McGill University March 14, 2013
Midterm examination

Advanced Calculus for Engineers

Math 264

March 14, 2013

Time: 8:35 - 9:55 AM

Prof. N. Sancho

Student name (last, first)	**Student number (McGill ID)**

INSTRUCTIONS

1. If you are not registered in this section, your grade will NOT count.
2. This is a closed book exam and calculators are NOT permitted.
3. Make sure you READ CAREFULLY the question before embarking on the solution.
4. EXCEPT for Question 1, you must SHOW ALL YOUR WORK.
5. This exam comprises 7 pages (including the cover page). Please provide all your answers on this exam.

Problem	1	2	3	4	5	6	Total
Mark							
Out of	10	10	10	10	10	10	60

Question 1 (10 pts) Answer true (T) or false (F) to the following statements:

(i) If $\mathbf{F}$ has zero divergence in a simply connected domain D, then $\mathbf{F}$ is conservative in D.

(ii) If div $\mathbf{F} = 0$ at all points in space, then the flux of $\mathbf{F}$ through the paraboloid $z = x^2 + y^2$, with orientation normal pointing upwards, is 0.

(iii) The curl of any vector field of the form $\langle 0, f_2(x, y, z), f_3(x, y, z)\rangle$ is always parallel to $\mathbf{i} =< 1, 0, 0 >$.

(iv) If $\mathbf{F}$ is a smooth vector field, then div curl $\mathbf{F} = 0$.

(v) If ϕ is a smooth scalar function, then div $\nabla\phi = 0$.

(vi) If $\mathcal{S}$ is the graph of $z = g(x, y)$ for $0 \le x \le 2$ and $0 \le y \le 2$ (with g smooth), then the surface area of $\mathcal{S}$ is given by

$$\int_0^2 \int_0^2 \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dx\, dy.$$

(vii) If $\mathbf{G}$ is a vector potential associated with a 3D vector field $\mathbf{F}$ then so is $\mathbf{G} + \langle 2, 4, 6\rangle$.

(viii) The area of region R bounded by a curve $\mathcal{C}$ (with the positive orientation) is given by the line integral

$$\int_{\mathcal{C}} y\, dx.$$

(ix) If a 3D vector field $\mathbf{F}$ is irrotational in $D = \left\{(x, y, z) \,\middle|\, x^2 + y^2 + z^2 < 2\right\}$, then for any closed curve $\mathcal{C}$ lying in D,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \;=\; 0.$$

(x) If a 2D vector field $\mathbf{F}$ is irrotational in $D = \left\{(x, y) \,\middle|\, 0 < x^2 + y^2 < 2\right\}$, then for any closed curve $\mathcal{C}$ lying in D,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \;=\; 0.$$

Question 2 a) (5 pts) Suppose a wire is bent into the shape of the parametrized curve

$$x(t) = 5t^3 \qquad y(t) = 2 + t^4 \qquad z(t) = 4t + 3,$$

from point $(0, 2, 3)$ to $(5, 3, 7)$. Suppose the mass density of the wire is given by $\rho(x, y, z) = xyz$. Write down an integral with respect to t whose value gives the mass of the wire (do **not** evaluate the integral).

b) (5 pts) Let $\mathbf{F} = \langle 2xy^2z^2, 2yx^2z^2, 2zx^2y^2 \rangle$. Evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ where $\mathcal{C}$ is curve from $(1, 0, 3)$ to $(0, 1, 2)$ lying on the intersection of the cylinder $x^2 + y^2 = 1$ and plane $y + z = 3$. Here, you must give a final numerical value to the integral.

Question 3 a) (5 pts) Let $\mathbf{F} = \langle 1, 2y+2z, y+z\rangle$. Consider the circle lying **in the plane** $y+z=3$ with centre $(0,1,2)$ and radius 1. Let $\mathcal{S}$ be part of the plane which lies **inside** this circle with normal pointing **upwards**. What is the flux

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S}?$$

b) (5 pts) Compute the flux integral

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S},$$

where $\mathbf{F} = \langle x, -xy, z\rangle$ and $\mathcal{S}$ is the **part of the sphere** $x^2+y^2+z^2=4$ **which lies in the cylinder** $x^2+y^2=1$ above the xy plane with unit normal pointing **upwards**. **You may leave your answer** as a iterated double integral with respect to x and y.

Question 4 (10 pts) Use the Divergence Theorem to find the flux of

$$\mathbf{F} = \langle\, 2x + y^4 + 2z^2 \,,\, ze^y + 2y \,,\, x\sin y + 3z \,\rangle,$$

out of the region bounded by the planes $z = 0$, $z = 2$, $x = 3$, $x = 5$, $y = 1$, and $y = 3$. That is, find

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S},$$

where $\mathcal{S}$ is the boundary of this region bounded (oriented with the outward normal).

Question 5 (10 pts) I want you to use Stokes's Theorem to evaluate the circulation

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r},$$

where $\mathcal{C}$ is the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the parabolic sheet $z = y^2$ with orientation induced by the upward normal to the parabolic sheet. I will not tell you what the $\mathbf{F}$ is but rather I will tell you its curl:

$$\mathbf{curl}\,\mathbf{F} = \langle z^3 y + x\,,\,-2y\,,\,5x^2 + z\rangle.$$

Question 6 a) (5 pts) Let $\mathcal{S}$ be the boundary of a square box of side length $\frac{1}{10}$ centred at the point $(1,2,3)$. Suppose $\mathbf{F}$ is a smooth vector field whose divergence at the point $(1,2,3)$ is given by 2. Use only this information to **approximate** the flux of **F out** of $\mathcal{S}$, i.e.

$$\int_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}.$$

b) (5 pts) Let $\mathbf{F}$ be a smooth 3D vector field and let $\mathcal{C}_\epsilon$ denote a circle with radius ϵ centred at the origin lying in the plane $x+y+z=0$. Give $\mathcal{C}_\epsilon$ the orientation induced by the normal $\langle 1,1,1 \rangle$. What is

$$\lim_{\epsilon \to 0^+} \frac{1}{2\pi\epsilon} \int_{\mathcal{C}_\epsilon} \mathbf{F} \cdot d\mathbf{r}\,?$$

Note here we are taking the limiting value of the **circulation per unit circumference**, not the **circulation per unit area**. You **must explain your reasoning**.