

July 7

An application of anafunctors

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Abstract. The category $p\text{Fun}$ is defined to have objects (X, A, Φ) where X, A are small categories, $\Phi: X \rightarrow A$ is a functor, and arrows $(X, A, \Phi) \xrightarrow{(\alpha, \Sigma, \theta)} (X', A', \Phi')$ where

$$\begin{array}{ccc}
 X & \xrightarrow{\Phi} & A \\
 \alpha \downarrow & \textcircled{\Sigma} & \downarrow \theta \\
 X' & \xrightarrow{\Phi'} & A'
 \end{array}$$

choice of orientation:

$$\Sigma: \theta \Phi \xrightarrow{\cong} \Phi' \alpha$$

($p\text{Fun}$ is a 'pseudo-version' of the arrow-category $\text{Cat}^{\mathcal{Q}}$) with composition the obvious pasting operation.

We prove that $p\text{Fun}$ is an accessible category, in fact $p\text{Fun} \simeq \text{Reg}(\mathcal{C}, \text{Set})$, for a small (countable) regular category \mathcal{C} .

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①

I define two categories, $pFun$ and $Sana$.

$pFun$: object: $(X, A, \bar{\Phi})$ where

X, A are small categories,

$\bar{\Phi}: X \rightarrow A$ is a functor;

arrow: $(X, A, \bar{\Phi}) \rightarrow (X', A', \bar{\Phi}')$

(τ, Σ, θ)

where

$$\begin{array}{ccc} X & \xrightarrow{\bar{\Phi}} & A \\ \tau \downarrow & \cong \Downarrow \Sigma & \downarrow \theta \\ X' & \xrightarrow{\bar{\Phi}'} & A' \end{array}$$

$$\Sigma: \theta \bar{\Phi} \xrightarrow{\cong} \bar{\Phi}' \tau \quad (*)$$

(natural isomorphism);

composition: usual pasting.

$Sana$: object: $F = (|F|, X, A, \varphi, \psi)$, a span:

$$\begin{array}{ccc} & |F| & \\ \varphi \swarrow & & \searrow \psi \\ X & & A \end{array}$$

of small categories and functors

such that: 1), 2) & 3) next page hold;

②

1) Whenever $s, t \in |F|$ (meaning: s, t objects of the category $|F|$) and $f: \varphi(s) \rightarrow \varphi(t)$ (in X),

there is a unique $g: s \rightarrow t$ (in $|F|$),

such that $\varphi(g) = f$; notation: $g = g_{s,t,f}$.

2) For... $x \in X$ and $b \in A$, define:

$$|F|(x, b) \stackrel{\text{def}}{=} \varphi^{-1}(x) \cap \varphi^{-1}(b)$$

$$\stackrel{\text{def}}{=} \{t \in |F| : \varphi(t) = x \ \& \ \varphi(t) = b\}$$

and define the map, for $s \in |F|, b \in A$:

$$F_{s,b} : |F|(\varphi(s), b) \rightarrow \text{Iso}(\varphi(s), b)$$

$$t \longmapsto \varphi(g_{s,t, \perp_{\varphi(s)}})$$

(well-defined by 1)).

