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On the strong attraction limit for a class of nonlocal interaction energies

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ABSTRACT

This note concerns the problem of minimizing a certain family of non-local energy functionals over measures on $\mathbb{R}^n$, subject to a mass constraint, in a strong attraction limit. In these problems, the total energy is an integral over pair interactions of attractive-repulsive type. The interaction kernel is a sum of competing power law potentials with attractive powers $\alpha \in (0,\infty)$ and repulsive powers associated with Riesz potentials. The strong attraction limit $\alpha \to \infty$ is addressed via Gamma-convergence, and minimizers of the limit are characterized in terms of an isodiametric capacity problem. We also provide evidence for symmetry-breaking of minimizers in high dimensions.
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1. Introduction and statement of the results

We consider mass-constrained variational problems of the form

$$\begin{cases} \text{Minimize} \quad \mathcal{E}_{\alpha,\lambda}(\mu) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_{\alpha,\lambda}(x-y)\, d\mu(x)d\mu(y) \\ \text{over } \mathcal{P} := \{\mu \text{ Borel measure on } \mathbb{R}^n : \mu(\mathbb{R}^n)=1\}, \end{cases} \tag{1}$$

where the interaction kernel is given by

$$K_{\alpha,\lambda}(x-y) := |x-y|^{\alpha} + |x-y|^{-\lambda} \qquad \text{with } \alpha \in (0,\infty) \text{ and } \lambda \in (0,n). \tag{2}$$

These kernels are attractive at long range, with the attraction controlled by the exponent α, and strongly repulsive at short range, with the repulsion controlled by λ, see Fig. 1. Since the kernels are lower semicontinuous, locally integrable, and grow at infinity, by the results of [5,14], Problem (1) has a global minimizer.

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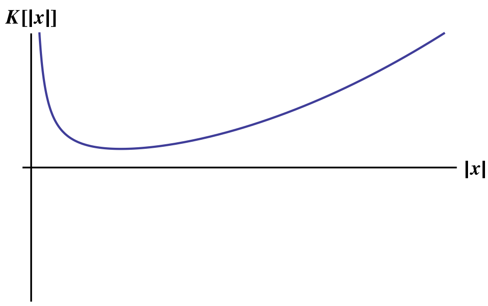

Fig. 1. Shape of the interaction kernel $K_{\alpha,\lambda}(|\cdot|)$ for $\alpha \in (0,\infty)$ and $\lambda \in (0,n)$.

Variational problems of the form (1) arise in connection with a class of models for aggregation and self-assembly that have recently received much attention (see for example, [2] and the references therein). In those models, a population density ρ evolves according to the equation

$$\rho_t + \nabla\cdot(\rho\mathbf{v}) = 0\,, \qquad \mathbf{v} = -\nabla K_{\alpha,\lambda} * \rho,$$

which is the gradient flow of the energy $\mathcal{E}_{\alpha,\lambda}(\mu)$ on absolutely continuous measures $\mu = \rho\, dx$ in the 2-Wasserstein metric (cf. [6]). Energy minimizers represent stable steady-states of the aggregation process.

Here, we study the minimization problem (1) in the strong attraction regime where $\alpha \to \infty$. In this limit, finite energy alone restricts the support of a measure to have diameter no larger than one.

In Fig. 2, we present a few particle simulations in dimension $n = 2$ which suggest that as α increases, minimizers concentrate on the boundary of the ball of diameter 1 for some values of λ; but spread out (non-uniformly) over the ball for larger values of λ. A broader range of behavior is expected for other parameters and in higher dimensions (see for example Fig. 3).

Our first result is that in the limit as $\alpha \to \infty$, Problem (1) approaches the problem of minimizing

$$\mathcal{E}_{\infty,\lambda}(\mu) := \begin{cases} \displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^{-\lambda}\, d\mu(x)d\mu(y) & \text{if}\,\mathrm{diam}(\mathrm{supp}\,\mu) \le 1 \\ +\infty & \text{otherwise} \end{cases} \tag{3}$$

over $\mathcal{P}$. The limit is understood in the sense of Gamma-convergence (cf. Section 3).

Theorem 1 (*Strong Attraction Limit*)**.** *Let $\lambda \in (0,n)$. Then $\mathcal{E}_{\alpha,\lambda} \xrightarrow{\Gamma} \mathcal{E}_{\infty,\lambda}$ as $\alpha \to \infty$ in the weak topology of measures.*

The limiting problem admits a solution:

Theorem 2 (*Existence*)**.** *The functional $\mathcal{E}_{\infty,\lambda}$ has a global minimizer in $\mathcal{P}$.*

The proofs of Theorems 1 and 2 are presented in Section 3.

Remark. In the literature, the interaction kernel is sometimes normalized to

$$\tilde{K}_{\alpha,\lambda}(x-y) := \frac{1}{\alpha}|x-y|^{\alpha} + \frac{1}{\lambda}|x-y|^{-\lambda}\,, \tag{4}$$

which assumes its minimum when $|x-y| = 1$ (cf. [4]). This normalization can be achieved by acting on $\mathcal{P}$ with a suitable dilation. For the normalized kernel, the conclusions of Theorem 1 hold with $\frac{1}{\lambda}\mathcal{E}_{\infty,\lambda}$ as the limiting functional, and Theorem 2 applies without change.

