EXAMPLE 5 Find the flux of $\mathbf{F} = \dfrac{2x\mathbf{i} + 2y\mathbf{j}}{x^2 + y^2} + \mathbf{k}$ downward through the surface $\mathcal{S}$ defined parametrically by

$$\mathbf{r} = u\cos v\mathbf{i} + u\sin v\mathbf{j} + u^2\mathbf{k}, \qquad (0 \le u \le 1,\ 0 \le v \le 2\pi).$$

Solution First we calculate $d\mathbf{S}$:

$$\frac{\partial \mathbf{r}}{\partial u} = \cos v\mathbf{i} + \sin v\mathbf{j} + 2u\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial v} = -u\sin v\mathbf{i} + u\cos v\mathbf{j}$$

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = -2u^2\cos v\mathbf{i} - 2u^2\sin v\mathbf{j} + u\mathbf{k}.$$

Since $u \ge 0$ on $\mathcal{S}$, the latter expression is an upward normal. We want a downward normal, so we use

$$d\mathbf{S} = (2u^2\cos v\mathbf{i} + 2u^2\sin v\mathbf{j} - u\mathbf{k})\,du\,dv.$$

On $\mathcal{S}$ we have

$$\mathbf{F} = \frac{2x\mathbf{i} + 2y\mathbf{j}}{x^2 + y^2} + \mathbf{k} = \frac{2u\cos v\mathbf{i} + 2u\sin v\mathbf{j}}{u^2} + \mathbf{k},$$

so the downward flux of $\mathbf{F}$ through $\mathcal{S}$ is

$$\iint_{\mathcal{S}} \mathbf{F} \bullet d\mathbf{S} = \int_0^{2\pi} dv \int_0^1 (4u - u)\,du = 3\pi.$$

EXERCISES 15.6

1. Find the flux of $\mathbf{F} = x\mathbf{i} + z\mathbf{j}$ out of the tetrahedron bounded by the coordinate planes and the plane $x + 2y + 3z = 6$.
2. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ outward across the sphere $x^2 + y^2 + z^2 = a^2$.
3. Find the flux of the vector field of Exercise 2 out of the surface of the box $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$.
4. Find the flux of the vector field $\mathbf{F} = y\mathbf{i} + z\mathbf{k}$ out across the boundary of the solid cone $0 \le z \le 1 - \sqrt{x^2 + y^2}$.
5. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ upward through the part of the surface $z = a - x^2 - y^2$ lying above plane $z = b < a$.
6. Find the flux of $\mathbf{F} = x\mathbf{i} + x\mathbf{j} + \mathbf{k}$ upward through the part of the surface $z = x^2 - y^2$ inside the cylinder $x^2 + y^2 = a^2$.
7. Find the flux of $\mathbf{F} = y^3\mathbf{i} + z^2\mathbf{j} + x\mathbf{k}$ downward through the part of the surface $z = 4 - x^2 - y^2$ that lies above the plane $z = 2x + 1$.
8. Find the flux of $\mathbf{F} = z^2\mathbf{k}$ upward through the part of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant of 3-space.
9. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ upward through the part of the surface $z = 2 - x^2 - 2y^2$ that lies above the xy-plane.
10. Find the flux of $\mathbf{F} = 2x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ upward through the surface $\mathbf{r} = u^2v\mathbf{i} + uv^2\mathbf{j} + v^3\mathbf{k}$, $(0 \le u \le 1, 0 \le v \le 1)$.
11. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$ upward through the surface $u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\,\mathbf{k}$, $(0 \le u \le 2, 0 \le v \le \pi)$.
12. Find the flux of $\mathbf{F} = yz\mathbf{i} - xz\mathbf{j} + (x^2 + y^2)\mathbf{k}$ upward through the surface $\mathbf{r} = e^u\cos v\,\mathbf{i} + e^u\sin v\,\mathbf{j} + u\,\mathbf{k}$, where $0 \le u \le 1$ and $0 \le v \le \pi$.
13. Find the flux of $\mathbf{F} = m\mathbf{r}/|\mathbf{r}|^3$ out of the surface of the cube $-a \le x, y, z \le a$.
14. Find the flux of the vector field of Exercise 13 out of the box $1 \le x, y, z \le 2$. *Note:* This problem can be solved very easily using the Divergence Theorem of Section 16.4; the required flux is, in fact, zero. However, the object here is to do it by direct calculation of the surface integrals involved, and as such it is quite difficult. By symmetry, it is sufficient to evaluate the net flux out of the cube through any one of the three pairs of opposite faces; that is, you must calculate the flux through only two faces, say $z = 1$ and $z = 2$. Be prepared to work very hard to evaluate these integrals! When they are done, you may find the identities
$2\arctan a = \arctan\left(2a/(1 - a^2)\right)$ and
$\arctan a + \arctan(1/a)\pi/2$ useful for showing that the net