Require: $F_{s,b}$ is a bijection. [$\text{Iso}(a, b)$ is

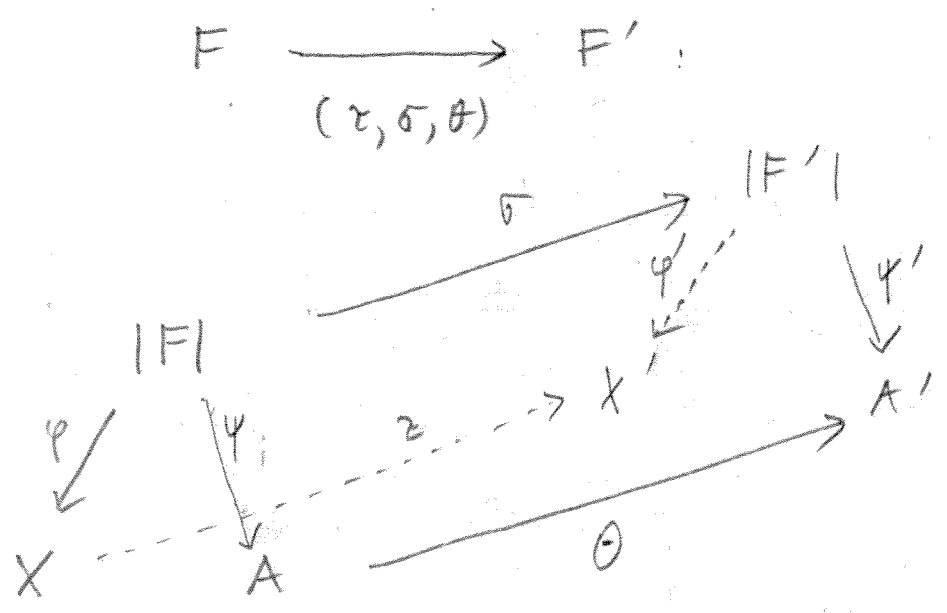
the set of all isomorphism $a \xrightarrow{\cong} b$.]

3) φ is surjective on objects:

for all $x \in X$, there is $s \in |F|$ s.t. $\varphi(s) = x$.

(end of definition of 'object' of Sara).

arrow of Sana: arrow of spans:



satisfying
$$\left. \begin{aligned}
 \tau\varphi &= \psi'\sigma, \\
 \theta\psi &= \psi'\sigma.
 \end{aligned} \right\} (*)$$

composition: usual. (componentwise).

Theorem pFun and Sana are equivalent categories.

Corollary pFun is a dual-regular category: for

some (countable) regular category \mathbb{C} ,

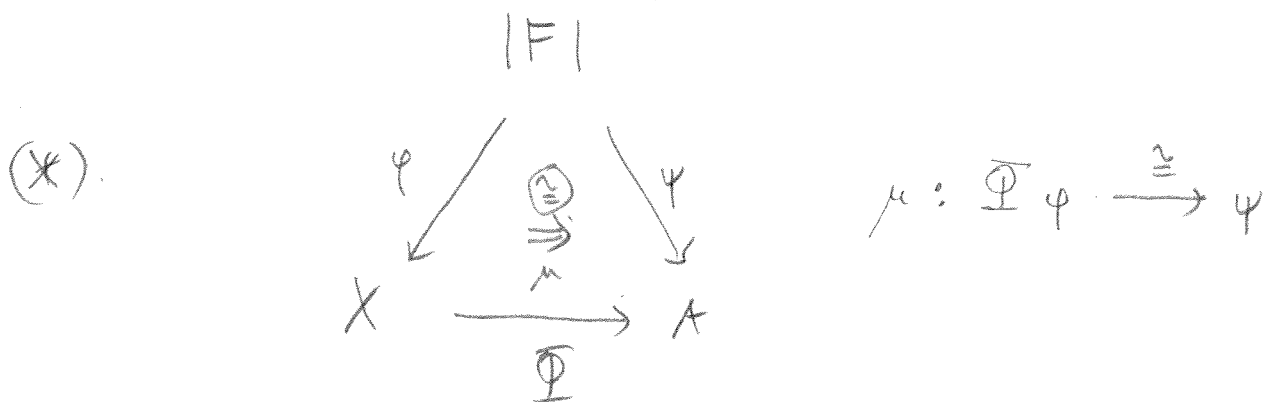
$\text{pFun} \cong \text{Reg}(\mathbb{C}, \text{Set}) =$ the category of all regular functors $\mathbb{C} \rightarrow \text{Set}$, a full subcategory of $[\mathbb{C}, \text{Set}]$. In particular, pFun is an accessible category.

Proof: Sana is easily seen to be dual-regular.