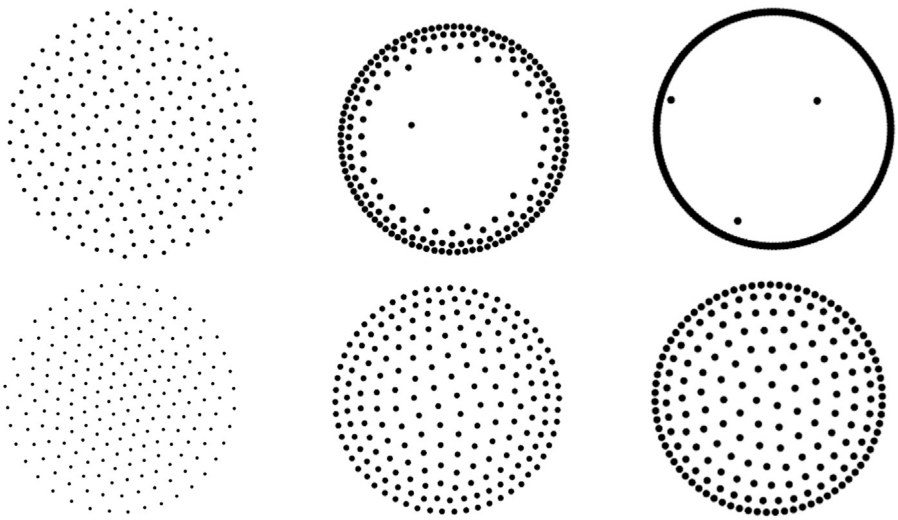

Fig. 2. Particle simulations associated with minimizers of (1) in dimension $n = 2$. Each particle $i = 1, \ldots, N$ is tracked via the system of ODEs

$$\frac{dX_i}{dt} = -\frac{1}{N}\sum_{j=1}^{N} \nabla K_{\alpha,\lambda}(X_i - X_j)$$

until the configuration stabilizes. The interaction kernel is given by Eq. (2), where the exponent of attraction ranges through $\alpha = 2, 20, 200$ (from left to right). *Top row:* Repulsive term replaced with the logarithmic term $-\log|x-y|$ that corresponds to the Newton potential ($\lambda = n-2$) in two dimensions. *Bottom row:* Exponent of repulsion $\lambda = 1$, which lies in the super-Newtonian regime.

We then consider the nature of minimizers for the limiting problem $\mathcal{E}_{\infty,\lambda}$. This turns out to be a rather subtle question; indeed, due to the diameter constraint, the functional $\mathcal{E}_{\infty,\lambda}$ is non-convex on $\mathcal{P}$. Our approach is to rephrase the limiting problem as an isodiametric capacity problem. More precisely, for $\lambda \in (0, n)$ and a set $A \subset \mathbb{R}^n$, we define the λ-*capacity* of A to be

$$C_\lambda(A) := \left(\inf_{\nu\in\mathcal{P}} \left\{ I_\lambda(\nu) \;\middle|\; \operatorname{supp}\nu \subset A \right\}\right)^{-1}, \tag{5}$$

where

$$I_\lambda(\nu) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^{-\lambda}\, d\nu(x)d\nu(y).$$

In the special case where $n = 3$ and $\lambda = 1$, C_λ agrees (up to a multiplicative constant) with the electrostatic capacity of A. It is straightforward (cf. Lemma 7) to show that

$$\inf_{\nu\in\mathcal{P}} \mathcal{E}_{\infty,\lambda}(\nu) = \left(\sup_{A\subset\mathbb{R}^n} \left\{ C_\lambda(A) \;\middle|\; \operatorname{diam}(A) \le 1 \right\}\right)^{-1},$$

with a direct relationship between the optimal measure ν on the left and optimal set A on the right. This allows us to exploit tools from potential theory (cf. [10]) to partially characterize the support of minimizers.

Theorem 3 (*Properties of Minimizers of the Limit Problem*)**.** *Let $n \geq 3$, $\lambda \in (0, n)$, and assume that μ minimizes $\mathcal{E}_{\infty,\lambda}$ on $\mathcal{P}$. Then there exists a convex body W of constant width 1 such that*

$$\begin{cases} \operatorname{supp}\mu \subset \partial W\,, & \lambda \in (0, n-2) \text{ (sub-Newtonian)},\\ \operatorname{supp}\mu = \partial W\,, & \lambda = n-2 \text{ (Newtonian)},\\ \operatorname{supp}\mu = W\,, & \lambda \in (n-2, n) \text{ (super-Newtonian)}. \end{cases}$$

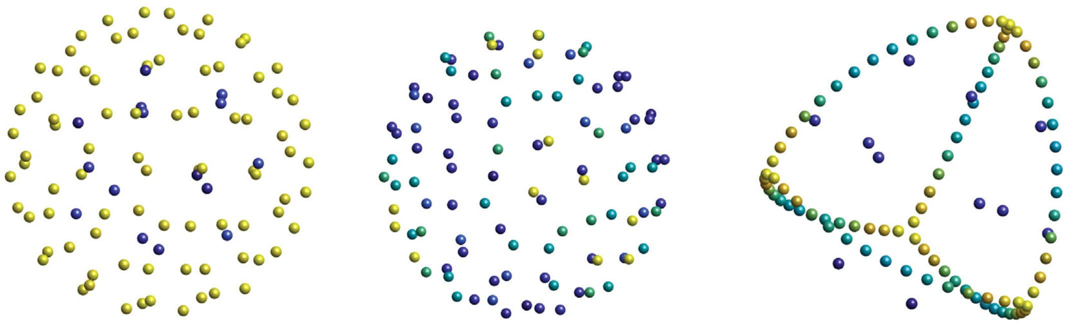

Fig. 3. Results of 3D particle simulations with $\lambda = 0.01$ and $\alpha = 2, 20, 200$ (from left to right).

