1. Use the Divergence theorem in 2D to prove (show) Greens Theorem. The Divergence theorem in 2D:

$$\iint_D (\nabla \cdot \mathbf{F})\, dA = \oint_C \mathbf{F} \cdot \hat{\mathbf{n}} ds \tag{1}$$

where $\hat{\mathbf{n}}$ is the outward-pointing unit normal to the boundary. Consider $d\mathbf{r} = (dx, dy)$ to be a vector tangential to the curve, C, then the outward normal vector, $\mathbf{n} = (dy, -dx)$. To normalize this, divide by the length $\sqrt{dx^2 + dy^2}$ which, by definition, is ds; these cancel, yielding $\hat{\mathbf{n}} ds = (dy, -dx)$. Then if we define $\mathbf{F} = (M, -L)$, expand the left side of Equation 1 and use the above result on the right side,

$$\iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dA = \oint_C (Ldx + Mdy)$$

Green's Theorem

2. Let $f(x)$ be defined for $-5 \leq x < 5$ by

$$f(x) = \begin{cases} x & \text{if } -5 \leq x \leq 0 \\ x^2 & \text{if } 0 < x \leq 2 \\ x - 2 & \text{if } 2 < x \leq 4 \\ 0 & \text{if } 4 < x < 5 \end{cases} \tag{2}$$

The complete Fourier series is given as:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \tag{3}$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Using $L = 5$ and splitting the domain of the function into separate integrals yields:

$$a_0 = \frac{1}{10} \int_{-5}^{5} f(x) dx = \frac{1}{10} \left(\int_{-5}^{0} x dx + \int_{0}^{2} x^2 dx + \int_{2}^{4} (x-2) dx + \int_{4}^{5} 0 dx \right)$$

$$a_0 = \left(\frac{1}{10}\right) \left(\frac{-25}{2} + \frac{8}{3} + 2 + 0 \right) = -\frac{47}{60} \simeq -0.7833$$

$$a_n = \frac{1}{5}\int_{-5}^{5} f(x)\cos\left(\frac{n\pi x}{5}\right)dx = \frac{1}{5}\left(\int_{-5}^{0} x\cos\left(\frac{n\pi x}{5}\right)dx + \int_{0}^{2} x^2\cos\left(\frac{n\pi x}{5}\right)dx + \int_{2}^{4}(x-2)\cos\left(\frac{n\pi x}{5}\right)dx\right)$$

$$a_n = \frac{5(1-\cos(n\pi))}{n^2\pi^2} + \frac{(4\pi^2n^2-50)\sin\left(\frac{2\pi n}{5}\right)+20\pi n\cos\left(\frac{2\pi n}{5}\right)}{n^3\pi^3} + \frac{2\pi n\sin\left(\frac{4\pi n}{5}\right)-5\cos\left(\frac{2\pi n}{5}\right)+5\cos\left(\frac{4\pi n}{5}\right)}{n^2\pi^2}$$

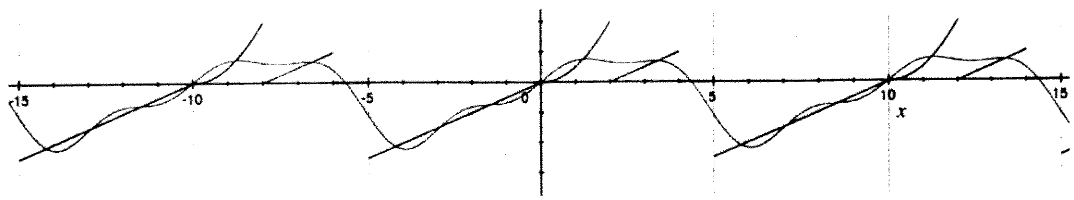
$$b_n = \frac{1}{5}\int_{-5}^{5} f(x)\sin\left(\frac{n\pi x}{5}\right)dx = \frac{1}{5}\left(\int_{-5}^{0} x\sin\left(\frac{n\pi x}{5}\right)dx + \int_{0}^{2} x^2\sin\left(\frac{n\pi x}{5}\right)dx + \int_{2}^{4}(x-2)\sin\left(\frac{n\pi x}{5}\right)dx\right)$$

$$b_n = \frac{-5\cos(n\pi)}{n\pi} + \frac{(50-4\pi^2n^2)\cos\left(\frac{2\pi n}{5}\right)+20\pi n\sin\left(\frac{2\pi n}{5}\right)-50}{n^3\pi^3} - \frac{2\pi n\cos\left(\frac{4\pi n}{5}\right)+5\sin\left(\frac{2\pi n}{5}\right)-5\sin\left(\frac{4\pi n}{5}\right)}{n^2\pi^2}$$

