Q: 15.4/Q.7

$$\bar{F} = (x+y)i + (y-z)j - (x+y)\bar{k}$$

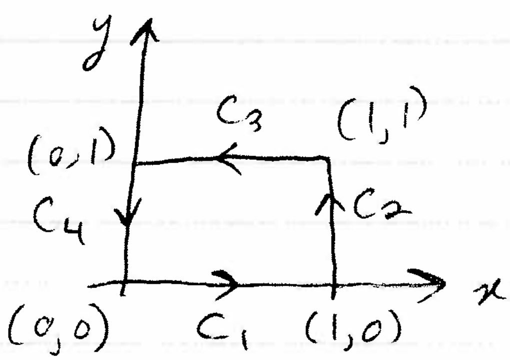
$$= \nabla\left(\frac{x^2+z^2}{2} + y(x-z)\right)$$

Work done by $\bar{F}$ in moving an object from $(1,0,-1)$ to $(0,-2,3)$:

$$W = \int_C \bar{F}.dr = \left(\frac{x^2+z^2}{2} + y(x-z)\right)_{1,0,-1}^{0,-2,3}$$

$$= \frac{19}{2}$$

Q: 15.4/Q.8



$$\int_{C_1} x^2y^2\,dx + x^3y\,dy = 0$$

$$\int_{C_2} x^2y^2\,dx + x^3y\,dy = \int_0^1 y\,dy = \frac{1}{2}$$

$$\int_{C_3} x^2y^2\,dx + x^3\,dy = \int_1^0 x^2\,dx = -\frac{1}{3}$$

$$\int_{C_4} x^2y^2\,dx + x^3y\,dy = 0$$

~~Fina~~ Therefore, $\int_C x^2y^2\,dx + x^3y\,dy = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

Q. 15.4/Q.11

$F = Ax\ln z\, i + By^2z\, j + \left(\frac{x^2}{z} + y^3\right)k$ is conservative

if: $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \iff 0 = 0$

$$\frac{\partial F_1}{\partial x} = \frac{\partial F_3}{\partial x} \iff A = 2$$

$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \iff B = 3$$

If $A=2$ and $B=3$, then $\bar{F} = \nabla\phi$ where

$$\phi = x^2 \ln z + y^3 z$$

If C is straight line $x = t+1, y = 1, z = t+1$ $(0 \le t \le 1)$ from $(1,1,1)$ to $(2,1,2)$ Then

$$\int_C 2x \ln z \, dx + 2y^2 z \, dy + y^3 dz$$

$$= \int_C \nabla\phi \cdot dr - \int_C y^2 z \, dy + \frac{x^2}{z} dz$$

$$= (x^2 \ln z + y^3 z)_{1,1,1}^{2,1,2} - \int_0^1 [(t+1)(0) + (t+1)] dt$$

$$= 4 \ln 2 + 2 - 1 - \left[\frac{t^2}{2} + t\right]_0^1$$

$$= 4 \ln 2 - \frac{1}{2}$$

Q.: 15.4/Q.13:

For $z = \ln(1+x), y = x$, from $x = 0$ to $x = 1$

$$\int_C [(2x \sin(\pi y) - e^z) dx + (\pi x^2 \cos(\pi y) - 3e^z dy - xe^z dz]$$

$$= \int_C \nabla(x^2 \sin(\pi y) - xe^z) d\bar{r} - 3\int_C e^z dy$$

$$\left[x^2\sin(\pi y) - xe^z\right]_{(0,0,0)}^{(1,1,\ln 2)} - 3\int_0^1 (1+x)\,dx$$

$$= -2 - 3\left[x + \frac{x^2}{2}\right]_0^1 = -\frac{13}{2}$$

<u>Q:</u> 15.4/Q.21

$$\nabla(fg) = \left(f\frac{\partial g}{\partial x} + \frac{\partial f}{\partial x}g\right)i + \left(f\frac{\partial g}{\partial y} + \frac{\partial f}{\partial y}g\right)j +$$

$$\left(f\frac{\partial g}{\partial z} + \frac{\partial f}{\partial z}g\right)k$$

Since C goes from P to Q:

$$\int_C f\nabla g \cdot dr + \int_C g\nabla f \cdot dr$$

$$= \int_C \nabla(fg) \cdot dr$$

$$= [fg]_P^Q$$

$$= f(Q)g(Q) - f(P)g(P)$$

Q: 15.4 | Q. 22

a) $C: x = a\cos t,\ y = a\sin t,\ 0 \le t \le 2\pi$

$$\frac{1}{2\pi}\oint_C \frac{x\,dy - y\,dx}{x^2+y^2} = \frac{1}{2\pi}\int_0^{2\pi} \frac{a^2\cos^2 t + a^2\sin^2 t}{a^2\cos^2 t + a^2\sin^2 t}\,dt = 1$$

b) $$\frac{1}{2\pi}\oint_C \frac{x\,dy - y\,dx}{x^2+y^2}$$

$$= \frac{1}{2\pi}\left[\int_1^1 \frac{dy}{1+y^2} + \int_1^{-1}\frac{dx}{x^2+1} + \int_{-1}^{1}\frac{-dy}{1+y^2} + \int_{-1}^{1}\frac{-dx}{x^2+1}\right]$$

$$= -\frac{2}{\pi}\int_{-1}^{1}\frac{dt}{1+t^2}$$

$$= -\frac{2}{\pi}\tan^{-1}(t)\Big|_{-1}^{1} = -\frac{2}{\pi}\left(\frac{\pi}{4}+\frac{\pi}{4}\right) = -1$$

c) $$\frac{1}{2\pi}\oint_C \frac{x\,dy - y\,dx}{x^2+y^2}$$

$$= \frac{1}{2\pi}\left[0 + \int_0^{\pi}\frac{4\cos^2 t + 4\sin^2 t}{4\cos^2 t + 4\sin^2 t}\,dt + 0 + \int_{\pi}^{0}\frac{\cos^2 t + \sin^2 t}{\cos^2 t + \sin^2 t}\right] = 0$$