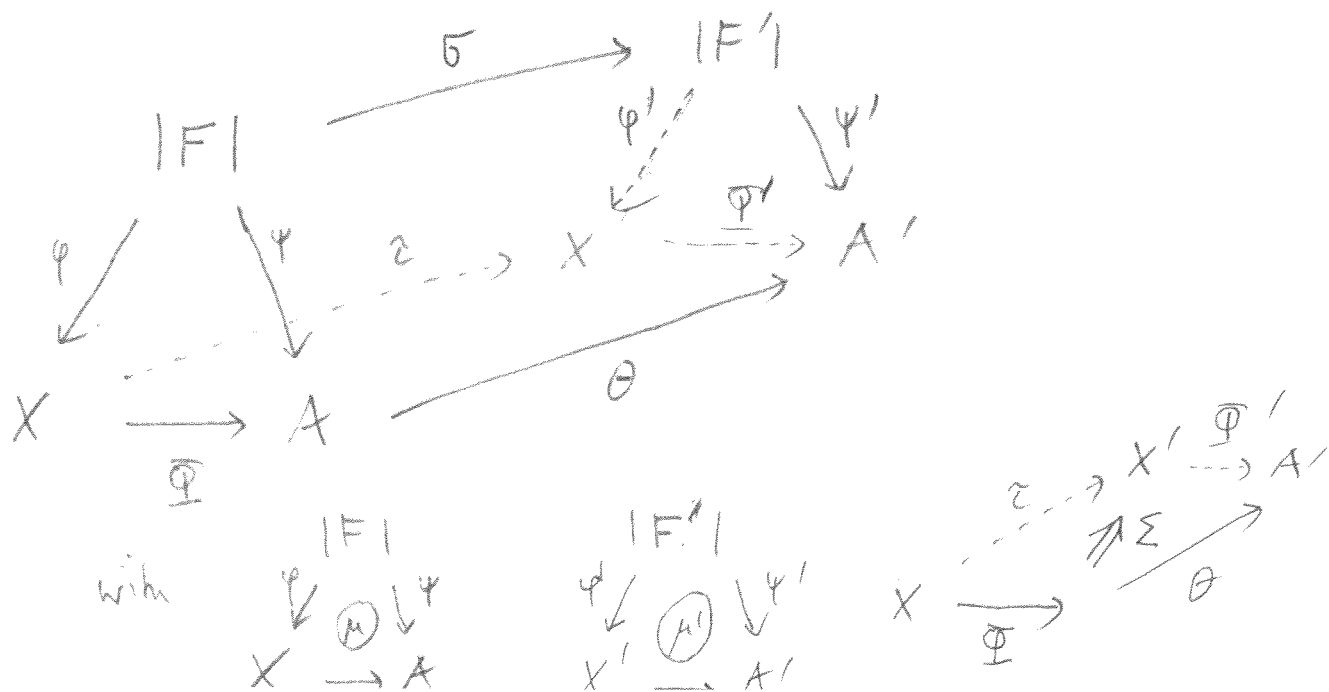
For the proof of the theorem, I introduce a third category, SanapFun, a 'combination' of the previous two:

object: $(F, \Phi, \mu) = (|F|, X, A, \varphi, \psi, \Phi, \mu)$:



such that: $F = (|F|, X, A, \varphi, \psi) \in \text{Sanap}$

arrow: $(F, \Phi, \mu) \longrightarrow (F', \Phi', \mu')$
 $(\tau, \sigma, \Sigma, \Theta)$



