a_{N+1} \right) \\ &\leq C\varepsilon \frac{L}{a} + C \sum_{n=1}^{\infty} \alpha^{1/2} \varepsilon^{1/2} a^{1/2} \left(\frac{1}{3^{n-1}} \right)^{1/2} + C\alpha \frac{a}{3^N} \\ &\leq C\varepsilon \frac{L}{a} + C\alpha^{1/2} \varepsilon^{1/2} a^{1/2} + C\alpha \frac{a}{3^N}. \end{aligned}$$

We choose N such the closure energy is of the same order as the interior energy, that is,

$$(2.3) \quad 3^N \sim \frac{\alpha^{1/2} a^{1/2}}{\varepsilon^{1/2}} \quad \text{so that} \quad E(a) \leq C \left(\varepsilon \frac{L}{a} + \alpha^{1/2} \varepsilon^{1/2} a^{1/2} \right).$$

Optimizing in a gives

$$(2.4) \quad a_{opt} = c_2 \frac{\varepsilon^{1/3} L^{2/3}}{\alpha^{1/3}}, \quad E_{opt} \sim \alpha^{1/3} \varepsilon^{2/3} L^{1/3},$$

for some constant c_2 .

A few points are in need of discussion. We should verify that $\theta_n < 1$ for $n = 1, \dots, N$, that $a_{opt} < 1$, and that the length of the branched region $h = \sum_1^N h_n$ is less than $L/2$. This may require adjustments of the constants c_1 and c_2 in the definitions of θ_n and a_{opt} , (2.2) and (2.4). Indeed, our construction has

$$\theta_n \leq \theta_N = \frac{c_1}{\sqrt{c_2}} \frac{\varepsilon^{1/6}}{L^{1/6} \alpha^{1/6}} \quad \text{and} \quad h = \frac{c_2^{3/2}}{c_1} L.$$

Our hypothesis (2.1) gives $\varepsilon^{1/6} L^{-1/6} \alpha^{-1/6} < 1$. So we can be sure that $a \ll 1$, $\theta_n \ll 1$, and $h \ll L$ by choosing $c_2^{3/2} \ll c_1 \ll c_2^{1/2} \ll 1$.

Finally, we must pay special attention to the case when a_{opt} , defined by (2.4), satisfies

$$(2.5) \quad \frac{\alpha^{1/2} a_{opt}^{1/2}}{\varepsilon^{1/2}} < 3.$$

This is because we implicitly assumed the opposite in (2.3). When (2.5) holds we do not need to branch; that is, the estimate for $E(a)$ in (2.3) remains valid since (2.5) implies that $\alpha a \leq 3\alpha^{1/2} \varepsilon^{1/2} a^{1/2}$. We have proved the following:

PROPOSITION 2.1 *Let E_0 denote the minimum energy associated with (P2). There is a constant C_0 such that*

$$E_0 \leq C_0 \alpha^{1/3} \varepsilon^{2/3} L^{1/3}$$

whenever $L > \varepsilon/\alpha$ and $\varepsilon^{1/3} L^{2/3} \alpha^{-1/3} < 1$.

2.2 A Lower Bound

Now we shall prove a lower bound with the same scaling law. The regime where our bound applies is more restrictive than that of the construction, because we assume $L > 1$ rather than $L > \varepsilon/\alpha$. In addition, for technical reasons it is convenient to replace the inequality $\varepsilon^{1/3}L^{2/3}\alpha^{-1/3} < 1$ with a slightly sharper condition (2.6).

We must consider an arbitrary admissible magnetization \mathbf{m} and show that it has (up to a constant) at least as much energy as the Privorotskiĭ construction \mathbf{m}_0 . It is convenient to work with the stream function for \mathbf{m} rather than \mathbf{m} itself. The constraint $\operatorname{div} \mathbf{m} = 0$ is equivalent to the existence of a function $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$\mathbf{m} = (\phi_y, -\phi_x) \quad \text{whence} \quad E(\mathbf{m}) = \int_{\Omega} \alpha \phi_x^2 + \varepsilon |\nabla \nabla \phi| dx dy .$$

We have $|\nabla \phi| = |\mathbf{m}| = 1$ in Ω , which implies that ϕ is Lipschitz continuous. We also have $\mathbf{m} \cdot \mathbf{n} = 0$ at $\partial\Omega$, which implies that ϕ is constant on each of the four segments comprising $\partial\Omega$. Since ϕ is only determined by \mathbf{m} up to an additive constant, we may assume without loss of generality that $\phi = 0$ on $\partial\Omega$. Thus (P2) has the equivalent formulation

$$(P2) \quad \min_{\substack{|\nabla \phi|=1 \\ \phi=0 \text{ at } \partial\Omega}} \int_{\Omega} \alpha \phi_x^2 + \varepsilon |\nabla \nabla \phi| dx dy .$$

We shall show the following:

THEOREM 2.2 *There exist constants c_0 and C_0 independent of ε , L , and α such that if*

$$(2.6) \quad \frac{\varepsilon^{1/3}L^{2/3}}{\alpha^{1/3}} < \frac{c_0}{4}$$

and $L > 1$, then the minimum energy of (P2) satisfies

$$c_0 \varepsilon^{2/3} \alpha^{1/3} L^{1/3} \leq E_0 \leq C_0 \varepsilon^{2/3} \alpha^{1/3} L^{1/3} .$$

PROOF: The upper bound is supplied by Proposition 2.1, so we need only address the lower bound. Let ϕ be a minimizer of (P2). It clearly suffices to give a lower bound for $\int_{\Omega} \alpha \phi_x^2 + \varepsilon |\phi_{yy}| dx dy$, since $|\phi_{yy}| \leq |\nabla \nabla \phi|$.

