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An application of anafunctors

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Abstract. The category pFun is defined to have objects (X, A, Φ) where X, A are small categories, $\Phi : X \to A$ is a functor, and arrows $(X, A, \Phi) \xrightarrow{(\tau,\Sigma,\theta)} (X', A', \Phi')$ where

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "\tau"'] \arrow[dr, phantom, "\Sigma \atop \cong"] & A \arrow[d, "\theta"] \\ X' \arrow[r, "\Phi'"'] & A' \end{tikzcd}$$

choice of orientation:

$$\Sigma : \theta\Phi \xrightarrow{\cong} \Phi'\tau$$

(pFun is a 'pseudo-version' of the arrow-category $Cat^{\mathbf{2}}$) with composition the obvious pasting operation.
We prove that pFun is an accessible category, in fact $pFun \simeq Reg(\mathbb{C}, Set)$, for a small (countable) regular category $\mathbb{C}$.

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I define two categories, pFun and Sana.

pFun: object: (X, A, Φ) where

X, A are small categories,

$\Phi : X \to A$ is a functor;

arrow: $(X, A, \Phi) \longrightarrow (X', A', \Phi')$

(τ, Σ, θ)

where

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "\tau"'] \arrow[dr, phantom, "\cong\ \Downarrow\Sigma"] & A \arrow[d, "\theta"] \\ X' \arrow[r, "\Phi'"'] & A' \end{tikzcd}$$

$$\Sigma : \theta\Phi \xrightarrow{\cong} \Phi'\tau \qquad (*)$$

(natural isomorphism);

composition: usual pasting.

Sana: object: $F = (|F|, X, A, \varphi, \psi)$, a span:

$$\begin{tikzcd} & {|F|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ X & & A \end{tikzcd}$$

of small categories and functors

such that: 1), 2) & 3) next page hold;

1) <u>Whenever</u> $s, t \in |F|$ (meaning: s, t <u>objects</u> of the category $|F|$) and $f : \varphi(s) \longrightarrow \varphi(t)$ (in X),

there is a unique $g : s \to t$ (in $|F|$),

such that $\varphi(g) = f$; <u>notation</u>: $g = g_{s,t,f}$.

2) For $x \in X$ and $b \in A$, <u>define</u>:

$$|F|(x, b) \underset{\text{def}}{=} \varphi^{-1}(x) \cap \psi^{-1}(b)$$

$$\underset{\text{def}}{=} \{ t \in |F| : \varphi(t) = x \ \& \ \psi(t) = b \}$$

and <u>define</u> the map, for $s \in |F|$, $b \in A$:

$$F_{s,b} : |F|(\varphi(s), b) \longrightarrow \mathrm{Iso}(\psi(s), b)$$

$$t \longmapsto \psi(g_{s,t,1_{\varphi(s)}})$$

(well-defined by 1)).

<u>Require</u>: $F_{s,b}$ is a <u>bijection</u>. [$\mathrm{Iso}(a, b)$ is the set of all isomorphism $a \xrightarrow{\cong} b$.]

3) φ is surjective on objects:

for all $x \in X$, there is $s \in |F|$ s.t. $\varphi(s) = x$.

(<u>end</u> of <u>definition</u> of 'object' of Sara).

arrow of Sana: arrow of spans:

$$F \xrightarrow{(\tau,\sigma,\theta)} F' :$$

$$\begin{tikzcd}
& {|F|} \arrow[dl, "\varphi"'] \arrow[d, "\psi"] \arrow[rrr, "\sigma"] & & & {|F'|} \arrow[dl, dashed, "\varphi'"'] \arrow[d, "\psi'"] \\
X \arrow[rrr, dashed, "\tau"] & A \arrow[rrr, "\theta"'] & & X' & A'
\end{tikzcd}$$

$$\text{satisfying} \quad \left.\begin{aligned} \tau\varphi &= \varphi'\sigma, \\ \theta\psi &= \psi'\sigma. \end{aligned}\right\} (*)$$

composition: usual. (componentwise).

$\boxed{\text{Theorem}}$ pFun and Sana are equivalent categories.

$\boxed{\text{Corollary}}$ pFun is a dual-regular category: for some (countable) regular category $\mathbb{C}$,

$\text{pFun} \simeq \text{Reg}(\mathbb{C}, \text{Set}) =$ the category of all regular functors $\mathbb{C} \to \text{Set}$, a full subcategory of $[\mathbb{C}, \text{Set}]$.

In particular, pFun is an accessible category.

Proof: Sana is easily seen to be dual-regular.