The set W may depend on μ as well as λ. We do not know whether minimizers are unique up to translation, and whether $\mathcal{E}_{\infty,\lambda}$ admits additional critical points, including local minima. The proof of Theorem 3 is presented in Section 4.

The theorem extends to lower dimensions as follows. For $n = 1$, the entire range $\lambda \in (0, 1)$ is super-Newtonian, and the support of any minimizing measure is an interval of length one. In dimension $n = 2$, the entire range $\lambda \in (0, 2)$ is super-Newtonian as well, and the support of any minimizing measure is a planar convex set W of constant width 1. The role of the Newton potential $|x - y|^{2-n}$ is played by the logarithmic kernel $-\log|x - y|$; in this case, the support of a minimizer is the boundary of a planar convex set of constant width 1.

In Section 5, we prove the following result pertaining to the asymmetry of minimizers in high space dimensions. Precisely, we prove

Theorem 4 (*Asymmetry of Minimizers in High Dimensions*)**.** *For every $\lambda > 0$ there exists N such that for all $n \geq N$,*

$$\sup\left\{C_\lambda(A) \;\middle|\; A \subset \mathbb{R}^n, \operatorname{diam}(A) \leq 1\right\} > C_\lambda(B^{(n)}_{1/2}),$$

where $B^{(n)}_{1/2} = \{x \in \mathbb{R}^n : |x| \leq \frac{1}{2}\}$.

This demonstrates that for any fixed value of $\lambda > 0$, the ball ceases to be optimal when n is sufficiently large. Thus, in this regime optimal measures are supported on the boundary of sets that are not radially symmetric. As a result, minimizers of $\mathcal{E}_{\alpha,\lambda}$ in high dimensions must also be asymmetric when α is large.

The prospect of symmetry-breaking presents an interesting, largely open, question. Even in low space dimensions, we suspect that when $0 < \lambda \ll n-2$ the maximal capacity among bodies of given diameter may be achieved by non-symmetric sets, and that the equilibrium measure may be supported on a proper subset of the boundary. For example, in Fig. 3 we present the results of 3D particle simulations for $\lambda = 0.01$ (which lies in the sub-Newtonian regime) and respectively, $\alpha = 2, 20, 200$. The simulations suggest that minimizers are asymmetric for large α. However, the number of particles is too small to draw conclusions about the supports of minimizing measures.

2. Related work and further questions

2.1. Comparisons with related work

According to Theorem 3, every minimizer μ of the functional $\mathcal{E}_{\infty,\lambda}$ is supported on a convex body W_μ of constant width 1, and the following relations, summarized in Table 1, hold true.

Table 1
Characteristics of minimizers of $\mathcal{E}_{\infty,\lambda}$ in terms of a body W_μ of constant width.

Repulsion	Geometry
$\lambda < n-2$	$\operatorname{supp}\mu \subset \partial W_\mu$
$\lambda = n-2$	$\operatorname{supp}\mu = \partial W_\mu$
$\lambda > n-2$	$\operatorname{supp}\mu = W_\mu$

In particular, the Hausdorff dimension of μ satisfies

$$\dim(\operatorname{supp}\mu) \begin{cases} \leq n-1\,, & \lambda \in (0, n-2)\,, \\ = n-1\,, & \lambda = n-2\,, \\ = n\,, & \lambda \in (n-2, n)\,. \end{cases}$$

To offer some perspective, note that classical results of geometric measure theory imply that $\dim(\operatorname{supp}\mu) \geq \lambda$ for every Borel measure μ with $\mathcal{E}_{\infty,\lambda}(\mu) < \infty$ (see for example Theorem 4.13 in [8]). For minimizers of energy functionals defined by attractive-repulsive pair interaction kernels, a stronger lower bound was obtained in [1, Theorem 1]. Specifically, minimizers of $\mathcal{E}_{\alpha,\lambda}$ in the sub-Newtonian regime $\lambda \in (0, n-2)$ satisfy

$$\dim(\operatorname{supp}\mu) \geq \lambda + 2\,. \tag{6}$$

When $\lambda \in (n-3, n-2)$ this lower bound exceeds $n-1$, and in particular exceeds the dimension of the support of the corresponding minimizer of $\mathcal{E}_{\infty,\lambda}$. The results of [1] apply more generally to *local minimizers*, in an optimal transport topology, for a larger class of attractive-repulsive functionals with integrable singularities at the origin. In light of (6), which holds for all $\alpha > 0$, the dimensional reduction of the support for $\lambda \leq n-2$ (cf. Theorem 3 or Table 1) is only achieved in the limit. In this limit, the dimension of minimizers are strictly smaller then those in the finite regime, a consequence of the strength of the diameter constraint.

Through Theorem 1 and Lemma 7 (below), the question of what the minimizers of the limiting functional look like is transformed into an isodiametric capacity problem: *For a given $\lambda \in (0, n)$, which sets of diameter 1 have the largest λ-capacity?* Although for any given set $W \subset \mathbb{R}^n$ the equilibrium measure that realizes the capacity is unique, there could be more than one capacity-maximizing set.