EXERCISES 15.4

In Exercises 1–6, evaluate the line integral of the tangential component of the given vector field along the given curve.

1. $\mathbf{F}(x, y) = xy\mathbf{i} - x^2\mathbf{j}$ along $y = x^2$ from $(0, 0)$ to $(1, 1)$
2. $\mathbf{F}(x, y) = \cos x\,\mathbf{i} - y\mathbf{j}$ along $y = \sin x$ from $(0, 0)$ to $(\pi, 0)$
3. $\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} - x\mathbf{k}$ along the straight line from $(0, 0, 0)$ to $(1, 1, 1)$
4. $\mathbf{F}(x, y, z) = z\mathbf{i} - y\mathbf{j} + 2x\mathbf{k}$ along the curve $x = t$, $y = t^2$, $z = t^3$ from $(0, 0, 0)$ to $(1, 1, 1)$
5. $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ from $(-1, 0, 0)$ to $(1, 0, 0)$ along either direction of the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = y$
6. $\mathbf{F}(x, y, z) = (x - z)\mathbf{i} + (y - z)\mathbf{j} - (x + y)\mathbf{k}$ along the polygonal path from $(0, 0, 0)$ to $(1, 0, 0)$ to $(1, 1, 0)$ to $(1, 1, 1)$
7. Find the work done by the force field

$$\mathbf{F} = (x + y)\mathbf{i} + (x - z)\mathbf{j} + (z - y)\mathbf{k}$$

in moving an object from $(1, 0, -1)$ to $(0, -2, 3)$ along any smooth curve.

8. Evaluate $\oint_{\mathcal{C}} x^2y^2\,dx + x^3y\,dy$ counterclockwise around the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$.
9. Evaluate

$$\int_{\mathcal{C}} e^{x+y}\sin(y + z)\,dx + e^{x+y}\Big(\sin(y + z) + \cos(y + z)\Big)\,dy + e^{x+y}\cos(y + z)\,dz$$

along the straight line segment from (0,0,0) to $(1, \frac{\pi}{4}, \frac{\pi}{4})$.

10. The field $\mathbf{F} = (axy + z)\mathbf{i} + x^2\mathbf{j} + (bx + 2z)\mathbf{k}$ is conservative. Find a and b, and find a potential for $\mathbf{F}$. Also, evaluate $\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$, where $\mathcal{C}$ is the curve from $(1, 1, 0)$ to $(0, 0, 3)$ that lies on the intersection of the surfaces $2x + y + z = 3$ and $9x^2 + 9y^2 + 2z^2 = 18$ in the octant $x \ge 0$, $y \ge 0$, $z \ge 0$.
11. Determine the values of A and B for which the vector field

$$\mathbf{F} = Ax\ln z\,\mathbf{i} + By^2z\,\mathbf{j} + \left(\frac{x^2}{z} + y^3\right)\mathbf{k}$$

is conservative. If $\mathcal{C}$ is the straight line from $(1, 1, 1)$ to $(2, 1, 2)$, find

$$\int_{\mathcal{C}} 2x\ln z\,dx + 2y^2z\,dy + y^3\,dz.$$

12. Find the work done by the force field

$$\mathbf{F} = (y^2\cos x + z^3)\mathbf{i} + (2y\sin x - 4)\mathbf{j} + (3xz^2 + 2)\mathbf{k}$$

in moving a particle along the curve $x = \sin^{-1}t$, $y = 1 - 2t$, $z = 3t - 1$, $(0 \le t \le 1)$.