For the proof of the theorem, I introduce a third category, SanapFun, a 'combination' of the previous two:

<u>object</u>: $(F, \overline{\Phi}, \mu) = (|F|, X, A, \overline{\Phi}, \varphi, \psi, \mu)$:

(*).
$$\begin{tikzcd} & {|F|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ X \arrow[rr, "\overline{\Phi}"'] \arrow[rr, phantom, bend left=40, "\overset{\simeq}{\Rightarrow}\mu"] & & A \end{tikzcd} \qquad \mu : \overline{\Phi}\varphi \xrightarrow{\simeq} \psi$$

such that: $F = (|F|, X, A, \varphi, \psi) \in$ Sana;

<u>arrow</u>: $(F, \overline{\Phi}, \mu) \xrightarrow[(\tau, \sigma, \Sigma, \theta)]{} (F', \overline{\Phi}', \mu')$

$$\begin{tikzcd} & & {|F'|} \arrow[dl, dashed, "\varphi'"'] \arrow[d, "\psi'"] \\ {|F|} \arrow[urr, "\sigma"] \arrow[d, "\varphi"'] \arrow[dr, "\psi"] & X' \arrow[r, dashed, "\overline{\Phi}'"] & A' \\ X \arrow[r, "\overline{\Phi}"'] \arrow[ur, dashed, "\tau"] & A \arrow[ur, "\theta"'] & \end{tikzcd}$$

with
$$\begin{tikzcd} & {|F|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ X \arrow[rr] \arrow[rr, phantom, bend left=40, "\mu"] & & A \end{tikzcd} \qquad \begin{tikzcd} & {|F'|} \arrow[dl, "\varphi'"'] \arrow[dr, "\psi'"] & \\ X' \arrow[rr] \arrow[rr, phantom, bend left=40, "\mu'"] & & A' \end{tikzcd} \qquad \begin{tikzcd} & X' \arrow[r, dashed, "\overline{\Phi}'"] & A' \\ X \arrow[r, "\overline{\Phi}"'] \arrow[ur, dashed, "\tau"] & \cdot \arrow[ur, "\theta"'] \arrow[u, phantom, "\Uparrow\Sigma"] & \end{tikzcd}$$

such that: $F \xrightarrow[(\tau,\sigma,\theta)]{} F'$ is an arrow

in Sana, and

$$\Phi \xrightarrow[(\tau,\Sigma,\theta)]{} \Phi' \quad \text{is an arrow}$$

in pFun,

and μ, Σ, σ and μ' are compatible:

$$
\left.
\begin{tikzcd}
 & {|F|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] \arrow[d, phantom, "\Leftarrow\mu^{-1}"] & \\
X \arrow[rr, "\Phi"'] \arrow[d, "\tau"'] & \; \arrow[d, phantom, "\Downarrow\Sigma"] & A \arrow[d, "\theta"] \\
X' \arrow[rr, "\Phi'"'] & \; & A'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & {|F|} \arrow[d, "\sigma"] & \\
 & {|F'|} \arrow[dl, "\varphi'"'] \arrow[dr, "\psi'"] \arrow[d, phantom, "\Leftarrow(\mu')^{-1}"] & \\
X' \arrow[rr, "\Phi'"'] & \; & A'
\end{tikzcd}
\right\} \quad (*)
$$

that is, $\boxed{\Sigma\varphi \circ \theta(\mu^{-1}) = (\mu')^{-1}\sigma\,.}$

Composition is defined so that we have

the forgetful functors (see next page) R & S:

$$\begin{tikzcd} & \mathrm{SanapFun} \arrow[dl, "S"'] \arrow[dr, "R"] & \\ \mathrm{Sana} & & \mathrm{pFun} \arrow[ul, bend right, "\hat{\Gamma}"'] \end{tikzcd}$$

I will also define, canonically, a functor $\hat{\Gamma}$ as shown, and show that: 1) $(R, \hat{\Gamma})$ form an equivalence of categories; and

2) S is a surjective equivalence.

Of course, the theorem will follow.

$\boxed{\text{Lemma 1.}}$ Let $(F, \Phi, \Lambda) \in$ Sanap Fun (for notation, see $(*)$, p 4).

Let $\Phi \| A$ denote the set all triples (x, a, ν) such that $x \in X$, $a \in A$ and $\nu : \Phi x \xrightarrow{\cong} a$.

We have the mapping $\boxed{sp : ob|F| \longrightarrow \Phi \| A}$ defined by $sp(s) = (\varphi(s), \psi(s), \Lambda_s : \Phi\varphi(s) \xrightarrow{\cong} \psi(s))$ $(s \in |F|)$.

Assertion: sp is a bijection.

<u>Proof</u>. 1) sp is <u>injective</u>: suppose $s, t \in |F|$ and $\varphi(s) = \varphi(t) = x$, $\psi(s) = \psi(t) = a$, and $\mu_s = \mu_t : \Phi x \xrightarrow{\cong} a$, to show $s \overset{?}{=} t$.