Our proof is by contradiction. We start by assuming that

$$(2.7) \quad \int_{\Omega} \alpha \phi_x^2 + \varepsilon |\phi_{yy}| dx dy \leq c_0 \varepsilon^{2/3} \alpha^{1/3} L^{1/3}$$

with c_0 an arbitrary constant. Assuming (2.6) with the *same* value of c_0 , we shall prove that c_0 cannot be too small. It follows that when c_0 is sufficiently small, (2.7) must fail, and this is the desired lower bound.

Step 1. We know that $\nabla\phi \in \text{BV}$. By Tonelli's theorem (see, e.g., theorem 5.3.5 of [18]), we have

$$\int_{\Omega} |\phi_{yy}| dx dy = \varepsilon \int_{-L}^L \text{TV}(\phi_y)(x) dx,$$

where $\text{TV}(\phi_y)(x)$ denotes the total variation of ϕ_y on the set $\{(x, y) \mid 0 \leq y \leq 1\}$. Thus inequality (2.7) implies the existence of $x_0 \in (-L, L)$ such that

$$(2.8) \quad \int_0^1 \alpha \phi_x^2 dy + \varepsilon \text{TV}(\phi_y)(x_0) \leq c_0 \frac{\alpha^{1/3} \varepsilon^{2/3} L^{1/3}}{2L}.$$

Step 2. We shall use (2.8) to prove a lower bound for

$$\int_0^1 \phi^2(x_0, y) + \frac{1}{4(1-s)} \phi_x^2(x_0, y) dy$$

for $s \in (\frac{1}{2}, 1)$. The idea is that if $\phi_x(x_0, y)$ is small, then $\phi_y^2(x_0, y) = 1 - \phi_x^2(x_0, y)$ is close to 1, so the graph of $y \rightarrow \phi(x_0, y)$ must look like a sawtooth. Furthermore, if $\int |\phi_{yy}(x_0, y)| dy$ is small, then the number of "teeth" must be small, so the function itself must be large, on average.

We make this argument quantitative. By the co-area formula (cf. [3] or [18]), for almost every $s \in (\frac{1}{2}, 1)$ and after possible redefinition of ϕ_y on a finite set, the set $S_{x_0} := \{(x_0, y) \mid y \in [0, 1], |\phi_y|(x_0, y) \geq s\}$ consists of a finite number of disjoint intervals. Moreover, the sets where $\phi_y \geq s$ and $\phi_y \leq -s$ are, in turn, finite collections of disjoint intervals. We group these intervals into new intervals $J_i = (y_i, y_{i+1})$ to form the following partition of $[0, 1]$. For simplicity, we use $\phi_y(y)$ to denote $\phi_y(x_0, y)$. Let $y_0 = 0$. If $(x_0, 0) \in S_{x_0}$ with $\phi_y \geq s$, let y_1 denote the first point where $\phi_y \leq -s$ (and similarly if $\phi_y(0) \leq -s$). If $(x_0, 0) \notin S_{x_0}$, let y_1 denote the first point y such that $(x_0, y) \in S_{x_0}$. Of course, if no such point exists, we let $y_1 = 1$ and stop. In general, if $\phi_y(y_i) \geq s$, we let y_{i+1} denote the first point where $\phi_y \leq -s$. Similarly, if $\phi_y(y_i) \leq -s$, we let y_{i+1} denote the first point where $\phi_y \geq s$. Thus on each interval (y_i, y_{i+1}) , either $-1 \leq \phi_y \leq s$ or $-s \leq \phi_y \leq 1$. In addition, we have

$$(2.9) \quad \text{TV}(\phi_y)(x_0) \geq 2s(\# \text{ intervals} - 2).$$

By Lemma 2.3, stated and proved below, we have

$$(2.10) \quad \int_0^1 \phi^2(x_0, y) + \frac{1}{4(1-s)} \phi_x^2(x_0, y) dy \geq \frac{1}{12} \sum |J_i|^3 \geq \frac{1}{12N^2},$$

where $N = N(x_0)$ is the number of intervals J_i . From (2.8) and (2.9), we have

$$N \leq \frac{1}{2s} \text{TV}(\phi_y)(x_0) + 2 \leq \frac{c_0}{4s} \frac{\alpha^{1/3}}{\varepsilon^{1/3} L^{2/3}} + 2.$$

Inequality (2.6) and the fact that $s \leq 1$ give

$$1 \leq \frac{c_0}{4s} \frac{\alpha^{1/3}}{\varepsilon^{1/3} L^{2/3}},$$

and so

$$N \leq \frac{3c_0}{4s} \frac{\alpha^{1/3}}{\varepsilon^{1/3} L^{2/3}}.$$

Combining this with (2.10) gives

$$(2.11) \quad \int_0^1 \phi^2(x_0, y) + \frac{1}{4(1-s)} \phi_x^2(x_0, y) dy \geq \frac{4s^2}{27c_0^2} \varepsilon^{2/3} L^{4/3} \alpha^{-\frac{2}{3}}.$$

Step 3. Now we use the fact that $\phi(\pm L, y) = 0$ to deduce a lower bound on the L^2 norm of ϕ_x . The basic idea is that if ϕ is large at $x = x_0$ but identically zero at $x = \pm L$, then ϕ_x must be large somewhere between x_0 and $\pm L$.