13. If $\mathcal{C}$ is the intersection of $z = \ln(1 + x)$ and $y = x$ from $(0, 0, 0)$ to $(1, 1, \ln 2)$, evaluate

$$\int_{\mathcal{C}}\Big(2x\sin(\pi y) - e^z\Big)\,dx + \Big(\pi x^2\cos(\pi y) - 3e^z\Big)\,dy - xe^z\,dz.$$

14. Is each of the following sets a domain? a connected domain? a simply connected domain?
    (a) the set of points (x, y) in the plane such that $x > 0$ and $y \ge 0$
    (b) the set of points (x, y) in the plane such that $x = 0$ and $y \ge 0$
    (c) the set of points (x, y) in the plane such that $x \ne 0$ and $y > 0$
    (d) the set of points (x, y, z) in 3-space such that $x^2 > 1$
    (e) the set of points (x, y, z) in 3-space such that $x^2 + y^2 > 1$
    (f) the set of points (x, y, z) in 3-space such that $x^2 + y^2 + z^2 > 1$

In Exercises 15–19, evaluate the closed line integrals

$$\text{(a)}\quad \oint_{\mathcal{C}} x\,dy, \qquad \text{(b)}\quad \oint_{\mathcal{C}} y\,dx$$

around the given curves, all oriented counterclockwise.

15. The circle $x^2 + y^2 = a^2$
16. The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
17. The boundary of the half-disk $x^2 + y^2 \le a^2$, $y \ge 0$
18. The boundary of the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$
19. The triangle with vertices $(0, 0)$, $(a, 0)$, and $(0, b)$
20. On the basis of your results for Exercises 15–19, guess the values of the closed line integrals

$$\text{(a)}\quad \oint_{\mathcal{C}} x\,dy, \qquad \text{(b)}\quad \oint_{\mathcal{C}} y\,dx$$

for any non–self-intersecting closed curve in the xy-plane. Prove your guess in the case that $\mathcal{C}$ bounds a region of the plane that is both x-simple and y-simple. (See Section 14.2.)

21. If f and g are scalar fields with continuous first partial derivatives in a connected domain D, show that

$$\int_{\mathcal{C}} f\nabla g \bullet d\mathbf{r} + \int_{\mathcal{C}} g\nabla f \bullet d\mathbf{r} = f(Q)g(Q) - f(P)g(P)$$

for any piecewise smooth curve in D from P to Q.

22. Evaluate

$$\frac{1}{2\pi}\oint_{\mathcal{C}} \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

    (a) counterclockwise around the circle $x^2 + y^2 = a^2$,
    (b) clockwise around the square with vertices $(-1, -1)$, $(-1, 1)$, $(1, 1)$, and $(1, -1)$,
    (c) counterclockwise around the boundary of the region $1 \le x^2 + y^2 \le 4$, $y \ge 0$.

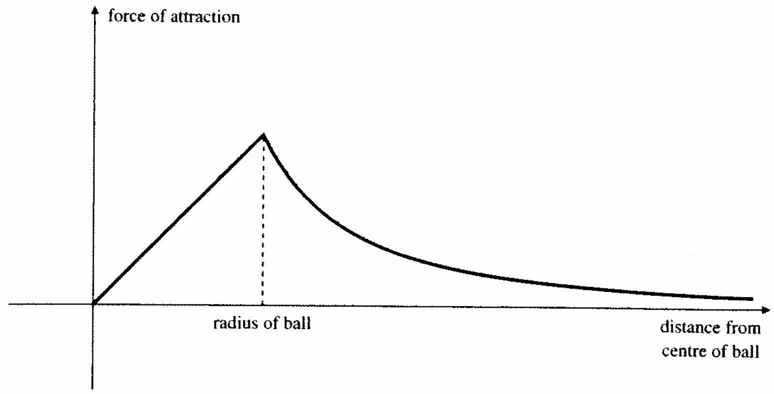

Figure 15.25 The force of attraction of a homogeneous solid ball on a particle located at varying distances from the centre of the ball

Remark All of the above discussion also holds for the electrostatic attraction or repulsion of a point charge by a uniform charge density over a spherical shell, which is also governed by an inverse square law. In particular, there is no net electrostatic force on a charge located inside the shell.