By condition 1), p. ②, there is $g : s \to t$ in $|F|$ such that $\varphi(g) = 1_x$. Apply naturality of μ against the arrow g:

$$\begin{tikzcd}
\Phi\varphi(s) \arrow[r, "\Phi\varphi(g)"] \arrow[d, "\mu_s"'] \arrow[dr, phantom, "\#"] & \Phi\varphi(t) \arrow[d, "\mu_t"] \\
\psi(s) \arrow[r, "\psi(g)"'] & \psi(t)
\end{tikzcd}$$

This reduces to

$$\begin{tikzcd}
& \Phi x \arrow[dl, "\cong"', "\mu_t = \mu_s"'] \arrow[dr, "\cong", "\mu_t = \mu_s"] & \\
a \arrow[rr, "\psi(g)"'] & & a
\end{tikzcd}$$

therefore $\psi(g) = 1_a$. Now, apply the uniqueness part of condition 2) on p. ②:

$$F_{s,a} : |F|(x, a) \longrightarrow \mathrm{Iso}(a, a)$$

$$u \longmapsto \psi(g_{s,u,1_{\varphi(s)}})$$

⑧

is injective. $F_{s,a}(s) = \psi(g_{s,s,1_{\varphi(s)}}) = \psi(1_s) = 1_{\psi(s)} = 1_a$,

and $F_{s,a}(t) = \psi(g_{s,t,1_{\varphi(s)}}) = \psi(g) = 1_a$

$\uparrow$ our g above — as shown on the prev. page

It follows that $s = t$.

2) s_1 is surjective: suppose $(x, a, \nu: \Phi x \xrightarrow{\cong} a) \in \Phi \| A$

Choose $s \in |F|$ such that $\varphi(s) = x$ (condition 3), p. ②).

Use condition 2), existence, to find $t \in |F|$ such that

for $g = g_{s,t,1_x} : s \to t$, we have $\psi(t) = a$, and

$$F_{s,a}(t) = \psi(g) = \nu \circ \mu_s^{-1} : \psi(s) \xrightarrow{\cong} a \quad (!).$$

Apply naturality of μ to $g : s \to t$:

$$\begin{tikzcd}
x = \Phi\varphi(s) \arrow[r, "\Phi\varphi(g) = 1_x"] \arrow[d, "\mu_s"', "\cong"] & \Phi\varphi(t) = x \arrow[d, "\mu_t"] \\
\psi(s) \arrow[r, "\cong", "\psi(g)"'] & \psi(t) = a
\end{tikzcd}$$

Given what $\psi(g)$ is, it follows that $\mu_t = \nu$.

We have found $t \in |F|$ such that

$$sp(t) = (x, a, v).$$

$\square$ Lemma 1

Lemma 2. The forgetful functors

$$R : \text{Sarap Fun} \longrightarrow \text{p Fun}$$

$$S : \text{Sarap Fun} \longrightarrow \text{Sara}$$

are full and faithful.

Remark. The Lemma is (essentially) a generalization of statement 9′ on p. 123 of "Avoiding the axiom of choice in general category theory", JPAA 108 (1996), 109–173. Said statement is the special case of Lemma 2 when τ and Θ are identity functors.

Proof. For $\underset{\sim}{F} = (F, \Phi, \mu)$, an object of Sanap Fun,

$$
\begin{tikzcd}
 & {|F|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\
X \arrow[rr, "\Phi"'] & \arrow[u, phantom, "\overset{\mu}{\Rightarrow}\cong"] & A
\end{tikzcd}
$$

$R(\underset{\sim}{F})$ is $R(\underset{\sim}{F}) = \Phi = (X, A, \Phi)$, and

$S(\underset{\sim}{F})$ is $S(\underset{\sim}{F}) = F = (|F|, X, A, \varphi, \psi)$.

Thus, S being full and faithful means that

if $\underset{\sim}{F} = (F, \Phi, \mu)$, $\underset{\sim}{F}' = (F', \Phi', \mu')$ are objects of Sanap Fun,

and $(\tau, \sigma, \theta): F \longrightarrow F'$ is an arrow in Sana,

then there is a unique Σ, denoted for short by $\Sigma = \hat{\sigma}$,

for which $(\tau, \sigma, \Sigma, \theta): \underset{\sim}{F} \to \underset{\sim}{F}'$.

R being full and faithful is a 'converse': for $\underset{\sim}{F}$

and $\underset{\sim}{F}'$ as before, if $(\tau, \Sigma, \theta): \Phi \longrightarrow \Phi'$,

there is a unique σ, denoted $\sigma = \check{\Sigma}$, such that

$(\tau, \check{\Sigma}, \Sigma, \theta): \underset{\sim}{F} \to \underset{\sim}{F}'$!