We may assume without loss of generality that $x_0 \in (-L, 0)$. For a.e. $y \in (0, 1)$,

$$\begin{aligned} |\phi(x_0, y)| &= |\phi(x_0, y) - \phi(-L, y)| \leq \int_{-L}^{x_0} |\phi_x|(x, y) dx \\ &\leq \left(\int_{-L}^{x_0} |\phi_x|^2 dx \right)^{1/2} (x_0 + L)^{1/2}, \end{aligned}$$

where we have used Hölder's inequality in the last step. Thus

$$|\phi(x_0, y)|^2 \leq L \int_{-L}^L \phi_x^2 dx.$$

Integrating over y and using (2.7) gives

$$\int_0^1 |\phi(x_0, y)|^2 dy \leq L \int_0^1 \int_{-L}^L \phi_x^2 dx dy \leq c_0 \frac{\varepsilon^{2/3} L^{4/3}}{\alpha^{2/3}}.$$

Combining this with (2.8), we have

$$\begin{aligned} \int_0^1 \phi^2(x_0, y) + \frac{1}{4(1-s)} \phi_x^2(x_0, y) dy &\leq c_0 \frac{\varepsilon^{2/3} L^{4/3}}{\alpha^{2/3}} + \frac{c_0}{8(1-s)} \frac{\varepsilon^{2/3}}{L^{2/3} \alpha^{2/3}} \\ &= c_0 \frac{\varepsilon^{2/3} L^{4/3}}{\alpha^{2/3}} \left(1 + \frac{1}{8(1-s)L^2} \right). \end{aligned}$$

Comparing this with (2.11), we conclude that

$$(2.12) \quad c_0^3 \geq \frac{4}{27} \frac{8(1-s)L^2 s^2}{1 + 8(1-s)L^2}.$$

We may choose s so that $1-s = 1/8L^2$. Then (2.12) becomes

$$c_0^3 \geq \frac{2}{27} \left(1 - \frac{1}{8L^2} \right)^2 \geq \frac{2}{27} \left(\frac{7}{8} \right)^2,$$

remembering the hypothesis that $L > 1$. The theorem holds for any c_0 small enough to violate this inequality, i.e., for c_0 less than about 0.38. ■

We made use of the following lemma:

LEMMA 2.3 *Let $\phi(y)$ be a Lipschitz-continuous function defined on an interval I such that $-s \leq \phi_y \leq 1$ for some positive $s < 1$. Then*

$$\int_I \phi^2(y) + \frac{|I|^2}{4(1-s)} (1 - \phi_y^2) dy \geq \frac{1}{12} |I|^3,$$

where $|I|$ is the length of I .

PROOF: We may assume without loss of generality that $I = (-a, a)$ for some $a > 0$. Since $-s \leq \phi_y \leq 1$, we have

$$\frac{a^2}{(1-s)} (1 - \phi_y^2) \geq a^2 (1 - \phi_y) \geq (a^2 - y^2) (1 - \phi_y)$$

for all $y \in (-a, a)$. Thus, first integrating by parts and then completing the square, we get

$$\begin{aligned} \int_{-a}^a \phi^2(y) + \frac{a^2}{(1-s)}(1-\phi_y^2)dy &\geq \int_{-a}^a \phi^2(y) + (a^2-y^2)(1-\phi_y)dy \\ &= \int_{-a}^a \phi^2(y) - 2y\phi + (a^2-y^2)dy \\ &\geq \int_{-a}^a a^2 - 2y^2dy = \frac{2}{3}a^3. \end{aligned}$$

■

REMARK 2.4 The paper [11] by Kohn and Müller gave a lower bound on $\int_{\Omega} \alpha \phi_x^2 + \varepsilon |\phi_{yy}| dx dy$ with boundary condition $\phi = 0$ at $\partial\Omega$ and constraint $\phi_y = \pm 1$. Here we have given a lower bound for the same energy and boundary condition with the different constraint $\phi_x^2 + \phi_y^2 = 1$. Our method extends that of [11], and, in particular, our Lemma 2.3 reduces essentially to lemma 2.7 of [11] when $\phi_y = \pm 1$.

3 The Case of Large Anisotropy: $\alpha \rightarrow \infty$

This section considers a different constrained version of (P1) in which we require \mathbf{m} to be aligned with the easy axis everywhere in Ω :

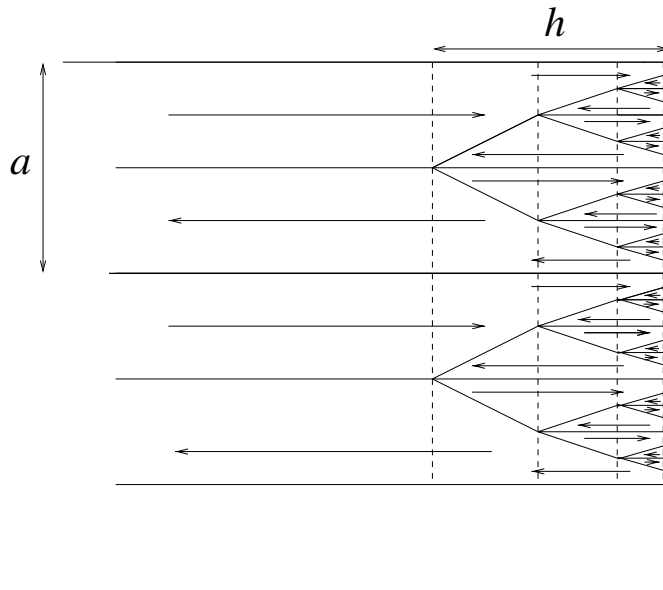
$$(P3) \quad \min_{\substack{\mathbf{m} \in \mathcal{A} \\ m_2=0 \text{ a.e.}}} \varepsilon \int_{\Omega} |\nabla \mathbf{m}| dx dy + \beta \int_{\mathbb{R}^2} |\nabla u|^2 dx dy$$

with the usual convention $\Delta u = \text{div } \mathbf{m}$. Our goal is to identify the optimal scaling law for (P3). We will show, roughly speaking, that when $L > 1$ and $\varepsilon^{1/3} L^{2/3} \beta^{-1/3} < 1$, the minimum energy scales like $\beta^{1/3} \varepsilon^{2/3} L^{1/3}$.