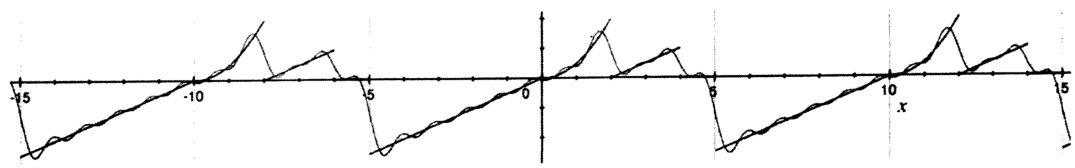
a_n (approximation of integrals)				
n	$\int_{-5}^{0} \frac{x}{5}\cos\left(\frac{n\pi x}{5}\right)dx$	$\int_{0}^{2} \frac{x^2}{5}\cos\left(\frac{n\pi x}{5}\right)dx$	$\int_{2}^{4} \frac{(x-2)}{5}\cos\left(\frac{n\pi x}{5}\right)dx$	sum
1	1.01321	0.303471	-0.192207	1.124474
2	0	-0.154138	-0.16113	-0.315268
3	0.112579	-0.396515	0.264754	-0.019182
4	0	-0.23963	-0.128949	-0.368579
5	0.0405285	0.0810569	0	0.1215854
6	0	0.212115	0.0466325	0.2587475
7	0.0206778	0.0706923	-0.0749352	0.0164349
8	0	-0.117313	0.0845327	-0.0327803
9	0.0125088	-0.124712	-0.0485699	-0.1607731
10	0	0.0202642	0	0.0202642

b_n (approximation of integrals)				
n	$\int_{-5}^{0} \frac{x}{5}\sin\left(\frac{n\pi x}{5}\right)dx$	$\int_{0}^{2} \frac{x^2}{5}\sin\left(\frac{n\pi x}{5}\right)dx$	$\int_{2}^{4} \frac{(x-2)}{5}\sin\left(\frac{n\pi x}{5}\right)dx$	sum
1	1.59155	0.419528	0.331011	2.342089
2	-0.795775	0.448164	-0.29326	-0.640871
3	0.530516	0.102969	0.0210453	0.6545303
4	-0.397887	-0.236226	0.140261	-0.493852
5	0.31831	-0.254648	-0.127324	-0.063662
6	-0.265258	-0.0171995	0.0807273	-0.2017302
7	0.227364	0.162957	-0.0440137	0.3463073
8	-0.198944	0.10445	-0.0124098	-0.1069038
9	0.176839	-0.0690386	0.0594983	0.1672987
10	-0.159155	-0.127324	-0.063662	-0.350141

i) Plot the first 3 terms ($n = 1, 2, 3$) of the Fourier series (blue) with the periodic extension of $f(x)$ (black)



ii) Plot the first 3 terms ($n = 1, 2, 3 \ldots 10$) of the Fourier series (red) with the periodic extension of $f(x)$ (black)