One candidate for a set that maximizes capacity among sets of diameter 1 is the ball of radius $\frac{1}{2}$, which uniquely maximizes volume under the diameter restriction. For each $\lambda \in (n-2, n)$, the equilibrium measure on the ball is a well-known positive, radially symmetric density, and for $\lambda \leq n-2$ it is the uniform measure on the boundary sphere [10, p. 163]. Note, however, that the ball *minimizes* capacity among sets of given volume, indicating competition between size and shape in the isodiametric problem.

There are a number of related results for the weak repulsion regime (corresponding to $\lambda < 0$) which imply that the support of minimizers has dimension zero [1, Theorem 2] provided that the pair interaction kernel vanishes more than quadratically as $|x-y| \to 0$. In particular, the variance is maximized, among probability measures on $\mathbb{R}^n$ whose support has diameter one, by the uniform measure on the vertices of the unit simplex [11].

2.2. Restricting problem (1) to densities and sets

In an interesting variant of Problem (1), the minimization is restricted to absolutely continuous probability measures $\mu = \rho dx$ with density bounded by $\rho \leq m^{-1}$ for some $m > 0$.

$$\begin{cases} \text{Minimize} \quad \mathcal{E}'_{\alpha,\lambda}(\rho) := \displaystyle\int_{R^n}\int_{R^n} K_{\alpha,\lambda}(x-y)\rho(x)\rho(y)\,dxdy \\ \text{over } \mathcal{A}_m := \left\{\rho \in L^1(\mathbb{R}^n) \;\middle|\; 0 \leq \rho \leq m^{-1}\,,\ \displaystyle\int_{\mathbb{R}^n} \rho\,dx = 1\right\}. \end{cases} \tag{7}$$

The density constraint plays the role of an additional repulsive term in the energy. This is relevant for biological aggregation problems, where the density of individuals cannot exceed a certain critical value. By rescaling, Problem (7) is equivalent to minimizing $\mathcal{E}'_{\alpha,\lambda}(\rho)$ among measures of mass m, subject to the density constraint $\rho \leq 1$. Unlike Problem (1), the mass m does not scale out of the problem. It is known that for each $\alpha > 0$ and $\lambda \in (0, n)$, the functional $\mathcal{E}'_{\alpha,\lambda}$ has a minimizer on $\mathcal{A}_m$ for any $m > 0$ (cf. [7]).

The proofs of Theorems 1 and 2 continue to hold over the set of probability measures of density at most m^{-1} in the weak topology on L^1 (which is strictly stronger than the topology of $\mathcal{P}$).

Corollary 5 (*Strong Attraction Limit with Density Constraint*). *For $\lambda \in (0, n)$ and $\mu \in \mathcal{P}$, let $\mathcal{E}'_{\alpha,\lambda}$ be as in Problem (7), and define $\mathcal{E}'_{\infty,\lambda}(\rho) := \mathcal{E}_{\infty,\lambda}(\rho\, dx)$ for $\rho \in L^1$. Then*

(1) $\mathcal{E}'_{\alpha,\lambda} \xrightarrow{\Gamma} \mathcal{E}'_{\infty,\lambda}$ as $\alpha \to \infty$ in the weak topology on L^1.
(2) For each $m \leq |B_{\frac{1}{2}}|$, the functional $\mathcal{E}'_{\infty,\lambda}$ attains a global minimum on $\mathcal{A}_m$.

The assumption on m guarantees that the energy of the uniform measure on $B_{\frac{1}{2}}$ remains bounded as $\alpha \to \infty$ (see the proof of Theorem 2). As $m \to 0$, the measures corresponding to a sequence of minimizers converge (up to translations, along suitable subsequences, weakly in L^1) to minimizers of $\mathcal{E}_{\infty,\lambda}$.

Problem (7) is of interest also when m is large. Under certain assumptions on α and λ, $\mathcal{E}'_{\alpha,\lambda}$ is minimized for m sufficiently large by the uniform probability density on a set S of volume m [4,9,12]. In the context of aggregation models, this indicates the formation of a swarm. A minimizing set is the solution of the purely geometric, non-local shape optimization problem

$$\begin{cases} \text{Minimize} \quad \mathcal{E}''_{\alpha,\lambda}(S) := \mathcal{E}_{\alpha,\lambda}(\nu_S) \\ \text{over } \mathcal{S}_m := \left\{ S \subset \mathbb{R}^n \;\middle|\; |S| = m \right\}, \end{cases} \tag{8}$$

where ν_S is the uniform probability measure on S. It turns out that the infimum in Problem (8) agrees with Problem (7), but it is not always attained. If the density of a minimizer of $\mathcal{E}'_{\alpha,\lambda}$ on $\mathcal{A}_m$ falls strictly between 0 and m^{-1} on all or part of its support, then the shape optimization problem (8) has no solution [4, Theorem 4.4], indicating a failure to fully aggregate. In this case, minimizing sequences for Problem (8) diverge due to oscillations. When m is too small, typically $\rho < m^{-1}$ everywhere (cf. [4,9,12]), preventing even partial aggregation.

All known solutions of the shape optimization problem (8) are radially symmetric, and in many cases they are large balls (cf. [4,9,12]). It may be possible to discover interesting examples of symmetry-breaking in the strong-attraction limit, using Corollary 5 and the known relation between Problems (7) and (8).