Remembering that the anisotropy energy $E_a = \alpha \int_{\Omega} m_2^2 dx dy$ prefers $m_2 = 0$, we see that (P3) is the formal limit of (P1) as $\alpha \rightarrow \infty$. Since α/β is the magnetic anisotropy κ , we are effectively considering the limit of large anisotropy (and correspondingly small exchange energy).

3.1 An Upper Bound

We shall prove an upper bound with the desired scaling law by displaying a suitable test field \mathbf{m}_{∞} . (We call it \mathbf{m}_{∞} because this magnetization is adapted to the limit of large anisotropy.) The construction is similar in some respects to that of Section 2; in particular, it involves branching of magnetic domains in

Figure 3.1. Domain structure of \mathbf{m}_∞ .

a qualitatively self-similar pattern. However, the details are entirely different from those of Section 2 because of the different constraint. Our construction has much less surface energy than the Kittel structure (Figure 1.2)—which is also admissible—because its basic domain width is $\varepsilon^{1/3}L^{2/3}\beta^{-1/3}$ rather than $\varepsilon^{1/2}L^{1/2}\beta^{-1/2}$. This time the branching costs field energy, and it is the interplay of field energy versus surface energy that drives the construction.

We assume throughout this subsection that

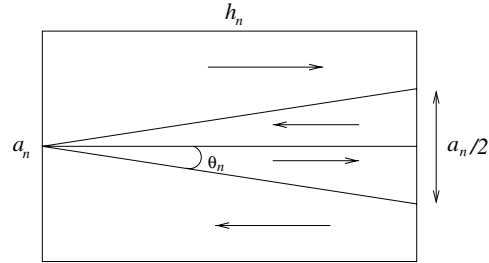
$$(3.1) \quad \frac{\varepsilon^{1/3}L^{2/3}}{\beta^{1/3}} < 1 \quad \text{and} \quad L > \frac{\varepsilon}{\beta}.$$

This is the analogue of (2.1), and it plays a similar role. The first inequality insures that the basic width of the branched construction is no larger than that of our unit-width domain Ω . The second inequality implies that the scaling law achieved by the branched construction is better than that of the Kittel structure.

The structure of \mathbf{m}_∞ is summarized by Figure 3.1, which shows just a quarter of Ω (the top right corner). For x near 0 it resembles the Kittel structure: $\mathbf{m} = (\pm 1, 0)$ in parallel strips of width a . As x approaches $\pm L$, the domains split repeatedly. Each generation of splitting halves the domain width, so after N steps the domain width is reduced to $a/2^N$. We will choose

N so that the field energy attributable to $\mathbf{m} \cdot \mathbf{n}$ being nonzero is of the same order as the interior energy.

The detailed geometry of the splitting is given by the following figure showing the n th basic cell:



The geometry dictates that $a_n = \frac{a}{2^{n-1}}$. Motivated by the Privorotskiĭ construction, we choose

$$\theta_n = c_1 \frac{\varepsilon^{1/2}}{\beta^{1/2} a_n^{1/2}},$$

for some constant c_1 . Assuming that $\theta_n < 1$, we have

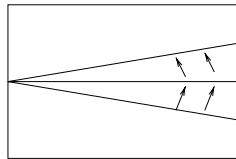
$$h_n \leq C \frac{a_n}{\theta_n}$$

for some constant C .

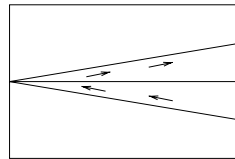
To estimate the field energy, we note that

$$(3.2) \quad \operatorname{div} \mathbf{m}_\infty = \operatorname{div}(\mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_3) \quad \text{as measures,}$$

where \mathbf{m}_1 and \mathbf{m}_2 are structures with domains identical to that of \mathbf{m}_∞ but with basic cells as illustrated below:

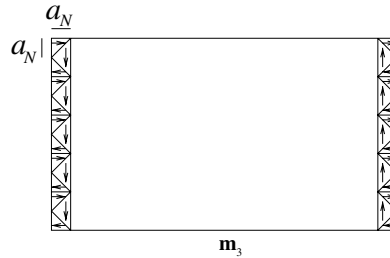


$$|\mathbf{m}_1| = 2 \sin \theta_n$$



$$|\mathbf{m}_2| = |\mathbf{m}_1| \tan \theta_n$$

In both cases the field vanishes outside the interior triangle. Inside the triangle the fields are, respectively, perpendicular and parallel to the domain walls with $|\mathbf{m}_1| = 2 \sin \theta_n$ and $|\mathbf{m}_2| = |\mathbf{m}_1| \tan \theta_n$. The domain structure associated with \mathbf{m}_3 is illustrated below:



Here $|\mathbf{m}_3| = 1$ in thin strips of width a_N near $x = \pm 1$ and $\mathbf{m}_3 = 0$ elsewhere.

