MATH 264 (Spring 2013)
Written Assignment 2
DUE IN CLASS April 11 (Choksi) April 12 (SANCHO)

1) (a) Use the Divergence theorem in 2D to prove (show) Green's Theorem.
(b) Use our version of Stokes's Theorem to prove (show) Green's Theorem.

2) Let $f(x)$ be defined for $-5 \le x < 5$ by

$$f(x) = \begin{cases} x & \text{if } -5 \le x \le 0 \\ x^2 & \text{if } 0 < x \le 2 \\ x-2 & \text{if } 2 < x \le 4 \\ 0 & \text{if } 4 < x < 5 \end{cases}$$

Find the Fourier Series of f (you can use computer software to compute the coefficients).
(i) Plot the first 3 terms of the Fourier series for $x \in [-15, 15]$. On the same graph, plot the periodic extension of the function $f(x)$.
(ii) Repeat (i) with the first 10 terms of the Fourier Series.
(iii) Finally **sketch** the function that the Fourier Series converges to.

3) Let

$$f(x) = \begin{cases} 1 & -2 \le x < -1 \\ 2 & -1 \le x < 0 \\ 3 & 0 \le x < 1 \\ 4 & 1 \le x < 2 \end{cases}$$

and extend $f(x)$ by periodicity to all $x \in (-\infty, \infty)$, i.e. let $f(x+4) = f(x)$ for all x. Consider the full Fourier series for f:

$$f(x) \;=\; a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{2}) + b_n \sin(\frac{n\pi x}{2}).$$

a) To what values will this Fourier series converge at $x = 0$, $x = 1$, $x = 4$, $x = 7.4$, and $x = 40$?
b) Find a_0.

4) Find the Fourier series for the following function defined on the interval $(-1, 1)$:

$$f(x) \;=\; \cos(26\pi x) \;-\; 4 \;-\; 3\sin(\pi x).$$

5) Let

$$f_1(x) \;=\; a_0 \;+\; \sum_{n=1}^{\infty} \; a_n \, \cos n\pi x, \qquad\qquad f_2(x) \;=\; \sum_{n=1}^{\infty} \; b_n \, \sin n\pi x,$$

and

$$f_3(x) \;=\; c_0 \;+\; \sum_{n=1}^{\infty} \; (\, c_n \, \cos n\pi x \;+\; d_n \, \sin n\pi x)$$

where

$$a_0 \;=\; \int_0^1 e^x \, dx, \quad a_n \;=\; 2 \int_0^1 e^x \cos n\pi x \, dx, \qquad b_n \;=\; 2 \int_0^1 e^x \sin n\pi x \, dx$$

and

$$c_0 = \frac{1}{2}\int_{-1}^{1} e^x\,dx \qquad c_n = \int_{-1}^{1} e^x \cos n\pi x\,dx \qquad d_n = \int_{-1}^{1} e^x \sin n\pi x\,dx.$$

By the Convergence Theorem, these Fourier series (i.e. the right hand sides) will all converge. **Sketch** the graphs of f_1, f_2, and f_3 for $x \in (-3, 3)$, that is, sketch the functions that the respective Fourier series converge to. Make sure your graph clearly indicates the values for $x = -2, -1, 0, 1, 2$. NOTE: all you need for these rough sketches is the shape of the graph of $y = e^x$ AND a basic understanding of our Fourier Convergence Theorem.

6a) Find all eigenvalues of the eigenvalue problem with mixed boundary conditions: $y(x)$ on $[0, \pi]$ solves

$$y'' + \lambda y = 0, \qquad y'(0) = 0 = y(\pi).$$

What are (all) the associated eigenfunctions.

b) Find all eigenvalues of the eigenvalue problem with periodic boundary conditions: $y(x)$ on $[0, 2]$ solves

$$y'' + \lambda y = 0, \qquad y(0) = y(2),\;\; y'(0) = y'(2).$$

What are (all) the associated eigenfunctions.

7) Solve the heat equation

$$u_t = u_{xx}, \qquad 0 < x < \pi,\;\; t > 0,$$
$$u'(0) = 0 = u(\pi),\;\; u(x, 0) = x.$$

Your answer will be an infinite series. What is happens to the temperature as t gets larger and larger. Explain your answer based upon what the boundary conditions mean.

8) Solve Laplace's equation

$$u_{xx} + u_{yy} = 0, \quad 0 < x < 2,\; 0 < y < 1,$$

with the Dirichlet boundary conditions:

$$u(x, 0) = 0 \;\text{ and }\; u(x, 2) = 0, \quad \text{for } 0 < x < 2,$$

$$u(0, y) = 0 \;\text{ and }\; u(2, y) = 3\sin(\pi y), \quad \text{for } 0 < y < 1.$$

9) Solve the wave equation

$$u_{tt} = 9u_{xx}, \qquad 0 < x < \pi,\;\; t > 0,$$
$$u(0, t) = 0 = u(\pi, t),\;\; t > 0,$$
$$u(x, 0) = 5\sin 7x, \quad u_t(x, 0) = 3\sin 2x \,+\, 4\sin 5x.$$