We are not aware of any explicit characterization of the minimizers for $\mathcal{E}'_{\infty,\lambda}$ on $\mathcal{A}_m$, even in the Newtonian case. Suppose that W maximizes capacity among sets of given diameter. Since the density constraint prevents minimizers to concentrate on a lower-dimensional set, one may wonder whether a thin neighborhood of ∂W might appear as a solution to Problem (8), and whether such a solution persists for sufficiently large finite values of α? When W is not a ball, this could give rise to symmetry-breaking in Problems (7) and (8).

3. Convergence

We begin by recalling a few definitions. Given a topological space X, let $(G_n)_n$ be a sequence of functions on X. We say that (G_n) **Gamma-converges** to a function G ($G_n \xrightarrow{\Gamma} G$) if the following two conditions hold for every $x \in X$:

- *Lower bound inequality:* for all sequences $(x_n)_n \subset X$ such that $x_n \to x \in X$,

$$\liminf_{n\to\infty} G_n(x_n) \geq G(x)\,;$$

- *Upper bound inequality:* for all $x \in X$ there exists a sequence $(x_n)_n \subset X$ such that $x_n \to x$ and

$$\limsup_{n\to\infty} G_n(x_n) \le G(x).$$

Gamma-convergence has many useful implications, the most important of which is that if x_n minimizes G_n over X, then every cluster point of the sequence (x_n) minimizes G over X (cf. [3]).

Given a sequence of measures $(\mu_n)_n \subset \mathcal{P}$, we say $(\mu_n)_n$ **converge weakly** to $\mu \in \mathcal{P}$ ($\mu_n \rightharpoonup \mu$) if

$$\lim_{n\to\infty} \int \phi \, d\mu_n = \int \phi \, d\mu$$

for every bounded continuous function ϕ on $\mathbb{R}^n$. This induces the weak topology on $\mathcal{P}$.

Proof of Theorem 1. Let $\mu \in \mathcal{P}$ be given. In the case where $\operatorname{diam}(\operatorname{supp}\mu) > 1$, choose two points $p, q \in \operatorname{supp}\mu$ with $|p-q| > 1$. By continuity of the distance function, there exist open neighborhoods U, V of p and q such that $\operatorname{dist}(U,V) > 1$. For any sequence of measures (μ_n) with $\mu_n \rightharpoonup \mu$ in $\mathcal{P}$, we have

$$\begin{aligned}\mathcal{E}_{\alpha,\lambda}(\mu_n) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^{\alpha} + |x-y|^{-\lambda}\, d\mu_n(x) d\mu_n(y) \\ &\ge \big(\operatorname{dist}(U,V)\big)^{\alpha} \mu_n(U)\mu_n(V)\,.\end{aligned}$$

Since $\liminf \mu_n(U) \ge \mu(U) > 0$ and likewise for V, it follows that $\mathcal{E}_{\alpha_n,\lambda}(\mu_n) \to \infty$ along every sequence (α_n) with $\alpha_n \to \infty$, verifying simultaneously the lower and upper bound inequalities for this case.

Otherwise, $\operatorname{diam}(\operatorname{supp}\mu) \le 1$. To see the lower bound inequality, let (μ_n) be a sequence in $\mathcal{P}$ that converges weakly to μ, and let $t > 0$. For every $\alpha > 0$,

$$\mathcal{E}_{\alpha,\lambda}(\mu_n) \ge \int_{\mathbb{R}^n}\int_{R^n} \min\{|x-y|^{-\lambda}, t\}\, d\mu_n(x) d\mu_n(y).$$

Since $\mathbb{R}^n \times \mathbb{R}^n$ is separable, the product measures $\mu_n \times \mu_n$ converge weakly to $\mu \times \mu$, and thus for any sequence (α_n),

$$\liminf_{n\to\infty} \mathcal{E}_{\alpha_n,\lambda}(\mu_n) \ge \int_{\mathbb{R}^n}\int_{R^n} \min\{|x-y|^{-\lambda}, t\}\, d\mu(x) d\mu(y).$$

By monotone convergence, taking $t \to \infty$ yields the lower bound inequality.

The upper bound inequality is achieved by a sequence of properly chosen dilations of μ. Given a sequence $\alpha_n \to \infty$, set $\beta_n = e^{\frac{1}{\sqrt{\alpha_n}}}$, and define a sequence of Borel measures by

$$\mu_n(A) = \mu(\beta_n A)\,, \qquad n \ge 1.$$

Since $\beta_n \to 1$, clearly $\mu_n \rightharpoonup \mu$. We estimate

$$\begin{aligned}\mathcal{E}_{\alpha_n,\lambda}(\mu_n) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^{\alpha_n} + |x-y|^{-\lambda}\, d\mu_n(x) d\mu_n(y) \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Big(\frac{|x-y|}{\beta_n}\Big)^{\alpha} + \Big(\frac{|x-y|}{\beta_n}\Big)^{-\lambda} d\mu(x) d\mu(y) \\ &\le e^{-\sqrt{\alpha_n}} + e^{\frac{\lambda}{\sqrt{\alpha_n}}} \mathcal{E}_{\infty,\lambda}(\mu)\,.\end{aligned}$$

We have used that $|x-y| \le 1$ on the support of μ to bound the first summand of the integrand, and inserted the definition of the limiting functional into the second summand. The desired inequality follows upon taking $n \to \infty$. □

The proof of Theorem 2 requires a compactness argument. To this end one often resorts to an application of Lions' concentration compactness principle for probability measures (cf. [15, Section 4.3]) which asserts that every sequence $(\mu_n)_n$ in $\mathcal{P}$ has a subsequence $(\mu_{n_k})_k$ satisfying one of the three following alternatives: (i) tightness up to translation (ii) vanishing (mass sent to infinity) or (iii) dichotomy (splitting). A standard technique is to show that (ii) and (iii) cannot happen, yielding (i) which, precisely, means: There exists a sequence $(y_k)_k \subset \mathbb{R}^n$ such that for all $\varepsilon > 0$ there exists $R > 0$ with the property that $\mu_{n_k}(B_R(y_k)) \geq 1 - \varepsilon$ for all k.