1) <u>Proof that</u> S <u>is</u> <u>full</u> <u>and</u> faithful : "given σ, define Σ", in the notation of the previous page.

For $x \in X$, we need to define the isomorphism arrow

$$\Sigma_x : \Theta\Phi(x) \xrightarrow{\cong} \Phi'\tau(x) \qquad (1)$$

such that : for any $s \in |F|$ with $\varphi(s) = x$, we have the commutativity

$$\begin{tikzcd}
\Phi'\tau(x) \arrow[rr, "\Sigma_x^{-1}"] \arrow[dr, "\mu'_{\sigma(s)}"', "\cong"] & \arrow[d, phantom, "\#"] & \Theta\Phi(x) \arrow[dl, "\Theta(\mu_s)", "\cong"'] \\
& \varphi'\sigma(s) = \theta\varphi(s) &
\end{tikzcd} \qquad (2)$$

It is immediate, therefore, that, if the natural transformation $\Sigma : \theta\Phi \xrightarrow{\cong} \Phi'\tau$ exists, it is unique, since by condition 3), there exist at least one $s \in |F|$ with $\varphi(s) = x$.

For any given $s \in |F|$, let

$$\hat{s} : \Theta\Phi\varphi(s) \xrightarrow{\cong} \Phi'\tau\varphi(s)$$

be the arrow for which

$$\begin{tikzcd}
\Phi'\tau\varphi(s) \arrow[rr, "\hat{s}^{-1}"] \arrow[dr, "\mu'_{\sigma(s)}"'] \arrow[rr, phantom, shift right=6ex, "\#"] & & \Theta\Phi\varphi(s) \arrow[dl, "\Theta(\mu_s)"] \\
& \psi'\sigma(s) = \Theta\psi(s) &
\end{tikzcd}$$

.

I claim that $\varphi(s_1) = \varphi(s_2)$ implies $\hat{s}_1 = \hat{s}_2$.

The claim is equivalent to the commutativity of the outside square of the following diagram of isomorphisms:

$$\begin{tikzcd}
& \psi'\sigma(s_1) = \Theta\psi(s_1) \arrow[dd, dashed, "\psi'\sigma(i) = \Theta\psi(i)" description] & \\
\Phi'\tau(x) \arrow[ur, "\mu'_{\sigma(s_1)}"] \arrow[dr, "\mu'_{\sigma(s_2)}"'] \arrow[r, phantom, "\#1"] & {} \arrow[r, phantom, "\#2"] & \Theta\Phi(x) \arrow[ul, "\Theta(\mu_{s_1})"'] \arrow[dl, "\Theta(\mu_{s_2})"] \\
& \psi'\sigma(s_2) = \Theta\psi(s_2) &
\end{tikzcd}$$

(Here we've put $x = \varphi(s_1) = \varphi(s_2)$, and defined $i : s_1 \longrightarrow s_2$ in $|F|$, an isomorphism (p. 2), such that $\varphi(i) = 1_x$ (condition 1)). The *left* triangle #1 commutes by the naturality of μ' tested by the arrow $\sigma(i) : \sigma(s_1) \longrightarrow \sigma(s_2)$ in $|F'|$:

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\Phi'\tau x = \Phi'\tau\varphi(s_1) = \Phi'\varphi'\sigma(s_1) \arrow[r, "\Phi'\varphi'\sigma(i)", "{\shortparallel \atop \Phi'\tau\varphi(i) = \Phi'\tau(1_x) = 1_{\Phi'\tau(x)}}"'] \arrow[d, "\mu'_{\sigma(s_1)}"'] \arrow[dr, phantom, "\#"] & \Phi'\varphi'\sigma(s_2) = \Phi'\tau(x) \arrow[d, "\mu'_{\sigma(s_2)}"] \\
\psi'\sigma(s_1) \arrow[r, "\psi'\sigma(i)"'] & \psi'\sigma(s_2)
\end{tikzcd}
$$

The *right* triangle #2 commutes by the naturality of μ tested by $i : s_1 \to s_2$, with an application of θ. This proves the *claim*.

The claim allows us to define, for any given $x \in X$, the arrow Σ_x (see (1), p. 11) satisfying (2) [p. 11] for all $s \in |F|$ with $\varphi(s) = x$.

It remains to show that $\Sigma = (\Sigma_x)_{x \in X}$ so defined is a natural transformation $\Sigma : \Theta\Phi \longrightarrow \Phi'\tau$.