We can use (3.2) to estimate the field energy of \mathbf{m}_∞ , since

$$\begin{aligned} E_f(\mathbf{m}_\infty) &= E_f(\mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_3) \\ &\leq \left(E_f(\mathbf{m}_1)^{1/2} + E_f(\mathbf{m}_2)^{1/2} + E_f(\mathbf{m}_3)^{1/2} \right)^2. \end{aligned}$$

Using Lemma 1.1 we have

$$\begin{aligned} E_f(\mathbf{m}_1) &\leq \beta \int_{\Omega} |\mathbf{m}_1|^2 dx dy \leq \frac{1}{a} \beta \sum_{n=1}^N 2^{n-1} 4 \sin^2 \theta_n \frac{a_n}{4} h_n \\ &\leq C \frac{1}{a} \beta \sum_{n=1}^{\infty} 2^{n-1} \theta_n a_n^2 \\ &= C \sum_{n=1}^{\infty} \beta^{1/2} \varepsilon^{1/2} \left(\frac{a}{2^{n-1}} \right)^{1/2} \\ &= C a^{1/2} \beta^{1/2} \varepsilon^{1/2}. \end{aligned}$$

Since for $\theta_n < 1$ we have $|\mathbf{m}_2| < C|\mathbf{m}_1|$, $E_f(\mathbf{m}_2)$ is also bounded above by $C a^{1/2} \beta^{1/2} \varepsilon^{1/2}$. Applying Lemma 1.1 once more, $E_f(\mathbf{m}_3)$ is bounded by $\beta a_N = \beta(a/2^N)$. We choose N so that $2^N \sim a^{1/2} \beta^{1/2} \varepsilon^{-1/2}$, so this term has the same order as the others. It follows that

$$E_f(\mathbf{m}_\infty) \leq C a^{1/2} \beta^{1/2} \varepsilon^{1/2}.$$

For the surface energy, we note that the interface perimeter of the basic cell is bounded above by $C\varepsilon(a_n/\theta_n)$. An analogous calculation shows

that the surface energy of the branched region of \mathbf{m}_∞ is bounded above by $Ca^{1/2}\beta^{1/2}\varepsilon^{1/2}$. Thus, for given a the total energy $E(a)$ of the structure just constructed is at most

$$(3.3) \quad Ca^{1/2}\beta^{1/2}\varepsilon^{1/2} + \frac{4\varepsilon(L-h)}{a} \leq C \left(a^{1/2}\beta^{1/2}\varepsilon^{1/2} + \frac{\varepsilon L}{a} \right),$$

where $h = h(a)$ denotes the width of the branched region, i.e.,

$$h(a) = \sum_{n=1}^{\infty} h_n \leq C \frac{a^{3/2}\beta^{1/2}}{\varepsilon^{1/2}}.$$

Optimizing the right-hand side of (3.3) with respect to a , we obtain

$$a = c_2 \frac{\varepsilon^{1/3}L^{2/3}}{\beta^{1/3}} \quad \text{and} \quad E \leq C\beta^{1/3}\varepsilon^{2/3}L^{1/3}$$

for some constant c_2 .

Finally, just as in the Privorotskiĭ construction of Section 2.1, we should adjust the choices of c_1 and c_2 to assure that $\theta_n < 1$, $a < 1$, and $h < L/2$. An argument analogous to that given in Section 2.1 applies for the case when $a^{1/2}\beta^{1/2}\varepsilon^{-1/2} < 1$. We have shown the following:

PROPOSITION 3.1 *Let E_0 denote the minimum energy of (P3). Then there is a constant C_0 such that*

$$E_0 \leq C_0\beta^{1/3}\varepsilon^{2/3}L^{1/3}$$

whenever $L > \varepsilon/\beta$ and $\varepsilon^{1/3}L^{2/3}\beta^{-1/3} < 1$.

3.2 A Lower Bound

We shall prove a lower bound with the same scaling law. The regime where our bound applies is more restrictive than that of the construction, because we assume $L > 1$ rather than $L > \varepsilon/\beta$. The situation is in many ways parallel to that of Section 2. A major difference is that the energy in Section 2 was local, while the one in this section is nonlocal.

The admissible magnetizations \mathbf{m} are no longer divergence free; rather they have the form $(m_1(x, y), 0)$ with $m_1 = \pm 1$ almost everywhere in Ω . So we cannot express \mathbf{m} as a curl, as we did in Section 2. But \mathbf{m} still determines a sort of stream function via its Helmholtz decomposition. Indeed, with our

usual convention that ∇u is the projection of \mathbf{m} onto the the space of gradients, we can define ϕ (up to an additive constant) by

$$(3.4) \quad (m_1, 0) - (u_x, u_y) = (\phi_y, -\phi_x).$$

Clearly $(\phi_y, -\phi_x)$ is the L^2 projection of \mathbf{m} onto the space of divergence-free vector fields. Notice that the norm of ∇u is controlled by the field energy, and if ∇u is small, then $\mathbf{m} \approx (\phi_y, -\phi_x)$.

The maps $\mathbf{m} \rightarrow \nabla u$ and $\mathbf{m} \rightarrow (\phi_y, -\phi_x)$ are classical singular integral operators, which map $L^p(\mathbb{R}^2) \rightarrow L^p(\mathbb{R}^2)$ for $1 < p < \infty$. Remembering that $\mathbf{m} = 0$ outside Ω , we see that ∇u and $\nabla \phi$ are in L^p for every p . Taking $p > 2$ it follows that u and ϕ are continuous.