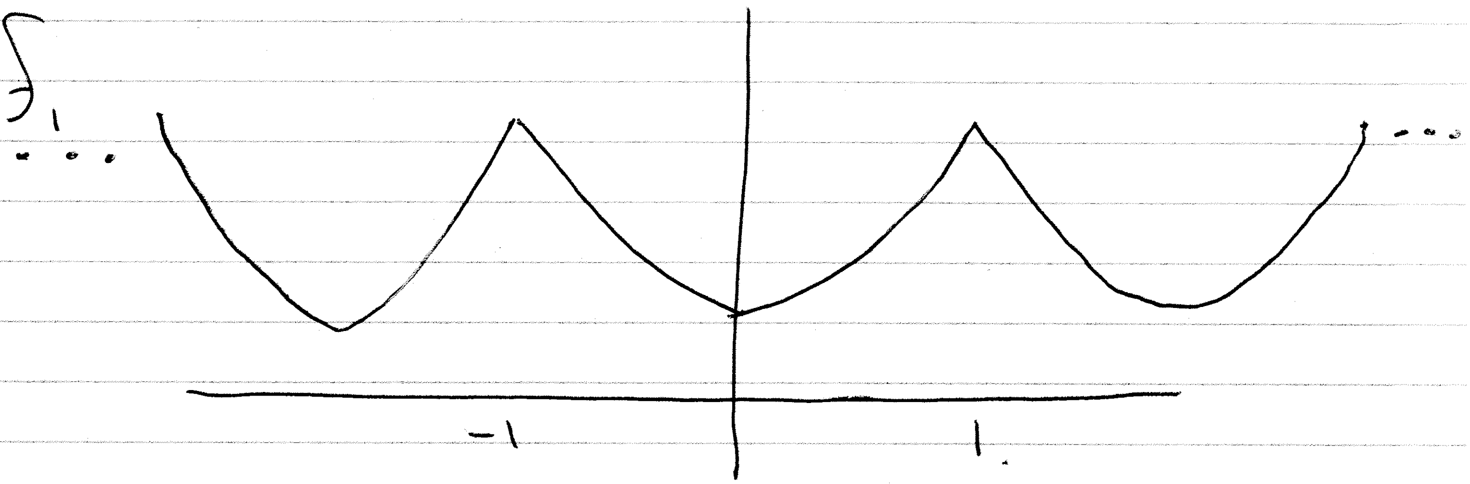
For part (iii)

- sketch $f(x)$ between $-5 \leq x < 5$
- repeat periodically in both direction
- when graph is discontinuous convergence is to the average.

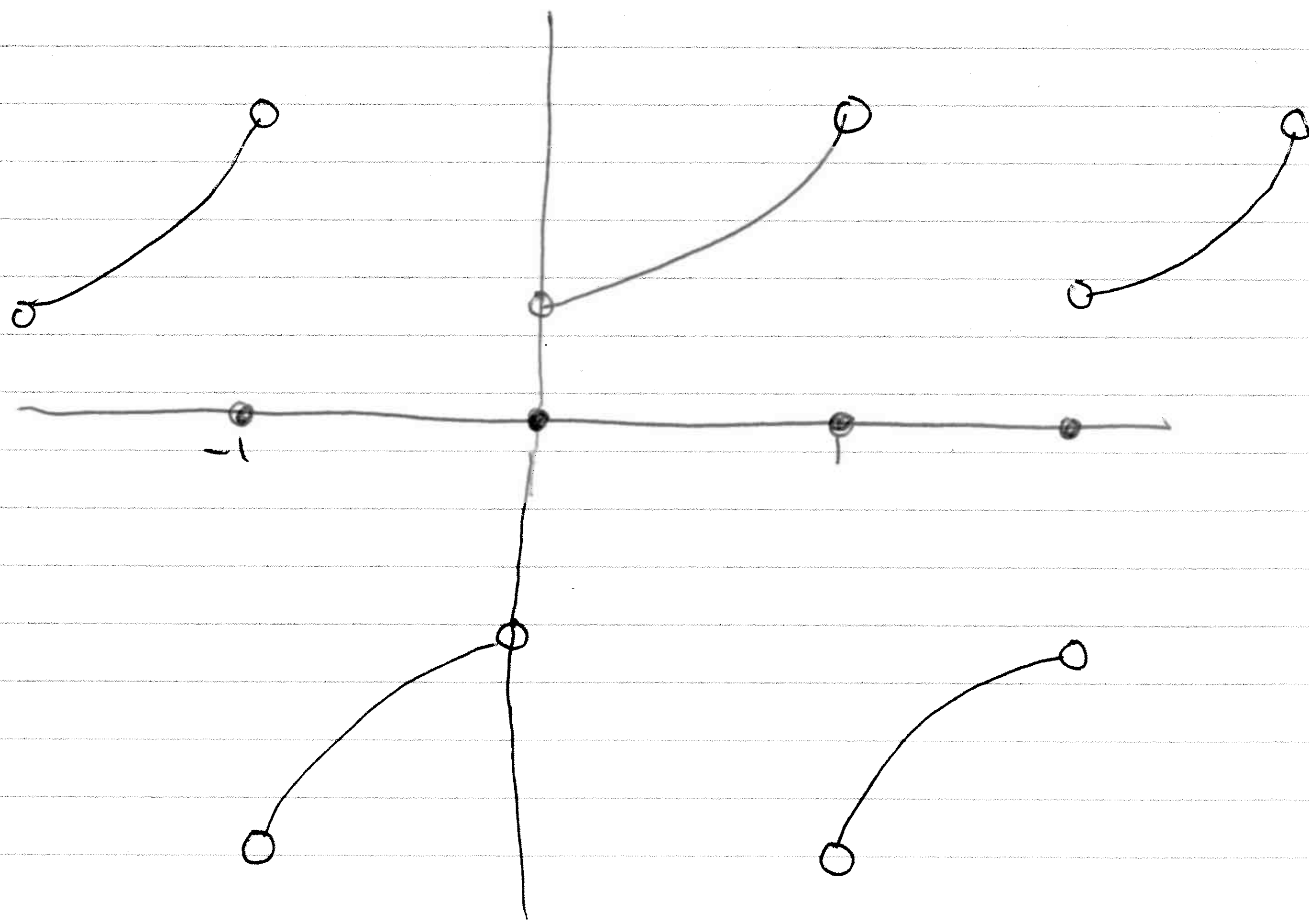
(use • vs ∘)