The naturality of Σ is shown by the commutativity of the outer square in the following diagram, where $f : x \to y$ is an arbitrary arrow in X, $s, t \in |F|$ and $g : s \to t$ in $|F|$ are such that $\varphi(s) = x$, $\varphi(t) = y$ (by using 3), p. 2), and $\varphi(g) = f$ (by using 1), p. 2), and thus $\Sigma_x = \hat{s}$, $\Sigma_y = \hat{t}$:

$$
\begin{tikzcd}
\Phi'\tau(x) = \Phi'\tau\varphi(s) \arrow[rr, "\Sigma_x^{-1}"] \arrow[dr, "\cong"] \arrow[ddd, "\Phi'\tau(x) = \Phi'\tau\varphi(s)"'] & & \Theta\Phi\varphi(s) = \Theta\Phi(x) \arrow[dl, "\cong"', "\Theta(\mu_s)"] \arrow[ddd, "\Theta\Phi\varphi(y) = \Theta\Phi(f)"] \\
& \psi'\sigma(s) = \Theta\psi(s) \arrow[d, "\psi'\sigma(g) = \Theta\psi(g)"] & \\
& \psi'\sigma(t) = \Theta\psi(t) & \\
\Phi'\tau(y) = \Phi'\tau\varphi(t) \arrow[rr, "\Sigma_y^{-1}"] \arrow[ur, "\mu'_{\sigma(t)}"] & & \Theta\Phi\varphi(t) = \Theta\Phi(y) \arrow[ul, "\Theta(\mu_t)"']
\end{tikzcd}
$$

where, on the two sides, the commutativities are ensured by the naturality of μ', and μ, respectively.

2) "Given Σ, define σ": proof of R being full and faithful:

To define the functor $\sigma : |F| \longrightarrow |F'|$,

first we define the object-function of σ. Let $s \in |F|$. The object $s' \underset{\mathrm{def}}{=} \sigma(s)$ of $|F'|$ has to satisfy the following three equalities:

$$\left.\begin{array}{lcl} \varphi'(s') & = & \tau\varphi(s) \\ \psi'(s') & = & \theta\psi(s) \\ \mu'_{s'} & = & \theta(\mu_s)\circ\Sigma^{-1}_{\varphi(s)} \end{array}\right\} \quad (*)$$

(see) see (*), p 3.

(note:
$$\begin{tikzcd} \Phi'\tau\varphi(s) \arrow[rr, "\mu'_{s'}"] \arrow[dr, "\Sigma^{-1}_{\varphi(s)}"'] & \arrow[d, phantom, "\#"] & \theta\psi(s) \\ & \theta\Phi\varphi(s) \arrow[ur, "\theta(\mu_s)"'] & \end{tikzcd}$$

is required by (*), p 5)

We apply Lemma 1 (p 6) to $(\mathbb{F}', \Phi', \mu') \in$ SanapFun. There is a unique $s' \in |F'|$ satisfying the three equalities (*), previous page. The uniqueness of the effect of σ on objects immediately follows. What remains is to define the effect of σ on arrows in $|F|$, and prove that σ is indeed a functor, and that the equalities (*), p 3 hold true for arrows in $|F|$ as arguments for σ as well (we have made sure that those equalities hold for objects in $|F|$; also that (*), p 5 holds; this latter equality involves only objects of $|F|$ as arguments of σ.)

Let $g: s \to t$ be an arrow in $|F|$. Look at the first equality in (*), p 3; this requires of $\sigma(g) \underset{def}{=} g'$ that it satisfies

$$\varphi'(g') = \tau\varphi(g) : \underset{\varphi'\sigma(s)}{\underset{\|}{\tau\varphi(s)}} \longrightarrow \underset{\varphi'\sigma(t)}{\underset{\|}{\tau\varphi(t)}}.$$

According to condition 1), p 2, on F', there is a unique such arrow $g': \sigma(s) \to \sigma(t)$; this we take to be $\sigma(g)$. Thus, we have for $s \xrightarrow{g} t$ in $|F|$, $\quad \sigma(s) \xrightarrow{\sigma(g)} \sigma(t)$ in $|F'|$

such that

$$\varphi'(\sigma(g)) = \tau\varphi(g);$$

in particular, the first of the two equations $(*)$ on p. 3 is taken care of.