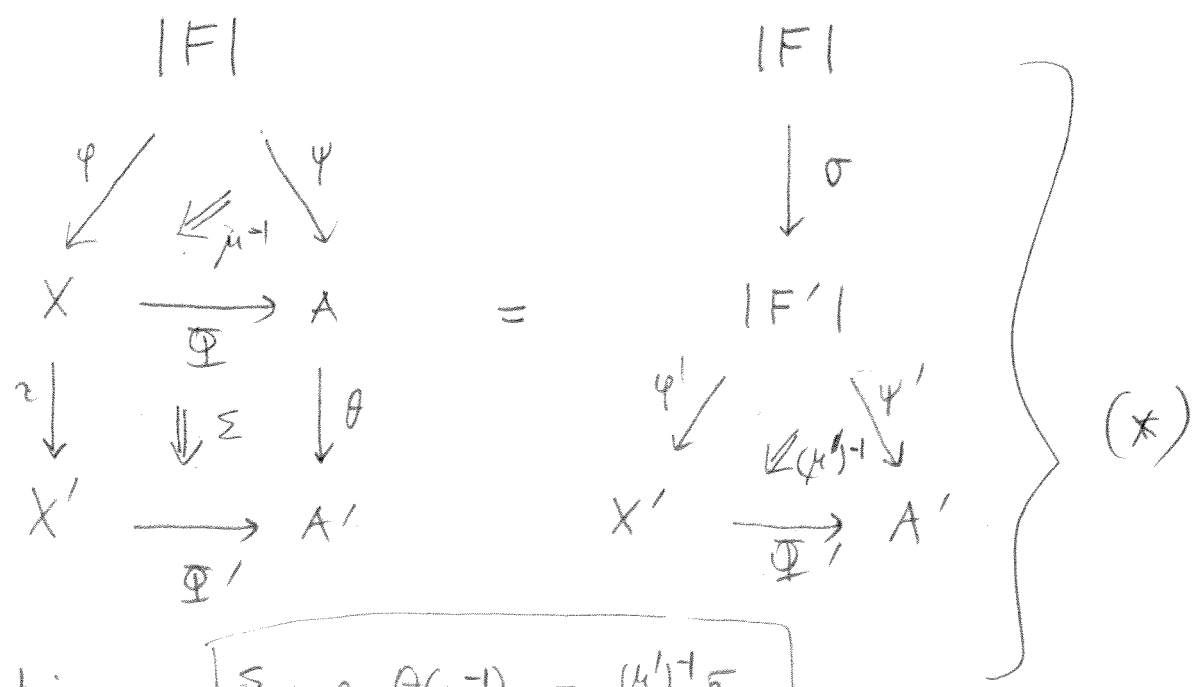
such that: $F \longrightarrow F'$ is an arrow
 (τ, σ, θ)

in Sana , and

$\Phi \longrightarrow \Phi'$ is an arrow
 (τ, Σ, θ)

in $p\text{Fun}$,

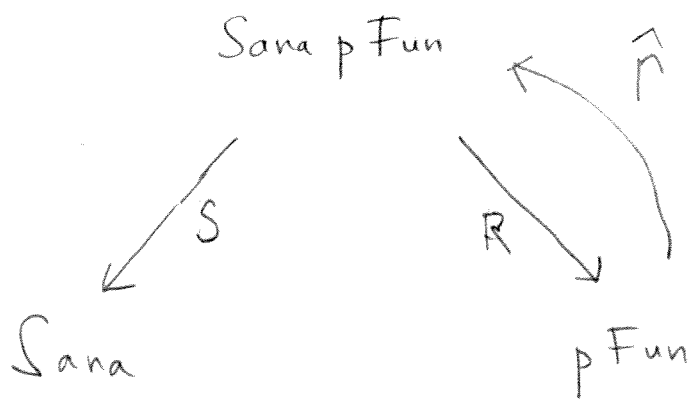
and μ, Σ, θ and μ' are compatible:



that is, $\Sigma \circ \theta \circ \mu^{-1} = (\mu')^{-1} \circ \sigma$.

Composition is defined so that we have

the forgetful functors (see next page) R & S :



I will also define, canonically, a functor $\hat{\Gamma}$ as shown, and show that: 1) $(R, \hat{\Gamma})$ form an equivalence of categories; and

2) S is a surjective equivalence.

Of course, the theorem will follow.

Lemma 1. Let $(F, \Phi, \lambda) \in \text{Sana p Fun}$
 (for notation, see (*), p 4).

Let $\Phi \parallel A$ denote the set all triples (x, a, ν) such that $x \in X$, $a \in A$ and $\nu: \Phi x \xrightarrow{\cong} a$.