5) e^x question

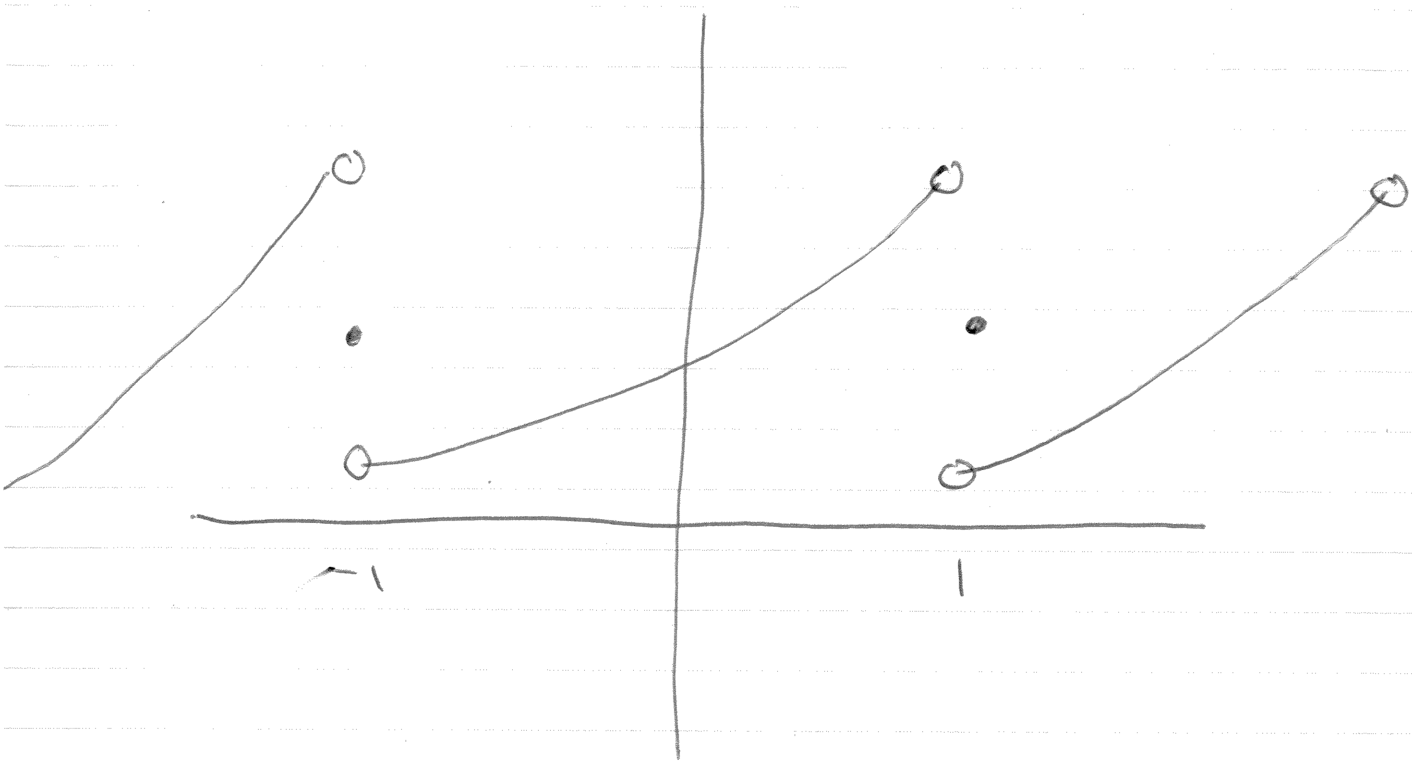
$\int_1$



$\int_2$



f_3



Prerequisites:

Look up the basics about the functions cos, sin, cosh, sinh on your all time favorite Wikipedia.
$\frac{d}{dx}\cosh(x) = \sinh(x)\ \forall x \in \mathbb{R}$ and $\frac{d}{dx}\sinh(x) = \cosh(x)\ \forall x \in \mathbb{R}$.
$x \longrightarrow \cos(x)$ and $x \longrightarrow \cosh(x)$ are even functions.
$x \longrightarrow \sin(x)$ and $x \longrightarrow \sinh(x)$ are odd functions.
$\cosh(x) \geq 1\ \forall x \in \mathbb{R}$ and $\cosh(x) = 1 \iff x = 0$.
$x \longrightarrow \sinh(x)$ is strictly increasing on $\mathbb{R}$ and $\sinh(x) = 0 \iff x = 0$.

Exercise 4 (a) Find the eigenfunctions and eigenvalues of $y''(x) = -\lambda y(x)$, defined on the interval $0 \leq x \leq L,\ y'(0) = 0 = y(L),\ \lambda \in \mathbb{R}$. We will solve for any nonzero L, in particular the result holds for L $= \pi$.

Remark One can show that the differential operator $\frac{d^2}{dx^2}$ operating on the collection of functions satisfying the above ODE and boundary conditions is self-adjoint with respect to the usual inner product. (See Sturm-Liouville theory). Therefore we may assume that the eigenvalues (λ) are real.