However, in our simpler case we may just as well directly prove tightness to obtain compactness.

Lemma 6. *Let $\mathcal{E}_{\alpha,\lambda}$ be as in Eq. (1), let (α_n) be a sequence with $\alpha_n \to \infty$, and fix $\lambda \in (0, n)$. Then every sequence (μ_n) in $\mathcal{P}$ such that $\mathcal{E}_{\alpha_n,\lambda}(\mu_n)$ is bounded has a subsequence that converges weakly, up to translations, to some $\mu \in \mathcal{P}$.*

Proof. Let (μ_n) be such that

$$\sup_{n\in\mathbb{N}} \mathcal{E}_{\alpha_n,\lambda}(\mu_n) < \infty.$$

Fix an $R > 1$. We have the lower bounds

$$\begin{aligned}\mathcal{E}_{\alpha_n,\lambda}(\mu_n) &\geq \iint_{|x-y|\geq R} R^{\alpha_n}\, d\mu_n(x) d\mu_n(y)\\ &\geq R^{\alpha_n} \int_{\mathbb{R}^n} \mu_n\big(\mathbb{R}^n \setminus B_R(y)\big)\, d\mu_n(y)\\ &\geq R^{\alpha_n}\big(1 - \sup_{y\in\mathbb{R}^n} \mu_n(B_R(y))\big)\,.\end{aligned}$$

Since the left hand side is bounded by assumption while $\alpha_n \to \infty$, it follows that $\sup_{y\in\mathbb{R}^n} \mu_n(B_R(y)) \to 1$. This establishes the first alternative of Lions' concentration compactness principle.

Choose a sequence $(y_n) \subset \mathbb{R}^n$ such that

$$\lim_{n\to\infty} \mu_n(B_2(y_n)) = 1.$$

Given $\varepsilon > 0$, let N be so large that $\mu_n(B_2(y_n)) \geq 1 - \varepsilon$ for all $n > N$. Then choose R so large that $\mu_n(B_R(y_n)) \geq 1 - \varepsilon$ for $n = 1, \ldots, N$. Taking $R \geq 2$ ensures that $\mu_n(B_R(y_n)) \geq \mu_n(B_2(y_n)) \geq 1 - \varepsilon$ also for $n > N$.

Let $(\tilde{\mu}_n)_n$ be the sequence of translates of μ_n defined by

$$\tilde{\mu}_n(A) = \mu_n(y_n + A)\,, \quad n \geq 1$$

for each Borel set $A \subset \mathbb{R}^n$. Since $(\tilde{\mu}_n)$ is tight. Prokhorov's theorem yields a subsequence $(\tilde{\mu}_{n_k})_k$ that converges weakly in $\mathcal{P}$. □

Proof of Theorem 2. Let (α_n) be a nonnegative sequence with $\alpha_n \to \infty$, and let (μ_n) be a sequence of measures such that each μ_n minimizes $\mathcal{E}_{\alpha_n,\lambda}$. We will prove that $(\mathcal{E}_{\alpha_n,\lambda}(\mu_{\alpha_n}))_n$ is bounded, and then apply Lemma 6.

Let ν be the uniform probability measure on the ball of radius $\frac{1}{2}$. Since μ_n minimizes $\mathcal{E}_{\alpha_n,\lambda}$ for each n, we have

$$\begin{aligned}\mathcal{E}_{\alpha_n,\lambda}(\mu_n) &\leq \mathcal{E}_{\alpha_n,\lambda}(\nu)\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^{\alpha} + |x-y|^{-\lambda}\, d\nu(x) d\nu(y)\end{aligned}$$

$$\leq 1 + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^{-\lambda}\, d\nu(x)d\nu(y)$$
$$< \infty\,.$$

In the last two inequalities, we have used that the support of ν has diameter one, and that the kernel is locally integrable.

By Lemma 6 there exists a subsequence μ_{n_k} that converges weakly up to translation, to some measure $\mu \in \mathcal{P}$. Since the functionals are translation invariant, we may assume that the sequence of minimizers itself that has a subsequence converging weakly to μ. By the properties of the Gamma-limit, μ is a global minimizer of $\mathcal{E}_{\infty,\lambda}$. □

4. Characterization of minimizers

We recall some classical results from potential theory. First, recall the λ-*capacity* of a set $A \subset \mathbb{R}^n$ previously defined in (5) as the reciprocal of the minimum of the repulsive energy I_λ over measures supported in A. If A is a compact set of positive Lebesgue measure, the λ-capacity is finite by the local integrability of the Riesz-potential, and the supremum is achieved by some measure $\mu \in \mathcal{P}$. Since I_λ is positive definite, the minimizer is unique.

The next lemma relates the minimization problem for $\mathcal{E}_{\infty,\lambda}$ to an isodiametric capacity problem.