We turn to verifying the second equation in $(*)$, p 3, for arrow-argument in $|F|$: for $g : s \to t$ in $|F|$, and $g' = \sigma(g)$,

$$(*) \qquad \theta\psi(g) \overset{?}{=} \psi'(g') \; : \; \begin{array}{c}\theta\psi(s) \\ \| \\ \psi'\sigma(s)\end{array} \longrightarrow \begin{array}{c}\theta\psi(t) \\ \| \\ \psi'\sigma(t)\end{array}$$

For any $g' : s' \to t'$ in $|F'|$, by the naturality of μ'

we have

$$\begin{tikzcd}
\Phi'\varphi'(s') \arrow[r, "\Phi'\varphi'(g')"] \arrow[d, "\mu'_{s'}"', "\cong"] \arrow[dr, phantom, "\#"] & \Phi'\varphi'(t') \arrow[d, "\mu'_{t'}", "\cong"'] \\
\psi'(s') \arrow[r, "\psi'(g')"'] & \psi'(t')
\end{tikzcd}$$

moreover, $\psi'(g')$ is the <u>unique</u> arrow $\psi'(s') \to \psi'(t')$ that makes the last diagram commute, since $\mu'_{s'}$ is an isomorphism.

Therefore, the questioned equality (*), previous page is equivalent to saying that the following commutes:

$$\begin{tikzcd}
\bar{\Phi}'\varphi'(s') \arrow[r, "\bar{\Phi}'\varphi'(g')"] \arrow[d, "\mu'_{s'}"'] \arrow[dr, phantom, "\#?"] & \bar{\Phi}'\varphi'(t') \arrow[d, "\mu'_{t'}"] \\
\psi'(s') \arrow[r, "\Theta\psi(g)"'] & \psi'(t')
\end{tikzcd} \quad ;$$

here we've put $s' = \sigma(s)$, $t' = \sigma(t)$, $g' = \sigma(g)$.
We bring in the defining equations for s' and t': see (*), p 15. Our diagram is the pasting of the following:

$$\begin{tikzcd}[column sep=large]
\bar{\Phi}'\varphi'(s') = \bar{\Phi}'\tau\varphi(s) \arrow[rr, "\bar{\Phi}'\tau\varphi(g) \,=\, \bar{\Phi}'\varphi'(g')"] \arrow[d, "\Sigma^{-1}_{\varphi(s)}"] \arrow[dd, bend right=60, "\mu'_{s'}"'] \arrow[drr, phantom, "\#1"] & & \bar{\Phi}'\tau\varphi(t) \arrow[d, "\Sigma^{-1}_{\varphi(t)}"'] \arrow[dd, bend left=60, "\mu'_{t'}"] \\
\Theta\bar{\Phi}\varphi(s) \arrow[rr, "\Theta\bar{\Phi}\varphi(g)"] \arrow[d, "\theta(\mu_s)"] \arrow[l, phantom, "\#3"] \arrow[drr, phantom, "\#2"] & & \Theta\bar{\Phi}\varphi(t) \arrow[d, "\theta(\mu_t)"'] \arrow[r, phantom, "\#4"] & {} \\
\Theta\psi(s) \arrow[rr, "\Theta\psi(g)"'] & & \Theta\psi(t)
\end{tikzcd}$$

whose cells commute, by the naturality of Σ (for #1), the naturality of μ (#2), and the third of the equations (*), p 15. This proves (*), p 17.

For σ being a functor: immediate from the fact that, for $s \xrightarrow{g} t$, $\bar\sigma(g)h : |F|\sigma(s) \longrightarrow \sigma(t)$ is the <u>unique</u> arrow $g' : \sigma(s) \longrightarrow \sigma(t)$ for which

$$\varphi'(g') = \tau\varphi(g)$$

(namely, if $s \xrightarrow{g} t \xrightarrow{h} u$ in $|F|$, we can show that the composite $\sigma(h)\circ\sigma(g) : \sigma(s) \longrightarrow \sigma(u)$, abbreviated $k' : \sigma(s) \to \sigma(u)$, satisfies

$$\varphi'(k') \overset{?}{=} \tau\varphi(h\circ g):$$

$$\begin{aligned}
\varphi'(k') &= \varphi'(\sigma(h)\circ\sigma(g)) \\
&= \varphi'\sigma(h)\circ\varphi'\sigma(g) \\
&= \tau\varphi(h)\circ\tau\varphi(g) \\
&= \tau\varphi(h\circ g)
\end{aligned}$$

therefore, $k' = \sigma(h\circ g)$. A similar, but simpler argument shows that $\sigma(1_s) = 1_{\sigma(s)}$.)

This completes the proof of Lemma 2.

$\square$ Lemma 2

I am returning to page 6, to defining the functor $\hat{\Gamma}$, and proving statements 1) and 2) on that page.