THEOREM 3.2 *There exist constants c_0 and C_0 independent of ε , L , and β such that if*

$$(3.5) \quad \frac{\varepsilon^{1/3} L^{2/3}}{\beta^{1/3}} < \frac{c_0}{4}$$

and $L > 1$, then the minimum energy of (P3) satisfies

$$c_0 \varepsilon^{2/3} \beta^{1/3} L^{1/3} \leq E_0 \leq C_0 \varepsilon^{2/3} \beta^{1/3} L^{1/3}.$$

PROOF: Our proof has the same overall structure as that of Theorem 2.2. The upper bound is supplied by Proposition 3.1, so we need only address the lower bound. Let $\mathbf{m} = (m_1, 0)$ be a minimizer of (P3), and note that $\mathbf{m} \in \text{BV}$. It clearly suffices to give a lower bound for $\varepsilon \int_{\Omega} |m_{1y}| dx dy + \beta \int_{\mathbb{R}^2} |\nabla u|^2 dx dy$ since $|m_{1y}| \leq |\nabla \mathbf{m}|$. We shall argue by contradiction, starting from the hypothesis that

$$(3.6) \quad E_0 \leq c_0 \varepsilon^{2/3} \beta^{1/3} L^{1/3}$$

for some $c_0 > 0$. Assuming (3.5) with the same value of c_0 , we shall prove that c_0 cannot be too small. It follows that when c_0 is smaller, (3.6) must fail, and this is the desired lower bound.

Step 1. By (3.6) and Tonelli's theorem (see, e.g., theorem 5.3.5 of [18]), we can find $x_0 \in (-L, L)$ such that

$$(3.7) \quad \varepsilon \int_0^1 |m_{1y}|(x_0, y) dy + \beta \int_0^1 u_x^2(x_0, y) dy \leq \frac{c_0}{2} \frac{\varepsilon^{2/3} \beta^{1/3}}{L^{2/3}}.$$

Step 2. In Section 2 we had $\phi = 0$ at $\partial\Omega$. Here we have no boundary condition for ϕ ; this step provides a substitute. We may assume without loss of generality that $x_0 \geq 0$. Using both (3.6) and (3.4), we have

$$\beta \int_0^1 \int_L^{2L} \phi_y^2(x, y) dx dy \leq c_0 \varepsilon^{2/3} \beta^{1/3} L^{1/3},$$

and hence there exists $x_1 \in (L, 2L)$ such that

$$\int_0^1 \phi_y^2(x_1, y) dy \leq c_0 \frac{\varepsilon^{2/3}}{\beta^{2/3} L^{2/3}}.$$

Adding a constant to ϕ if necessary, we may assume that $\int_0^1 \phi(x_1, y) dy = 0$. It follows that

$$(3.8) \quad \int_0^1 \phi^2(x_1, y) dy \leq \frac{c_0}{\pi^2} \frac{\varepsilon^{2/3}}{\beta^{2/3} L^{2/3}},$$

since the first nonzero Neumann eigenvalue of $-\phi'' = \lambda\phi$ on $(0, 1)$ is π^2 .

Step 3. In proving Theorem 2.2, we gave a lower bound for $\int_0^1 \phi^2 + \frac{1}{4(1-s)} \phi_x^2 dy$ at $x = x_0$. This step provides a substitute for that estimate.

Rather than focus on $\phi(x_0, y)$, it is convenient to work with the primitive of m_1 , i.e., the function $F : [0, 1] \rightarrow \mathbb{R}$ defined by $F'(y) = m_1(x_0, y)$ and $\int_0^1 F(y) dy = \int_0^1 \phi(x_0, y) dy$. Clearly $F' = \pm 1$ almost everywhere on $[0, 1]$. Moreover, the number of times $F' = m_1$ changes sign is controlled by the energy. Indeed, denoting this number by $N(x_0)$, we see from (3.7) that

$$(3.9) \quad N(x_0) \leq \frac{c_0}{4} \frac{\beta^{1/3}}{\varepsilon^{1/3} L^{2/3}}.$$

Applying Lemma 2.7 of [11], we obtain

$$\int_0^1 F^2(y) dy \geq \frac{1}{12(N(x_0) + 1)^2}.$$

Combining this with (3.9) and (3.5) gives

$$(3.10) \quad \int_0^1 F^2(y) dy \geq \frac{4}{12} \frac{\varepsilon^{2/3} L^{4/3}}{c_0^2 \beta^{2/3}}.$$

Step 4. We just obtained a lower bound on $\int F^2 dy$. Now we shall give an upper bound. Taken together, these estimates supply the desired inequality for c_0 .

By the triangle inequality

$$(3.11) \quad |F(y)|_{L^2} \leq |F(y) - \phi(x_0, y)|_{L^2} + |\phi(x_0, y) - \phi(x_1, y)|_{L^2} \\ + |\phi(x_1, y)|_{L^2}.$$

To estimate the first term on the right, we observe that $F - \phi$ has mean value 0, and $(F - \phi)_y = m_1 - \phi_y = u_x$ by (3.4). It follows using (3.7) that

$$\int_0^1 (F(y) - \phi(x_0, y))^2 dy \leq \frac{1}{\pi^2} \int_0^1 (m_1(x_0, y) - \phi_y(x_0, y))^2 dy \\ \leq \frac{c_0}{2\pi^2} \frac{\varepsilon^{2/3}}{\beta^{2/3} L^{2/3}}.$$

To estimate the second term, we observe from (3.4) that $\phi_x = u_y$. Thus

$$\int_0^1 (\phi(x_0, y) - \phi(x_1, y))^2 dy \leq 2L \int_0^1 \int_0^{2L} \phi_x^2 dx dy \\ \leq 2L \int_0^1 \int_0^{2L} u_y^2 dx dy \\ \leq 2L c_0 \frac{\varepsilon^{2/3} L^{1/3}}{\beta^{2/3}}$$

using Hölder's inequality and (3.6). The third term on the right-hand side of (3.11) is estimated by (3.8). And the left-hand side of (3.11) is estimated by (3.10). Assembling these results, (3.11) implies

$$\frac{1}{\sqrt{3}} \leq c_0^{3/2} \left(\sqrt{2} + \frac{1}{\pi L} \left(\frac{1}{\sqrt{2}} + 1 \right) \right).$$