- If $\lambda = 0$, the general solution to the differential equation is $y(x) = Ax + B$ for any constants A and B. $y'(0) = 0 \Longrightarrow A = 0$ and $y(L) = 0 \Longrightarrow B = 0$. So $\lambda = 0$ is not an eigenvalue.

- If $\lambda < 0$, let $\lambda = -\mu^2$. Then the general solution to the differential equation is $y(x) = A\cosh(\mu x) + B\sinh(\mu x)$ for any constants A and B, or equivalently $\tilde{A}e^{\mu x} + \tilde{B}e^{-\mu x}$ for any constants $\tilde{A}$ and $\tilde{B}$.

  $y'(0) = 0 \Longrightarrow B\mu = 0$ and since we are assuming $\lambda < 0, \mu \neq 0$ and we must have $B = 0$.

  $y(L) = 0 \Longrightarrow A\cosh(\mu L) = 0$. Since $x \longrightarrow \cosh(x)$ is a strictly positive function on $\mathbb{R}$, we conclude $A = 0$. So there aren't any eigenvalues for $\lambda < 0$.

- If $\lambda > 0$, let $\lambda = \mu^2$. Then the general solution to the differential equation is $y(x) = A\cos(\mu x) + B\sin(\mu x)$ for any constants A and B, or equivalently $\tilde{A}e^{i\mu x} + \tilde{B}e^{-i\mu x}$ for any constants $\tilde{A}$ and $\tilde{B}$.