Lemma 7. *Let $n \geq 1$, $\lambda \in (0,n)$. Then*

$$\inf_{\nu\in\mathcal{P}} \mathcal{E}_{\infty,\lambda}(\nu) = \Big(\sup_{A\subset\mathbb{R}^n} \big\{C_\lambda(A) \;\big|\; \mathrm{diam}(A) \leq 1\big\}\Big)^{-1}.$$

Furthermore, the infimum on the left hand side is attained for some measure μ with $\mathrm{diam}(\mathrm{supp}\,\mu) = 1$, *and the supremum on the right hand side is attained for some convex body $W \subset \mathbb{R}^n$ of constant width 1 containing the support of μ. Conversely, if W maximizes λ-capacity among bodies of constant width, then the equilibrium measure on W attains the minimum on the left hand side.*

Proof. We split the minimization problem for $\mathcal{E}_{\infty,\lambda}$ into two steps,

$$\inf_{\nu\in\mathcal{P}} \mathcal{E}_{\infty,\lambda}(\nu) = \inf_{A\subset\mathbb{R}^n}\Big\{\inf_{\nu\in\mathcal{P}}\big\{I_\lambda(\nu) \;\big|\; \mathrm{supp}\,\nu \subset A\big\} \;\Big|\; \mathrm{diam}(A) \leq 1\Big\}$$
$$= \Big(\sup_{A\subset\mathbb{R}^n} \big\{C_\lambda(A) \;\big|\; \mathrm{diam}(A) = 1\big\}\Big)^{-1}.$$

By Theorem 2, the infimum on the left hand side is attained for some measure $\mu \in \mathcal{P}$. Clearly, $\mathrm{diam}(\mathrm{supp}\,\mu) = 1$, since otherwise μ could be rescaled to lower the value of $\mathcal{E}_{\infty,\lambda}$. Moreover, $A = \mathrm{supp}\,\mu$ achieves the supremum on the right hand side, and μ is the equilibrium measure for the capacity $C_\lambda(A)$. Since the capacity increases monotonically under inclusion, we may replace A by its convex hull. The last claim follows since every closed convex set of diameter 1 is contained in a convex body W of constant width 1 (cf. [13]). Since $C_\lambda(W) = C_\lambda(\mathrm{supp}\,\mu)$, if follows that μ is the equilibrium measure also for W. □

We can now appeal to known properties of equilibrium measures in classical potential theory. Given a probability measure μ on $\mathbb{R}^n$ and $\lambda \in (0,n)$, we define the corresponding potential by

$$\phi_\lambda^\mu(x) := \int_{\mathbb{R}^n} |x-y|^{-\lambda}\, d\mu(y)\,.$$

For any $x \in \mathbb{R}^n$, the integral is well-defined and strictly positive, though possibly infinite. The function has the following regularity property outside the support of μ.

Lemma 8. *Let μ be a probability measure on $\mathbb{R}^n$. On $\mathbb{R}^n \setminus \operatorname{supp}\mu$, the potential ϕ^λ_μ is smooth and*

$$\begin{cases} \text{strictly subharmonic} & \lambda \in (0, n-2)\,, \\ \text{harmonic} & \lambda = n-2\,, \\ \text{strictly superharmonic} & \lambda \in (n-2, n)\,. \end{cases}$$

Proof. By direct computation,

$$\Delta\phi^\mu_\lambda(x) \;=\; \lambda(\lambda+2-n)\int_{\mathbb{R}^n} |x-y|^{-\lambda-2}\,d\mu(y)$$

away from the support of μ. □

In the super-Newtonian regime, the equilibrium measure has the following property.

Lemma 9 (*[10, p. 137]*)**.** *Let $\lambda \geq n-2$, and let $W \subset \mathbb{R}^n$ be a compact set of positive capacity. If $\mu \in \mathcal{P}$ minimize I_λ among probability measures supported on W, then*

$$\begin{aligned} &\phi^\mu_\lambda(x) = I_\lambda(\mu) \quad \text{approximately everywhere on } W \\ &\phi^\mu_\lambda(x) \leq I_\lambda(\mu) \quad \text{throughout } \mathbb{R}^n \end{aligned}$$

where approximately everywhere means everywhere except on a set of capacity zero.

We are ready for the proof of Theorem 3.

Proof. Let μ be a minimizer of $\mathcal{E}_{\infty,\lambda}$. By Lemma 7, μ is the equilibrium measure that achieves the λ-capacity of some convex body W of constant width 1. When $\lambda \leq n-2$, classical results of potential theory (cf. [10, p. 162]) ensure that $\operatorname{supp}\mu \subset \partial W$. This proves the claim in the sub-Newtonian regime.

Let now $\lambda \geq n-2$, and $p \in \partial W$. Since W is a convex body, every neighborhood of p intersects the interior of W in a set of positive volume (and hence positive capacity). Again by classical results of potential theory (cf. [10, p. 164]), p lies in the support of μ. Therefore $\partial W \subset \operatorname{supp}\mu$. Together with the result for $\lambda \leq n-2$, this completes the proof in the Newtonian case.

For $\lambda > n-2$ Lemma 8 yields that the potential ϕ^λ_μ is strictly subharmonic outside the support of μ. By the strong maximum principle, ϕ^λ_μ is non-constant on every non-empty open set U with $\mu(U) = 0$. On the other hand, ϕ^λ_μ is constant on the interior of W by Lemma 9. Therefore $\mu(U) > 0$ for every non-empty open subset of the interior of W, and we conclude that $W \subset \operatorname{supp}\mu$. This proves the claim in the super-Newtonian regime. □

5. Capacity estimates

We close with some simple capacity estimates which will prove Theorem 4.