We have the mapping $sp: \text{obj } F \longrightarrow \Phi \parallel A$ defined

by $sp(s) = (\varphi(s), \psi(s), \lambda_s: \Phi \varphi(s) \xrightarrow{\cong} \psi(s))$. ($s \in |F|$).

Assertion: sp is a bijection.

Proof. 1) φ is injective: suppose $s, t \in |F|$

and $\varphi(s) = \varphi(t) = x$, $\psi(s) = \psi(t) = a$, and

$\mu_s = \mu_t : \Phi x \xrightarrow{\cong} a$, to show $s \stackrel{?}{=} t$.

By condition 1), p. ②, there is $g : s \rightarrow t$ in $|F|$

such that $\varphi(g) = 1_x$. Apply naturality of μ against

the arrow g :

$$\begin{array}{ccc}
 \Phi_{\varphi(s)} & \xrightarrow{\Phi_{\varphi(g)}} & \Phi_{\varphi(t)} \\
 \mu_s \downarrow & \# & \downarrow \mu_t \\
 \psi(s) & \xrightarrow[\psi(g)]{} & \psi(t)
 \end{array}$$

This reduces to

$$\begin{array}{ccc}
 & \Phi_x & \\
 \mu_t = \mu_s & \xrightarrow[\cong]{} & \mu_t = \mu_s \\
 & \xrightarrow[\psi(g)]{} & a
 \end{array}$$

Therefore $\psi(g) = 1_a$. Now, apply the uniqueness part of condition 2) on p. ②:

$$\begin{array}{ccc}
 F_{s,a} : |F|(x, a) & \longrightarrow & \text{Iso}(a, a) \\
 u \longmapsto & \psi(g_s, u, 1_{\varphi(s)}) & = \dots
 \end{array}$$

is injective. $F_{s,a}(s) = \psi(g_{s,s,1_{\psi(s)}}) = \psi(1_s) = \nu^{-1}_{\psi(s)} = 1_a$

and $F_{s,a}(t) = \psi(g_{s,t,1_{\psi(s)}}) = \psi(g) = 1_a$

↑
our g above as shown on the prev. page

It follows that $s = t$.

2) ψ is surjective: suppose $(x, a, \nu: \mathbb{P}_x \xrightarrow{\cong} \mathbb{P}_a) \in \mathbb{P}/A$

Choose $s \in |F|$ such that $\psi(s) = x$ (condition 3), p. 2).

Use condition 2), existence, to find $t \in |F|$ such that

for $g = g_{s,t,1_x} : s \rightarrow t$, we have $\psi(t) = a$, and

$$F_{s,a}(t) = \psi(g) = \nu \circ \mu_s^{-1} : \psi(s) \xrightarrow{\cong} a \quad (1).$$

Apply naturality of μ to $g : s \rightarrow t$:

$$\begin{array}{ccc}
 x = \mathbb{P}_{\psi(s)} & \xrightarrow{\mathbb{P}g = 1_x} & \mathbb{P}_{\psi(t) = a} \\
 \mu_s \downarrow \cong & & \downarrow \mu_t \\
 \psi(s) & \xrightarrow[\psi(g)]{\cong} & \psi(t) = a
 \end{array}$$

Given what $\psi(g)$ is, it follows that $\mu_t = \nu$.

We have found $t \in |F|$ such that

$$sp(t) = (x, a, v).$$

□ Lemma 1

Lemma 2.

The forgetful functors

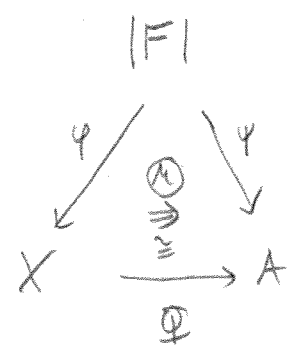
$$R : \text{SanaFun} \longrightarrow \text{pFun}$$

$$S : \text{SanaFun} \longrightarrow \text{Sana}$$

are full and faithful.