  $y'(0) = 0 \Longrightarrow B\mu = 0$ and since we are assuming $\lambda > 0$, we must have $B = 0$.

  $y(L) = 0 \Longrightarrow A\cos(\mu L) = 0$. In order to avoid the trivial solution, we must have $\mu L = (2n+1)\frac{\pi}{2}, n \in \mathbb{Z} \Longrightarrow \lambda = \mu^2 = (\frac{(2n+1)\pi}{2L})^2, n \in \mathbb{Z}$. In order to avoid repetition we restrict $n \in \mathbb{N}$ (including 0).

We conclude that the eigenvalues are $\lambda_n = (\frac{(2n+1)}{2})^2, n = 0, 1, 2, ...$ and the eigenfunctions corresponding to the eigenvalues λ_n are $y_n(x) = A_n \cos(\frac{(2n+1)\pi}{2L}x)$.

(b) Find the eigenfunctions and eigenvalues of $y''(x) = -\lambda y(x)$, defined on the interval $0 \leq x \leq 2L,\ y(0) = y(2L),\ y'(0) = y'(2L),\ \lambda \in \mathbb{R}$. We will solve for any nonzero L, in particular the result holds for 2L = 2. It turns out to be trickier to work on [0, 2L] for periodic boundary conditions, and it is easier to work on an interval that is symmetric around 0, i.e. [-L, L]. We essentially shift to the left by L. So the boundary conditions become $y(-L) = y(L)$ and $y'(-L) = y'(L)$.

The general solutions are the same as in (a) since it is the same ODE.

- For $\lambda = 0$, $y(-L) = -AL + B = y(L) = AL + B \implies A = 0$. Moreover, $y'(-L) = y'(L)$ is satisfied by the constant function. So $\lambda = 0$ is an eigenvalue with eigenfunction $y(x) = B$ for any constant B.

- If $\lambda < 0$, $y(-L) = A\cosh(-\mu L) + B\sinh(-\mu L) = y(L) = A\cosh(\mu L) + B\sinh(\mu L) \implies 2B\sinh(\mu L) = 0$. Since $\mu \neq 0, L \neq 0$, we must have $B = 0$.

  $y'(-L) = \mu A\sinh(-\mu L) + \mu B\cosh(-\mu L) = y'(L) = \mu A\sinh(\mu L) + \mu B\cosh(\mu L) \implies 2\mu A\sinh(\mu L) = 0$. Since $\mu \neq 0, L \neq 0$, we must have $A = 0$.
  So there aren't any eigenvalues for $\lambda < 0$.

- If $\lambda > 0$, $y(-L) = A\cos(-\mu L) + B\sin(-\mu L) = y(L) = A\cos(\mu L) + B\sin(\mu L) \implies 2B\sin(\mu L) = 0$. $y'(-L) = -\mu A\sin(-\mu L) - \mu B\cos(-\mu L) = y'(L) = -\mu A\sin(\mu L) - \mu B\cos(\mu L) \implies 2\mu A\sin(\mu L) = 0$. Since $\mu \neq 0$, we must have $\sin(\mu L) = 0$ in order to avoid the trivial solution. Thus $\mu L = n\pi, n \in \mathbb{Z}\setminus\{0\}$, and so the eigenvalues are $\lambda_n = \frac{n^2\pi^2}{L^2}$, $n \in \mathbb{N}\setminus\{0\}$ (again avoiding repetition) and the eigenfunctions are $A_n\cos(\frac{n\pi x}{L})$ and $B_n\sin(\frac{n\pi x}{L})$.

To conclude, on [-L, L], the eigenvalues are 0 with eigenfunction $y(x) =$ constant, and for n = 1,2,3, there are 2 eigenfunctions for the eigenvalue $\lambda_n = \frac{n^2\pi^2}{L^2}$, namely, $A_n\cos(\frac{n\pi x}{L})$ and $B_n\sin(\frac{n\pi x}{L})$

On [0, 2L], the eigenvalues are 0 with eigenfunction $y(x) =$ constant, and for n = 1,2,3..., there are 2 eigenfunctions per eigenvalue, namely, $A_n\cos(\frac{n\pi(x-L)}{L})$ and $B_n\sin(\frac{n\pi(x-L)}{L})$.