3. $\boxed{\text{\underline{The} \underline{definition} of } \hat{\Gamma} : \text{pFun} \longrightarrow \text{SanapFun}.}$

Given any functor $\Phi : X \longrightarrow A$, we define

$\hat{\Gamma}(\Phi) = (\Gamma\Phi, \Phi, \mu[\Phi])$; where $\Gamma\Phi = (|\Gamma\Phi|, \varphi[\Phi], \psi[\Phi])$,

$$
\begin{tikzcd}
 & {|\Gamma\Phi|} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\
X \arrow[rr, "\Phi"'] & \arrow[u, phantom, "\overset{\Rrightarrow}{\mu}"] & A
\end{tikzcd}
\qquad
\begin{array}{l}
\varphi = \varphi[\Phi] \\
\psi = \psi[\Phi] \\
\mu = \mu[\Phi]
\end{array}
$$

as follows:

<u>object</u> of $|\Gamma\Phi|$: triple (x, a, ν),

where $x \in X$, $a \in A$, $\nu : \Phi x \xrightarrow{\cong} a$;

<u>arrow</u> of $|\Gamma\Phi|$:

$$(x, a, \nu) \xrightarrow[(f,g)]{} (y, b, \rho)$$

where $f : x \to y$, $g : a \to b$ such that

$$\begin{array}{ccc} \Phi(x) & \xrightarrow{\nu} & a \\ {\scriptstyle \Phi(f)}\downarrow & \# & \downarrow{\scriptstyle g} \\ \Phi(y) & \xrightarrow[\rho]{} & b \end{array} \qquad ; \qquad (*)$$

composition: component-wise.

φ and ψ: obvious forgetful functors

$\mu : \Phi\varphi \xrightarrow{\cong} \psi$ has components

$$\mu_{(x,a,\nu)} = \nu : \Phi x \xrightarrow{\cong} a .$$

This is a very general construction, defining the so-called (here) pseudo-graph of any functor.

The main point is that

$$\Gamma\Phi = (|\Gamma\Phi|, \varphi, \psi) \in \mathrm{Sana},$$

i.e., $\Gamma\Phi$ satisfies conditions 1), 2) and 3) on page 2. Condition 1) holds, since in the diagram $(*)$, there always is a unique $g : a \to b$

once the other items are given.

Concerning condition 2): for $x \in X$, $b \in A$, we now have $|F|(x,b) = \{(x,b,\nu) : \nu \in \mathrm{Iso}(\Phi x, b)\}$.

For $s = (x, a, \mu)$, $\mathrm{Iso}(\varphi(s), b) = \mathrm{Iso}(a, b)$.

For $s = (x, a, \mu)$ and $t \in |F|(x,b)$, $t = (x, b, \nu)$, the arrow $g_{s,t,1_{\varphi(s)}}$ is the arrow $(1_x, g) : s \longrightarrow t$ where g is defined by the commutativity

$$\begin{tikzcd} & a \arrow[d, "g"] \arrow[d, phantom, shift right=3ex, "\#"] \\ \Phi x \arrow[ur, "\mu", "\cong"'] \arrow[r, "\nu"', "\cong"] & b \end{tikzcd} \qquad (*)$$

The mapping $F_{s,b} : F(\varphi(s), b) \longrightarrow \mathrm{Iso}(\varphi(s), b)$ maps (x, b, ν) to the arrow g determined as in $(*)$ preceding. $F_{s,b}$ is a bijection, since if $s = (x, a, \mu)$ is given and if g is any isomorphism $g : a \xrightarrow{\cong} b$, then there is a unique $\nu : \Phi x \xrightarrow{\cong} b$, an isomorphism, and hence a unique $t = (x, b, \nu) \in |F|(x, b)$ such that $F_{s,b}(t) = g$.

Condition 3): given $x \in X$, we have $s = (x, \Phi x, id_{\Phi x}) \in |\Gamma \Phi|$, such that $\varphi(s) = x$.

<u>Remark</u>: $\Gamma\Phi$ is the <u>saturated</u> <u>ana</u><u>functor</u> associated with the functor Φ, according to the paper cited on p. (16).

We have defined the <u>object function</u> $\vec{\hat{\Gamma}} : \mathrm{Ob}(p\,\mathrm{Fun}) \to$ $\to \mathrm{Ob}(\mathrm{Sana}\,p\mathrm{Fun})$ such that, with functor $R : \mathrm{Sanap\,Fun} \to p\,\mathrm{Fun}$, we have that the composite

$$R \circ \hat{\Gamma} : \mathrm{Ob}(p\,\mathrm{Fun}) \longrightarrow \mathrm{Ob}(p\,\mathrm{Fun})$$

is the identity. Given that R is full and faithful, it immediately follows that there is a unique functor $\hat{\Gamma} : p\,\mathrm{Fun} \longrightarrow \mathrm{Sana}\,p\,\mathrm{Fun}$ with the given object function such that $R \circ \hat{\Gamma} : p\,\mathrm{Fun} \to p\mathrm{Fun}$ is the identity functor. Moreover, $\hat{\Gamma}$ is the quasi-inverse of R, i.e. $\hat{\Gamma} \circ R \cong \mathrm{Id}_{\mathrm{Sanap\,Fun}}$, as is immediately seen by using again the fact that R is full and faithful.