Remembering our hypothesis $L > 1$, we conclude that

$$\frac{1}{\sqrt{3}} \leq c_0^{3/2} \left(\sqrt{2} + \frac{1}{\pi} \left(\frac{1}{\sqrt{2}} + 1 \right) \right)$$

(this is the *only* place in the proof where we use the assumption $L > 1$). The theorem holds for any c_0 small enough to violate this inequality, i.e., for c_0 less than about 0.44. ■

REMARK 3.3 In [11], Kohn and Müller briefly considered a “soft boundary condition” variational problem of the form

$$(3.12) \quad \int_{\Omega} a \phi_x^2 + \varepsilon |\phi_{yy}| dx dy + b \|\phi(L, \cdot)\|_{H^{1/2}}^2$$

subject to $\phi_y = \pm 1$. The discussion here is closely related to this problem with $a = b = \beta$. Indeed, if \mathbf{m} is admissible for (P3) and ϕ is determined from \mathbf{m} by (3.4), then $\phi_x = u_y$ so

$$\beta \int_{\Omega} \phi_x^2 dx dy = \beta \int_{\Omega} u_y^2 dx dy \leq \beta \int_{\mathbb{R}^2} |\nabla u|^2 dx dy.$$

Also, $(\phi_y, -\phi_x) = (-u_x, -u_y)$ for $x > L$, so

$$\beta \|\phi(L, \cdot)\|_{H^{1/2}}^2 \leq \beta \int_0^1 \int_L^\infty |\nabla \phi|^2 dx dy \leq \beta \int_{\mathbb{R}^2} |\nabla u|^2 dx dy.$$

Thus the first and third terms of (3.12) are controlled by the energy (P3). The surface energy is a little different, since $\phi_{yy} = m_{1y} - u_{xy} \neq m_{1y}$. The constraint is also a little different, since $\phi_y = \pm 1 - u_x \neq \pm 1$. These differences are minor, however. The argument given above can also be used to prove a lower bound for (3.12), supplementing the results in [11].

4 Finite Anisotropy and the Full Problem

The preceding sections addressed constrained versions of (P1), representing the limits of small and large anisotropy. We turn now to the full, unconstrained problem:

$$\min_{\mathbf{m} \in \mathcal{A}} \alpha \int_{\Omega} m_2^2 dx dy + \varepsilon \int_{\Omega} |\nabla \mathbf{m}| dx dy + \beta \int_{\mathbb{R}^2} |\nabla u|^2 dx dy$$

in which $\Delta u = \operatorname{div} \mathbf{m}$ and $\Omega = (-L, L) \times (0, 1)$. Let E_0 denote the minimum energy.

THEOREM 4.1 *There are constants c_0 and C_0 independent of ε , L , α , and β such that if*

$$(4.1) \quad \frac{\varepsilon^{1/3} L^{2/3}}{(\min\{\alpha, \beta\})^{1/3}} \leq \frac{c_0}{4}$$

and $L > 1$, then

$$c_0 (\min\{\alpha, \beta\})^{1/3} \varepsilon^{2/3} L^{1/3} \leq E_0 \leq C_0 (\min\{\alpha, \beta\})^{1/3} \varepsilon^{2/3} L^{1/3}.$$

PROOF: The constructions of Sections 2 and 3 are both admissible, so the upper bound is an immediate consequence of Propositions 2.1 and 3.1. To prove the lower bound, we shall combine the arguments used for Theorems 2.2 and 3.2.

Our argument actually gives a lower bound for

$$\eta \int_{\Omega} m_2^2 dx dy + \varepsilon \int_{\Omega} |m_{1y}| dx dy + \eta \int_{\mathbb{R}^2} |\nabla u|^2 dx dy$$

with $\eta = \min\{\alpha, \beta\}$. Suppose

$$(4.2) \quad E_0 \leq c_0 \varepsilon^{2/3} \eta^{1/3} L^{1/3}.$$

Let $\mathbf{m} = (m_1, m_2)$ be the magnetization achieving energy E_0 , and consider its Helmholtz decomposition

$$(4.3) \quad (\phi_y, -\phi_x) = (m_1, m_2) - (u_x, u_y).$$

Step 1. By assumption (4.2) and Tonelli's theorem, there exists $x_0 \in (-L, L)$ such that

$$(4.4) \quad \frac{c_0}{2} \frac{\varepsilon^{2/3} \eta^{1/3}}{L^{2/3}} \geq \int_0^1 \eta m_2^2(x_0, y) + \varepsilon |m_{1y}|(x_0) dy \\ + \eta \int_0^1 u_x^2(x_0, y) + u_y^2(x_0, y) dy,$$

where $|m_{1y}|(x_0)$ denotes the total variation measure (with respect to y) at fixed $x = x_0$. We may assume $x_0 \geq 0$ without loss of generality.