Exercise ~~6~~ 8 Solve $\nabla^2 u(x,y) = 0$ on the square $[0,2]\times[0,1]$ with Dirichlet boundary conditions:
$\forall x \in [0,2]$, $u(x,0) = u(x,1) = 0$; $\forall y \in [0,1]$, $u(0,y) = 0$, $u(2,y) = 3\sin(\pi y)$.

Assuming that $u(x,y) = X(x)Y(y)$, we get $-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} = \lambda$ constant. To decide which ODE to solve first, it is advised to look at the boundary conditions. $u(x,0) = u(x,1) = 0 \implies X(x)Y(0) = X(x)Y(1) = 0$. Since this must be so for all x, we conclude $Y(0) = 0 = Y(1)$. So we have a well posed Sturm Liouville eigenvalue problem for the equation in y. Meanwhile, $u(0,y) = 0, u(2,y) = 3\sin(\pi y) \implies X(0) = 0$, but no information about $X(1)$ nor $X'(1)$. So the ODE in x is not a well posed Sturm Liouville problem.

Solving the equation in y with the above boundary conditions yields the eigenvalues $\lambda_n = -n^2\pi^2$ and eigenfunctions $Y_n(y) = B_n\sin(n\pi y)$, B_n are arbitrary constants and $n = 1,2,3...$ The general solution to $-\frac{X''(x)}{X(x)} = -n^2\pi^2$ is $X_n(x) = C_n\cosh(n\pi x) + D_n\sinh(n\pi x)$. $X(0) = 0 \implies X_n(x) = D_n\sinh(n\pi x)$.

For $n = 1,2,3...$, $X_n(x)Y_n(y)$ solves the PDE and 3 out of the 4 boundary conditions. By linearity, $\sum_{n=1}^{\infty} B_n\sin(n\pi y)\sinh(n\pi x)$ also solves the PDE and satisfies 3 out the 4 boundary conditions.

Finally, $u(2,y) = 3\sin(\pi y) \implies \sum_{n=1}^{\infty} B_n\sin(n\pi y)\sinh(2n\pi) = 3\sin(\pi y)$, which we recognize as a Fourrier sine series. By inspection, $B_n\sinh(2n\pi) = 3$ if $n = 1$, $B_n = 0$ otherwise.

The solution to the problem is $u(x,y) = \frac{3\sin(\pi y)\sinh(\pi x)}{\sinh(2\pi)}$.

7) $u_t = u_{xx}$ $\qquad 0 < x < \pi$

$u_x(0,t) = 0 \qquad u(\pi, t) = 0$

$u(x,0) = x$

Separation of Variables

$\Rightarrow \quad X'' + \lambda X = 0$

$X'(0) = 0 \qquad X(\pi) = 0$

by perious question

solutions are:

$$\lambda_n = \left(\frac{2n+1}{2}\right)^2 \qquad X_n(x) = \cos\left(\frac{2n+1}{2}x\right)$$

$$n = 0, 1, \ldots$$

or $\lambda_n = \left(\frac{2n-1}{2}\right)^2 \qquad X_n(x) = \cos\left(\frac{2n-1}{2}x\right)$

for $n = 1, 2, 3, \ldots$

T equation is as usual

$$T_n(t) = e^{-\left(\frac{2n-1}{2}\right)^2 t}.$$

So general solution form is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \cos\left(\frac{2n-1}{2} x\right) e^{-\left(\frac{2n-1}{2}\right)^2 t}.$$

to find c_n we must write

$$x = \sum_{n=1}^{\infty} c_n \cos\left(\frac{2n-1}{2} x\right)$$

this is a different type of Fourier series but we can still find c_n as

$\left\{ \cos\left(\frac{2n-1}{2} x\right) \right\}$ are

mutually orthogonal on $[0, L]$

$$\therefore \quad c_n = \frac{2}{\pi} \int_0^{\pi} x \cos\left(\frac{2n-1}{2} x\right) dx$$

As $t \to \infty$ the solution converges to 0.