Remark. The Lemma is ^(essentially) a generalization of statement 9' on p. 123 of "Avoiding the axiom of choice in general category theory", JPA 108 (1996), 109-173. Said statement is the special case of the Lemma 2 when Σ and Θ are identity functors.

Proof. For $\underline{F} = (F, \Phi, \mu)$, an object of SanaFun ,



$R(\underline{F})$ is $R(\underline{F}) = \Phi = (X, A, \Phi)$, and
 $S(\underline{F})$ is $S(\underline{F}) = F = (|F|, X, A, \Phi, \psi)$.

Thus, S being full and faithful means that

if $\underline{F} = (F, \Phi, \mu)$, $\underline{F}' = (F', \Phi', \mu')$ are objects of SanaFun ,

and $(\tau, \sigma, \theta): F \rightarrow F'$ is an arrow in Sana ,

then there is a unique Σ , denoted for short by $\Sigma = \hat{\sigma}$,
for which $(\tau, \sigma, \Sigma, \theta): \underline{F} \rightarrow \underline{F}'$.

R being full and faithful is a 'converse': for \underline{F}

and \underline{F}' as before, if $(\tau, \Sigma, \theta): \Phi \rightarrow \Phi'$,

there is a unique $\hat{\sigma}$, denoted $\hat{\sigma} = \check{\Sigma}$, such that

$(\tau, \check{\Sigma}, \Sigma, \theta): \underline{F} \rightarrow \underline{F}'$

1) Proof that S is full and faithful: "given σ , define Σ ", in the notation of the previous page.

For $x \in X$, we need to define the isomorphism
 arrow

$$\Sigma_x : \Theta \Phi(x) \xrightarrow{\cong} \Phi'_\sigma(x) \tag{1}$$

such that: for any $s \in |F|$ with $\varphi(s) = x$, we have the commutativity

$$\begin{array}{ccc} \Phi'_\sigma(x) & \xleftarrow{\Sigma_x^{-1}} & \Theta \Phi(x) \\ \mu'_\sigma(s) \searrow \cong & \# & \cong \swarrow \Theta(\mu_s) \\ & \varphi'_\sigma(s) = \Theta \varphi(s) & \end{array} \tag{2}$$

It is immediate, therefore, that, if the natural transformation $\Sigma : \Theta \Phi \xrightarrow{\cong} \Phi'_\sigma$ exists, it is unique, since by condition 3), there exist at least one $s \in |F|$ with $\varphi(s) = x$.

For any given $s \in |F|$, let $\hat{s} : \mathbb{F}'_s$

(12)

$$\hat{s} : \Theta \mathbb{F}'_s \xrightarrow{\cong} \mathbb{F}'_{\tau(\varphi(s))}$$

be the arrow for which

$$\mathbb{F}'_{\tau(\varphi(s))} \xleftarrow{\hat{s}^{-1}} \Theta \mathbb{F}'_s$$