Step 2. Using the same argument as in Step 2 of Theorem 3.2, we see that for some $x_1 \in (L, 2L)$,

$$(4.5) \quad \int_0^1 \phi^2(x_1, y) dy \leq \frac{c_0}{\pi^2} \frac{\varepsilon^{2/3}}{\eta^{2/3} L^{2/3}}.$$

Step 3. As in step 3 of Theorem 3.2, consider the continuous function $F : [0, 1] \rightarrow \mathbb{R}$ satisfying $F'(y) = m_1(x_0, y)$ and $\int_0^1 F(y) dy = \int_0^1 \phi(x_0, y) dy$. We use the arguments of step 2 of Theorem 2.2 with $F(y)$ replacing $\phi(x_0, y)$ and m_2 replacing ϕ_x to see that for $s \in (\frac{1}{2}, 1)$,

$$(4.6) \quad \int_0^1 F^2(y) + \frac{1}{4(1-s)} m_2^2(x_0, y) dy \geq \frac{4s^2}{27c_0^2} \frac{\varepsilon^{2/3} L^{4/3}}{\eta^{2/3}}.$$

Step 4. CASE 1. Suppose

$$\int_0^1 \frac{1}{4(1-s)} m_2^2(x_0, y) dy \geq \frac{2s^2}{27c_0^2} \frac{\varepsilon^{2/3} L^{4/3}}{\eta^{2/3}}.$$

From (4.4) we have

$$\int_0^1 m_2^2(x_0, y) dy \leq \frac{c_0}{2} \frac{\varepsilon^{2/3}}{L^{2/3} \eta^{2/3}}.$$

Therefore

$$c_0^3 \geq \frac{16s^2(1-s)L^2}{27}.$$

Choosing s such that $1-s = 1/8L^2$ and remembering that $L > 1$, we conclude that

$$(4.7) \quad c_0^3 \geq \frac{2}{27} \left(\frac{7}{8}\right)^2.$$

Step 4. CASE 2. Suppose

$$\int_0^1 F^2(y) dy \geq \frac{2s^2}{27c_0^2} \frac{\varepsilon^{2/3} L^{4/3}}{\eta^{2/3}}.$$

The triangle inequality implies that

$$(4.8) \quad |F(y)|_{L^2} \leq |F(y) - \phi(x_0, y)|_{L^2} + |\phi(x_0, y) - \phi(x_1, y)|_{L^2} + |\phi(x_1, y)|_{L^2}.$$

Since $(F - \phi)_y = u_x$ and $F - \phi$ has mean value 0, (4.4) implies that

$$|F(y) - \phi(x_0, y)|_{L^2} \leq \frac{\sqrt{c_0}}{\sqrt{2\pi}} \frac{\varepsilon^{1/3}}{\eta^{1/3} L^{1/3}}.$$

Since $\phi_x = u_y - m_2$, we may argue as for Theorem 3.2 to get

$$|\phi(x_0, y) - \phi(x_1, y)|_{L^2} \leq \sqrt{2c_0} \frac{\varepsilon^{1/3} L^{2/3}}{\eta^{1/3}}.$$

Combining these estimates with (4.5) and (4.8) and using the hypothesis $L \geq 1$, we get

$$\left(\frac{1}{\pi} \left(\frac{1}{\sqrt{2}} + 1\right) + \sqrt{2}\right) c_0^{3/2} \geq s \sqrt{\frac{2}{27}}.$$

We must keep the choice $1-s = 1/8L^2$ for consistency, so our conclusion in this case is

$$(4.9) \quad \left(\frac{1}{\pi} \left(\frac{1}{\sqrt{2}} + 1\right) + \sqrt{2}\right) c_0^{3/2} \geq \frac{7}{8} \sqrt{\frac{2}{27}}.$$

Thus the theorem holds whenever c_0 is small enough to violate both (4.7) and (4.9), i.e., when c_0 is less than about 0.24. ■

In summary, our variational problem (P1) has two distinct regimes, corresponding to $\alpha \ll \beta$ and $\alpha \gg \beta$, with crossover where $\alpha \sim \beta$. These regimes correspond to small and large anisotropy, respectively, since $\alpha = \kappa|\mathbf{M}|^2$ and $\beta = |\mathbf{M}|^2$ where κ is the anisotropy. The branched structure of Section 2 is preferred when $\alpha \ll \beta$ and that of Section 3 when $\alpha \gg \beta$; the two are comparable when $\alpha \sim \beta$. In each case the better of two constructions achieves the optimal energy up to a constant factor (independent of $\alpha, \beta, \varepsilon$, and L). These assertions apply, however, only if the surface tension is sufficiently small relative to the other parameters, i.e., if (4.1) holds, and we have proved the lower bounds only when $L > 1$.

We close by checking that our results are physically meaningful, in the sense that the smallest length scale of the construction is large compared to the width ε/α of a Bloch wall. If $\alpha \leq \beta$ (equivalent to $\kappa \leq 1$), then we are assuming

$$(4.10) \quad \frac{\varepsilon^{1/3} L^{2/3}}{\alpha^{1/3}} \leq C,$$

and the construction is that of Section 2. Its smallest length scale is of order $\varepsilon^{2/3} L^{1/3} \alpha^{-2/3}$. This is larger than ε/α if $\varepsilon/\alpha < L$. But (4.10) gives $\varepsilon/\alpha < C^3 L^{-2}$. So our result is meaningful if L is large enough. If $\alpha \geq \beta$ (equivalent to $\kappa \geq 1$), then we are assuming

$$(4.11) \quad \frac{\varepsilon^{1/3} L^{2/3}}{\beta^{1/3}} \leq C,$$

and the construction is that of Section 3. Its smallest length scale is of order $\varepsilon^{2/3} L^{1/3} \beta^{-2/3}$. This is larger than ε/α if $\varepsilon/\beta < L\kappa^3$. But (4.11) gives $\varepsilon/\beta < C^3 L^{-2}$. So our result is meaningful if $L\kappa$ is large enough (and in particular if L is large enough, since $\kappa > 1$).

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