This makes sense since one end is insulated <u>but</u> the other is exposed to the "outside" with fixed temp. 0.

Thus all initial temp will eventually "escape".

Solutions

Q. Solve wave equation

$$\psi_{tt} = 9\psi_{xx} \quad , \; 0<x<\pi, \; t>0$$

$$\psi(0,t)=0=\psi(\pi,t), \; t>0$$

$$\psi(x,0) = 5\sin 7x \quad \& \quad \psi_t(x,0) = 3\sin 2x + 4\sin 5x$$

<u>Small displacement of a string</u>

<u>Wave equation</u> $\dfrac{\partial^2 \psi}{\partial t^2} - a^2 \nabla^2 \psi = F$

matching $a = 3 \; m.s^{-1}$ and $F=0$

$\psi(0,t)=0=\psi(\pi,t)$: ends are fixed

$\psi(x,0)$ and $\psi_t(x,0) \neq 0$ there are initial displacement and initial velocity

<u>Let $\psi(x,t) = X(x)\,T(t)$</u>

Substitute in PDE, $XT'' - 9X''T = 0$

Divide by XT, $\dfrac{T''}{T} = 9\dfrac{X''}{X} = -\lambda$

$$\frac{X''}{X} = \frac{1}{9}\frac{T''}{T} = -\lambda = \text{constant}$$

① $\dfrac{X''}{X} = -\lambda$ i.e. $X'' = -\lambda X$: eigenvalue problem

- $\psi(0,t)=0$ and $\psi(\pi,t)=0$

$X(0)=0$ $\qquad X(\pi)=0$

hermition operator, $X_n(x) = A_n \sin\left(\dfrac{n\pi x}{L}\right)$, $\lambda = +\dfrac{n^2\pi^2}{L^2}$ $\forall n=1,2,\ldots$

In our case, $L=\pi$ so $\underline{X_n(x) = A_n \sin(nx)}$

$$\underline{\lambda = +n^2}$$

(2) $\frac{1}{9}\frac{T''}{T} = -n^2$

$T'' = -9n^2$

let $T = e^{rt}$

$r^2 = -9n^2$

$r = \pm i\,3n$

Hence $\underline{T_n(x,t) = B_n \cos(3nt) + C_n \sin(3nt)}$

$\psi_n(x) = X_n(x)T_n(t)$

$\psi_n(x) = A_n \sin(nx)\,(B_n \cos(3nt) + C_n \sin(3nt))$

$\underline{\psi_n(x) = \sin(nx)\,(D_n \cos(3nt) + E_n \sin(3nt))}$

Our original equation can be re-written as $\underbrace{\left(9\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)}_{\text{Linear operator}}\psi = 0$

In order to find the nullspace of L we use Principle of Superposition

$\psi(x,t) = \sum_{n=1}^{\infty} \sin(nx)\,(D_n \cos(3nt) + E_n \sin(3nt))$

<u>let's use IC's</u>

- $\psi(x,0) = \sum_{n=1}^{\infty} \sin(nx)\,D_n = 5\sin 7x = g(x)$

→ sine Fourier Series

$D_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{\pi n x}{L}\right)dx$

or <u>matching</u> $D_7 = 5$, all other D's are 0.

$$\therefore \quad \psi_t(x,0) = \sum_{n=1}^{\infty} \sin(nx)\left(-D_n\, 3n \sin(3nt) + 3n E_n \cos(3nt)\right)\Big|_{t=0}$$

$$= \sum_{n=1}^{\infty} \sin(nx)\,(3n E_n) = 3\sin 2x + 4\sin 5x$$

Hence $6E_2 = 3$ so $\underline{E_2 = \frac{1}{2}}$

and $3\cdot 5\, E_5 = 4$ so $\underline{E_5 = \frac{4}{15}}$

all other E's are 0.

HENCE

$$\therefore \psi(x,t) = \sin(2x)\,\frac{1}{2}\sin 6t + \sin(5x)\,\frac{4}{15}\sin(15t)$$

$$+ \sin(7x)\, 5\cos(21t)$$