EXERCISES 15.5

1. Verify that on the curve with polar equation $r = g(\theta)$ the arc length element is given by
$$ds = \sqrt{(g(\theta))^2 + (g'(\theta))^2}\, d\theta.$$
What is the area element on the vertical cylinder given in terms of cylindrical coordinates by $r = g(\theta)$?

2. Verify that on the spherical surface $x^2 + y^2 + z^2 = a^2$ the area element is given in terms of spherical coordinates by $dS = a^2 \sin\phi\, d\phi\, d\theta$.

3. Find the area of the part of the plane $Ax + By + Cz = D$ lying inside the elliptic cylinder
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

4. Find the area of the part of the sphere $x^2 + y^2 + z^2 = 4a^2$ that lies inside the cylinder $x^2 + y^2 = 2ay$.

5. State formulas for the surface area element dS for the surface with equation $F(x, y, z) = 0$ valid for the case where the surface has a one-to-one projection on (a) the xz-plane and (b) the yz-plane.

6. Repeat the area calculation of Example 8 by projecting the part of the surface shown in Figure 15.23 onto the yz-plane and using the formula in Exercise 5(b).

7. Find $\iint_{\mathcal{S}} x\, dS$ over the part of the parabolic cylinder $z = x^2/2$ that lies inside the first octant part of the cylinder $x^2 + y^2 = 1$.

8. Find the area of the part of the cone $z^2 = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 2ay$.

9. Find the area of the part of the cylinder $x^2 + y^2 = 2ay$ that lies outside the cone $z^2 = x^2 + y^2$.

10. Find the area of the part of the cylinder $x^2 + z^2 = a^2$ that lies inside the cylinder $y^2 + z^2 = a^2$.

❸ 11. A circular cylinder of radius a is circumscribed about a sphere of radius a so that the cylinder is tangent to the sphere along the equator. Two planes, each perpendicular to the axis of the cylinder, intersect the sphere and the cylinder in circles. Show that the area of that part of the sphere between the two planes is equal to the area of the part of the cylinder between the two planes. Thus, the area of the part of a sphere between two parallel planes that intersect it depends only on the radius of the sphere and the distance between the planes, and not on the particular position of the planes.

■ 12. Let $0 < a < b$. In terms of the elliptic integral functions defined in Exercise 19 of Section 15.3, find the area of that part of each of the cylinders $x^2 + z^2 = a^2$ and $y^2 + z^2 = b^2$ that lies inside the other cylinder.

13. Find $\iint_{\mathcal{S}} y\, dS$, where $\mathcal{S}$ is the part of the plane $z = 1 + y$ that lies inside the cone $z = \sqrt{2(x^2 + y^2)}$.

14. Find $\iint_{\mathcal{S}} y\, dS$, where $\mathcal{S}$ is the part of the cone $z = \sqrt{2(x^2 + y^2)}$ that lies below the plane $z = 1 + y$.

15. Find $\iint_{\mathcal{S}} xz\, dS$, where $\mathcal{S}$ is the part of the surface $z = x^2$ that lies in the first octant of 3-space and inside the paraboloid $z = 1 - 3x^2 - y^2$.

16. Find the mass of the part of the surface $z = \sqrt{2xy}$ that lies above the region $0 \le x \le 5$, $0 \le y \le 2$, if the areal density of the surface is $\sigma(x, y, z) = kz$.

17. Find the total charge on the surface
$$\mathbf{r} = e^u \cos v\mathbf{i} + e^u \sin v\mathbf{j} + u\mathbf{k}, \quad (0 \le u \le 1,\ 0 \le v \le \pi),$$
if the charge density on the surface is $\delta = \sqrt{1 + e^{2u}}$.

Exercises 18–19 concern **spheroids**, which are ellipsoids with two of their three semi-axes equal, say $a = b$:
$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1.$$