Q : 15.5/Q.10

$\frac{1}{8}$ of required area lies in first ~~quadrant~~ octant, above the triangle T with vertices $(0,0,0)$, $(a,0,0)$ and $(a,a,0)$

The surface $x^2+z^2=a^2$ has normal $\bar{n}=x\bar{i}+z\bar{k}$,

so an area element on it can be written

$$dS = \frac{|\bar{n}|}{|\bar{n}\cdot\bar{k}|}\,dx\,dy = \frac{a}{z}\,dx\,dy = \frac{a\,dx\,dy}{\sqrt{a^2-x^2}}$$

The area of part of cylinder lying inside the cylinder $y^2+z^2=a^2$:

$$S = 8\iint_T \frac{a\,dx\,dy}{\sqrt{a^2-x^2}} = 8a\int_0^a \frac{dx}{\sqrt{a^2-x^2}}\int_0^x dy$$

$$= 8a\int_0^a \frac{x\,dx}{\sqrt{a^2-x^2}}$$

$$= -8a\left[\sqrt{a^2-x^2}\right]_0^a$$

$$= 8a^2$$

Q: 15.5/Q.16

surface $z = \sqrt{2xy}$ has area element:

$$dS = \sqrt{1 + \frac{y}{2x} + \frac{x}{2y}}\, dx\, dy$$

$$= \sqrt{\frac{2xy + y^2 + x^2}{2xy}}\, dx\, dy = \frac{|x+y|}{\sqrt{2xy}}\, dx\, dy$$

If density is kz, mass of specified part of surface is:

$$m = \int_0^5 dx \int_0^2 k\sqrt{2xy}\, \frac{x+y}{\sqrt{2xy}}\, dy$$

$$= k \int_0^5 dx \int_0^2 (x+y)\, dy$$

$$= k \int_0^5 (2x+2)\, dx$$

$$= 35k$$

Q: 15.6 / 10.6:

For $z = x^2 - y^2$, the upward surface element is

$$\hat{N}\, dS = \frac{-2x\hat{i} + 2y\hat{j} + \hat{k}}{1}\, dx\, dy$$

Flux of $\bar{F} = x\hat{i} + x\hat{j} + z\hat{k}$ upward through S,

$$\cancel{\iint_S \bar{F} \cdot \hat{N}\, dS}$$

the part of $z = x^2 - y^2$ inside $x^2 + y^2 = a^2$:

$$\iint_S \bar{F} \cdot \hat{N}\, dS = \iint_{x^2+y^2 \le a^2} (-2x^2 + 2xy + 1)\, dx\, dy$$

$$= -2\int_0^{2\pi} \cos^2\theta\, d\theta \int_0^a r^3\, dr + 0 + \pi a^2$$

$$= \pi a^2 - 2\pi \frac{a^4}{4}$$

$$= \frac{\pi}{2} a^2 (2 - a^2)$$

Q: 15.6/Q.8:

upward vector surface element on top half of $x^2+y^2+z^2=a^2$

$$\hat{N}\,dS = \frac{\partial x\hat{i}+\partial y\hat{j}+\partial z\hat{k}}{\partial z}\,dx\,dy$$

$$= \left(\frac{x\hat{i}+y\hat{j}}{z}+\hat{k}\right)dx\,dy$$

The flux of $\bar{F}=z^2\hat{k}$ upward through first octant part S of sphere is:

$$\iint_S \bar{F}\cdot\hat{N}\,dS = \int_0^{\pi/2} d\theta \int_0^a (a^2-r^2)\,r\,dr = \frac{\pi a^4}{8}$$

Q: 15.6/Q.10

S: $\bar{r}: u^2v\hat{i}+uv^2\hat{j}+v^3\hat{k}$ $(0\le u\le 1, 0\le v\le 1)$

has upward surface element

$$\hat{N}\,dS = \frac{\partial\bar{r}}{\partial u}\times\frac{\partial\bar{r}}{\partial v}\,du\,dv$$

$$= (2uv\hat{i}+v^2\hat{j})\times(u^2\hat{i}+2uv\hat{j}+3v^2\hat{k})\,du\,dv$$

$$= (3v^4\hat{i}-6uv^3\hat{j}+3u^2v^2\hat{k})\,du\,dv$$

Flux of $\bar{F} = 2x\hat{i} + y\hat{j} + z\hat{k}$ upward through S is

$$\iint_S \bar{F} \cdot \hat{N}\, dS$$

$$= \int_0^1 du \int_0^1 (6u^2v^5 - 6u^2v^5 + 3u^2v^5)\, dv$$

$$= \frac{1}{2} \int_0^1 u^2\, du$$

$$= \frac{1}{6}$$