Lemma 10. *Let $n \geq 1$, $\lambda \in (0, n)$. Then*

$$\sup_{A\subset\mathbb{R}^n} \left\{ C_\lambda(A) \;\middle|\; \operatorname{diam}(A) = 1 \right\} < 1.$$

Proof. By Lemma 7, there is a set $A \subset \mathbb{R}^n$ that maximizes the capacity C_λ among sets of diameter 1. Let μ be the equilibrium measure on A that achieves the capacity. We estimate

$$\mathcal{E}_{\infty,\lambda}(\mu) - 1 \geq \int \Big(\int_{B_{\frac{1}{2}}(x)} (|x-y|^{-\lambda} - 1)\,d\mu(y) \Big) d\mu(x) > 0,$$

where the first inequality holds since the integrand is nonnegative for every pair of points $x, y \in A$, and the second inequality uses that $\mu(B_{\frac{1}{2}}(x)) > 0$ for x in the support of μ. By Lemma 7, $C_\lambda(A) = (\mathcal{E}_{\infty,\lambda}(\mu))^{-1} < 1$, as claimed. □

We next consider the capacity of balls in high dimensions.

Lemma 11. *For every* $\lambda > 0$

$$\lim_{n\to\infty} C_\lambda(B^{(n)}_{1/2}) = 2^{-\frac{\lambda}{2}}.$$

Proof. This follows by directly computing $C_\lambda(B^{(n)}_{1/2})$ (cf. [10, p. 163]) and applying Stirling's approximation. □

Finally, we construct sets of larger capacity in high dimensions.

Lemma 12. *For every* $\lambda > 0$,

$$\lim_{n\to\infty} \left(\sup\left\{C_\lambda(A) \;\middle|\; A \subset \mathbb{R}^n, \operatorname{diam}(A) \le 1\right\}\right) = 1.$$

Proof. Since $C_\lambda(A) < 1$ for all n by Lemma 10, it suffices to establish the corresponding lower bound on the capacity.

We will construct a family of subsets $(A_n)_n$ of diameter 1 in the sphere of radius $\frac{1}{2}\sqrt{2}$ in $\mathbb{R}^n$ that achieves this limit. For each $n > \lambda + 1$, the spherical cap of diameter 1 in this sphere has positive λ-capacity. Let A_n be a set of maximal capacity among such subsets, and let μ_n be the equilibrium measure on A_n that attains the capacity.

For $m, n > \lambda + 1$, consider a convex combination

$$\mu = (1-t)(\mu_m \otimes \delta) + t(\delta \otimes \mu_n)$$

on $\mathbb{R}^{m+n}$, where δ denotes the unit mass at 0 in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and $t \in (0, 1)$ will be chosen below. By definition, μ is supported on $(A_m \times \{0\}) \cup (\{0\} \times A_n)$, which lies in the sphere of radius $\frac{1}{2}\sqrt{2}$ in $\mathbb{R}^{m+n}$ and has diameter 1. We estimate

$$\begin{aligned}
\mathcal{E}_{\infty,\lambda}(\mu_{m+n}) - 1 &\le \mathcal{E}_{\infty,\lambda}(\mu) - 1 \\
&= \int\!\!\int (|x-y|^{-\lambda} - 1)\, d\mu(x) d\mu(y) \\
&= (1-t)^2(\mathcal{E}_{\infty,\lambda}(\mu_m) - 1) + t^2(\mathcal{E}_{\infty,\lambda}(\mu_n) - 1)\,;
\end{aligned}$$

the mixed terms vanish because $|x - y| = 1$ whenever $x \in A_m \times \{0\}$ and $y \in \{0\} \times A_n$. Minimization over t yields

$$\mathcal{E}_{\infty,\lambda}(\mu_{m+n}) - 1 \le \frac{(\mathcal{E}_{\infty,\lambda}(\mu_m) - 1)(\mathcal{E}_{\infty,\lambda}(\mu_n) - 1)}{\mathcal{E}_{\infty,\lambda}(\mu_m) + \mathcal{E}_{\infty,\lambda}(\mu_n) - 2}.$$

Since $\mathcal{E}_{\infty,\lambda}(\mu_n) > 1$ for all n by Lemma 10, we can pass to reciprocals and conclude that $(\mathcal{E}_{\infty,\lambda}(\mu_n) - 1)^{-1}$ is superadditive in n. By Fekete's superadditivity lemma

$$\lim_{n\to\infty} \frac{1}{n}\left(\mathcal{E}_{\infty,\lambda}(\mu_n) - 1\right)^{-1} = \sup_n \frac{1}{n}\left(\mathcal{E}_{\infty,\lambda}(\mu_n) - 1\right)^{-1} > 0.$$

It follows that $\lim_{n\to\infty} C_\lambda(A_n) = (\lim_{n\to\infty} \mathcal{E}_{\infty,\lambda}(\mu_n))^{-1} = 1$. □

The proof of Theorem 4 is an immediate corollary of Lemmas 11 and 12 since $2^{-\frac{\lambda}{2}} < 1$ for every $\lambda > 0$. Note that the near-maximizers constructed in the proof of Lemma 12 have dimension much below n, but this need not be true for actual maximizers.

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