<u>Conclusion</u>: $R : \mathrm{SanapFun} \longrightarrow \mathrm{pFun}$

is an equivalence of categories, with quasi-inverse the canonical (choice-free) functor $\hat{\Gamma} : \mathrm{pFun} \longrightarrow \mathrm{SanapFun}$

Lemma 3 $\boxed{S : \text{SanapFun} \longrightarrow \text{Sana} \text{ is surjective on objects}}$

Proof. Let $F \in$ Sana; we use the notation started on p. 1. We are to define $\Phi : X \to A$ and μ as in $(*)$, p. 4.

Let, for any $x \in X$, $s_x \in |F|$ be chosen so that $\varphi(s_x) = x$ (see condition 3), p. 2).

Define $\Phi(x) \underset{\text{def}}{=} \psi(s_x)$.

For $f : x \to y$ in X, we have

$g_f \underset{\text{def}}{=} g_{s_x, s_y, f}$ (see p. 2, condition 1)); we let $\Phi(f) \underset{\text{def}}{=} \psi(g_f) : \Phi(x) \longrightarrow \Phi(y)$. The uniqueness part in condition 1), p. 2, ensures that if $x \xrightarrow{f} y \xrightarrow{e} z$, then $g_{ef} = g_e g_f$, and $g_{1_x} = 1_{s_x}$, which ensures that $\Phi : X \to A$ is a functor.

For $s \in |F|$, we have:

$$\begin{tikzcd}[column sep=large, row sep=large]
S_{\varphi(s)} \arrow[r, dashed, "g_s = g_{\varphi(s),s,1_{\varphi(s)}}"] \arrow[d, mapsto, "\varphi"'] \arrow[dr, phantom, "\overset{\varphi}{\downarrow}"] & s \arrow[d, mapsto, "\varphi"] \\
\varphi(s) \arrow[r, "1_{\varphi(s)}"'] & \varphi(s)
\end{tikzcd}$$

Thus, $\psi(g_s) : \psi(S_{\varphi(s)}) \longrightarrow \psi(s)$

is an arrow, an isomorphism,

$\psi(g_s) = \mu_s \quad : \quad \Phi\varphi(s) \xrightarrow{\cong} \psi(s)$

Moreover, μ_s is natural in s: let $h: s \to t$ in $|F|$, and consider the diagram

$$\begin{tikzcd}[column sep=large, row sep=large]
S_{\varphi(s)} \arrow[r, "g_s"] \arrow[d, "g_{\varphi(h)}"'] & s \arrow[d, "h"] \\
S_{\varphi(t)} \arrow[r, "g_t"'] & t
\end{tikzcd} \qquad (*)$$

The functor $\varphi : |F| \to X$ maps this to the commutative diagram

$$\begin{tikzcd} \varphi(s) \arrow[r, "1_{\varphi(s)}"] \arrow[d, "\varphi(h)"'] & \varphi(s) \arrow[d, "\varphi(h)"] \\ \varphi(t) \arrow[r, "1_{\varphi(t)}"'] & \varphi(t) \end{tikzcd}$$

therefore $(*)$, previous page, is commutative (by 1), p ②).

It's image under $\psi : |F| \to A$,

$$\begin{tikzcd} \psi(s_{\varphi(s)}) \arrow[r, "\psi(g_s)"] \arrow[d, "\psi(g_{\varphi(h)})"'] \arrow[dr, phantom, "\#"] & \psi(s) \arrow[d, "\psi(h)"] \\ \psi(s_{\varphi(t)}) \arrow[r, "\psi(g_t)"'] & \psi(t) \end{tikzcd}$$

is the same as

$$\begin{tikzcd} \Phi\varphi(s) \arrow[r, "\mu_s"] \arrow[d, "\Phi\varphi(h)"'] \arrow[dr, phantom, "\#"] & \psi(s) \arrow[d, "\psi(h)"] \\ \Phi\varphi(t) \arrow[r, "\mu_t"'] & \psi(t) \end{tikzcd}$$

We have shown that $\mu = (\mu_s)_{s \in |F|} : \Phi_\varphi \xrightarrow{\cong} \varphi$ is a natural isomorphism, and this completes the proof that the functor S is surjective on objects.

□ Lemma 3

Lemma 2 (p. 9), the Conclusion on p 23.1, and Lemma 3, p 24 establish the Theorem, and the Corollary, p 3.