McGill University March 15, 2013
Midterm examination

Advanced Calculus for Engineers

Math 264

March 15, 2013

Time: 10:05 - 11:25 AM

Prof. R. Choksi

Student name (last, first)	**Student number (McGill ID)**

INSTRUCTIONS

1. If you are not registered in this section, your grade will NOT count.
2. This is a closed book exam and calculators are NOT permitted.
3. Make sure you READ CAREFULLY the question before embarking on the solution.
4. EXCEPT for Question 1, you must SHOW ALL YOUR WORK.
5. This exam comprises 7 pages (including the cover page). Please provide all your answers on this exam.

Problem	1	2	3	4	5	6	Total
Mark							
Out of	10	10	10	10	10	10	60

Question 1 (10 pts) Answer true (T) or false (F) to the following statements:
(i) If ϕ is a smooth scalar function, then div curl $\nabla\phi = 0$.

(ii) If $\mathbf{F}$ is a smooth conservative vector field, then div $\mathbf{F} = 0$.

(iii) If a vector field $\mathbf{F}$ has zero divergence in a simply connected domain D, then $\mathbf{F}$ is conservative in D.

(iv) The curl of any vector field of the form $\langle f_1(x,y,z), f_2(x,y,z), 0\rangle$ is always parallel to $\mathbf{k} =< 0, 0, 1 >$.

(v) If $\mathbf{G}$ is a vector potential associated with a vector field $\mathbf{F}$ then so is $\mathbf{G} + \nabla\phi$, where ϕ is any smooth function.

(vi) If $\mathcal{S}$ is the graph of $z = f(x,y)$ for $0 \leq x \leq 2$ and $0 \leq y \leq 2$ (with f smooth), then

$$\iint_{\mathcal{S}} 5\, dS \;=\; 5\int_0^2\int_0^2 \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy.$$

(vii) If a 3D vector field $\mathbf{F}$ is irrotational in $D = \left\{(x,y,z) \,\middle|\, 1 < x^2 + y^2 + z^2 < 2\right\}$, then for any closed curve $\mathcal{C}$ lying in D,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \;=\; 0.$$

viii) If a 2D vector field $\mathbf{F}$ is irrotational in $D = \left\{(x,y) \,\middle|\, 1 < x^2 + y^2 < 2\right\}$, then for any closed curve $\mathcal{C}$ lying in D,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \;=\; 0.$$

(ix) The area of region R bounded by a curve $\mathcal{C}$ (with the positive orientation) is given by the line integral

$$\int_{\mathcal{C}} x\, dx.$$

(x) If div $\mathbf{F} = 0$ at all points in space, then the flux of $\mathbf{F}$ through the plane $x + y + z = 1$ with orientation normal pointing upwards is 0.

Question 2 a) (5 pts) Suppose a wire is bent into the shape of the parametrized curve

$$x(t) = 3t^2 \qquad y(t) = 2t^3 \qquad z(t) = t^4,$$

from point $(3, 2, 1)$ to $(12, 16, 16)$. Suppose the mass density of the wire is given by $\rho(x, y, z) = x^2yz$. Write down an integral with respect to t whose value gives the mass of the wire (do **not** evaluate the integral).

b) (5 pts) Let $\mathbf{F} = \langle yz, xz, xy \rangle$. Evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ where $\mathcal{C}$ is curve from $(1, 0, 2)$ to $(0, 1, 3)$ lying on the intersection of the cylinder $x^2 + y^2 = 1$ and plane $x + z = 3$. Here, you must give a final numerical value to the integral.

Question 3 a) (5 pts) Let $\mathbf{F} = \langle 1, 2, x + y + z \rangle$. Consider the **triangle** lying **in the plane** $x + y + z = 3$ with **vertices** $(0, 0, 3), (0, 3, 0), (0, 0, 3)$. Let $\mathcal{S}$ be part of the plane which lies **inside** this triangle with normal pointing **upwards**. What is the flux

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}?$$

b) (5 pts) Compute the flux integral

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F} = \langle y, -x, z^2 \rangle$ and $\mathcal{S}$ is the part of the conical surface $2z - \sqrt{x^2 + y^2} = 0$ between $z = 2$ and $z = 4$ with unit normal pointing **downwards**. **You may leave your answer** as a iterated double integral with respect to x and y.

Question 4 (10 pts) Use the Divergence Theorem to find the flux of

$$\mathbf{F} = \langle x + y^3 + 4z^4 \,,\, ze^x + y \,,\, x \sin y + 3z \rangle,$$

out of the region bounded below by the plane $z = 0$ and above by the sphere $x^2 + y^2 + z^2 = 1$. That is, find

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S},$$

where $\mathcal{S}$ is the boundary of the region bounded below by the plane $z = 0$ and above by the sphere $x^2 + y^2 + z^2 = 1$ (oriented with the outward normal).

Question 5 (10 pts) I want you to use Stokes's Theorem to evaluate the circulation

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

where $\mathcal{C}$ is the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the parabolic sheet $z = x^2$ with orientation induced by the upward normal to the parabolic sheet. I will not tell you what $\mathbf{F}$ is but rather I will tell you its curl:

$$\textbf{curl}\,\mathbf{F} = \langle -2x, \, , \, x^3 z^2 + y \, , \, 5y^2 + z \rangle .$$

Question 6 a) (5 pts) Let $\mathcal{C}$ be the curve which is the boundary of the **square lying in the plane** $x + y + z = 6$ **centred** at the point $(1, 2, 3)$ with side **length** $\frac{1}{10}$. Give $\mathcal{C}$ the orientation induced by upward normal $\langle 1, 1, 1\rangle$. Suppose $\mathbf{F}$ is a smooth vector field whose curl at the point $(1, 2, 3)$ is given by $\langle 2, 3, 1\rangle$. Use only this information to **approximate** the circulation integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$

b) (5 pts) Let $\mathbf{F}$ be a smooth 3D vector field and let $\mathcal{S}_\epsilon$ denote the sphere of radius ϵ centred at the origin. What is

$$\lim_{\epsilon \to 0^+} \frac{1}{4\pi\epsilon^2} \iint_{\mathcal{S}_\epsilon} \mathbf{F} \cdot d\mathbf{S}\,?$$

Note here we are taking the limiting value of the **flux per unit surface area**, not the **flux per unit enclosed volume**. You **must explain your reasoning**.