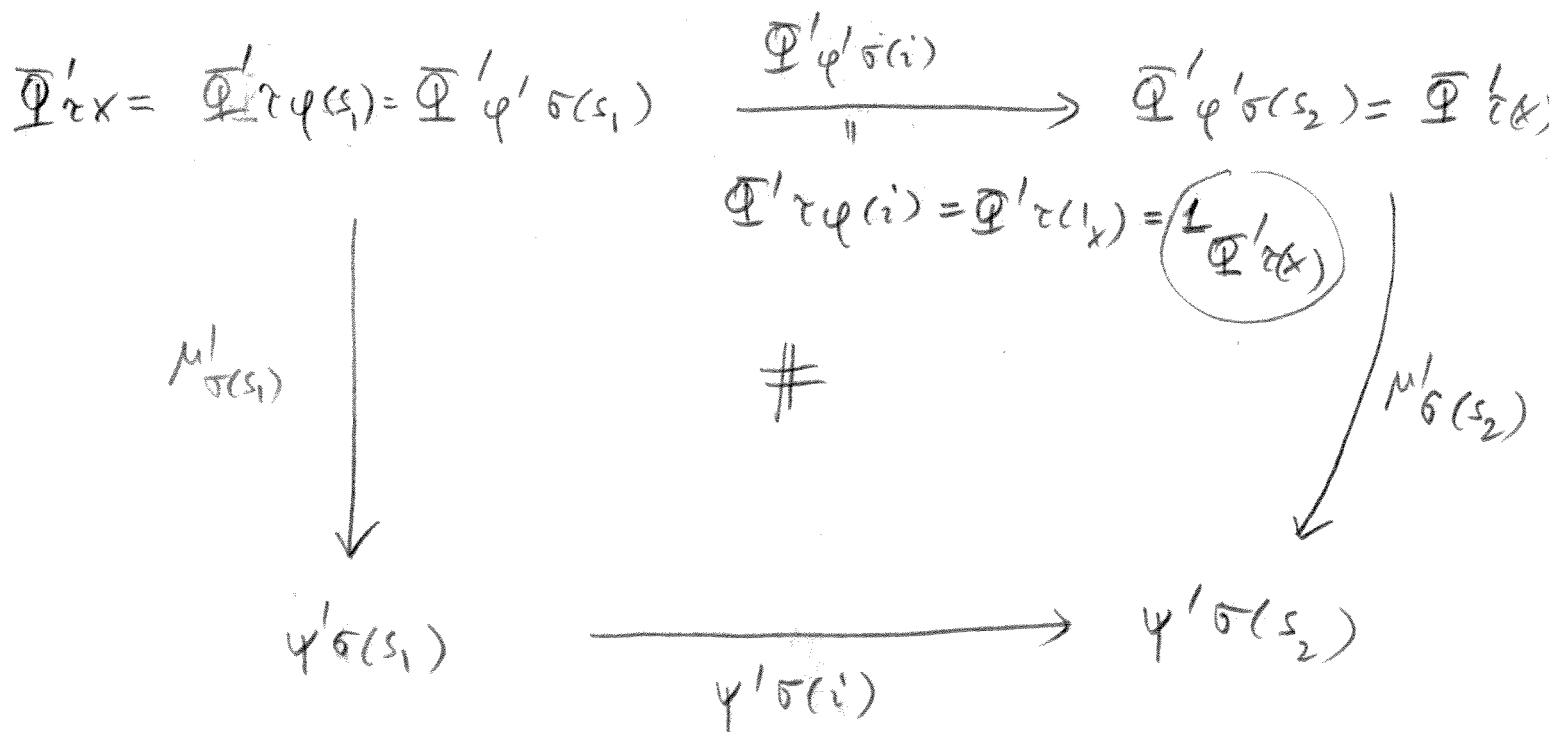
$$\begin{array}{ccc} & \# & \\ \mu'_s & \searrow & \swarrow \Theta(\mu_s) \\ & \psi'_s = \Theta \psi & \end{array}$$

I claim that $\varphi(s_1) = \varphi(s_2)$ implies $\hat{s}_1 = \hat{s}_2$.

The claim is equivalent to the commutativity of the outside square of the following diagram of isomorphisms:

$$\begin{array}{ccccc} & & \psi'_s(s_1) = \Theta \psi(s_1) & & \\ & \nearrow \mu'_s & & \nwarrow \Theta(\mu_{s_1}) & \\ \mathbb{F}'_x & \text{(#1)} & \psi'_s(x) = \Theta \psi(x) & \text{(#2)} & \Theta \mathbb{F}'_x \\ & \searrow \mu'_s & \downarrow & \swarrow \Theta(\mu_{s_2}) & \\ & & \psi'_s(s_2) = \Theta \psi(s_2) & & \end{array}$$

(Here we've put $x = \varphi(s_1) = \varphi(s_2)$, and defined $i: s_1 \rightarrow s_2$ in $[F]$, an isomorphism, such that $\varphi(i) = 1_x$ (condition 1)). The left triangle #1 commutes by the naturality of μ' tested by the arrow $\sigma(i): \sigma(s_1) \rightarrow \sigma(s_2)$ in $[F']$:



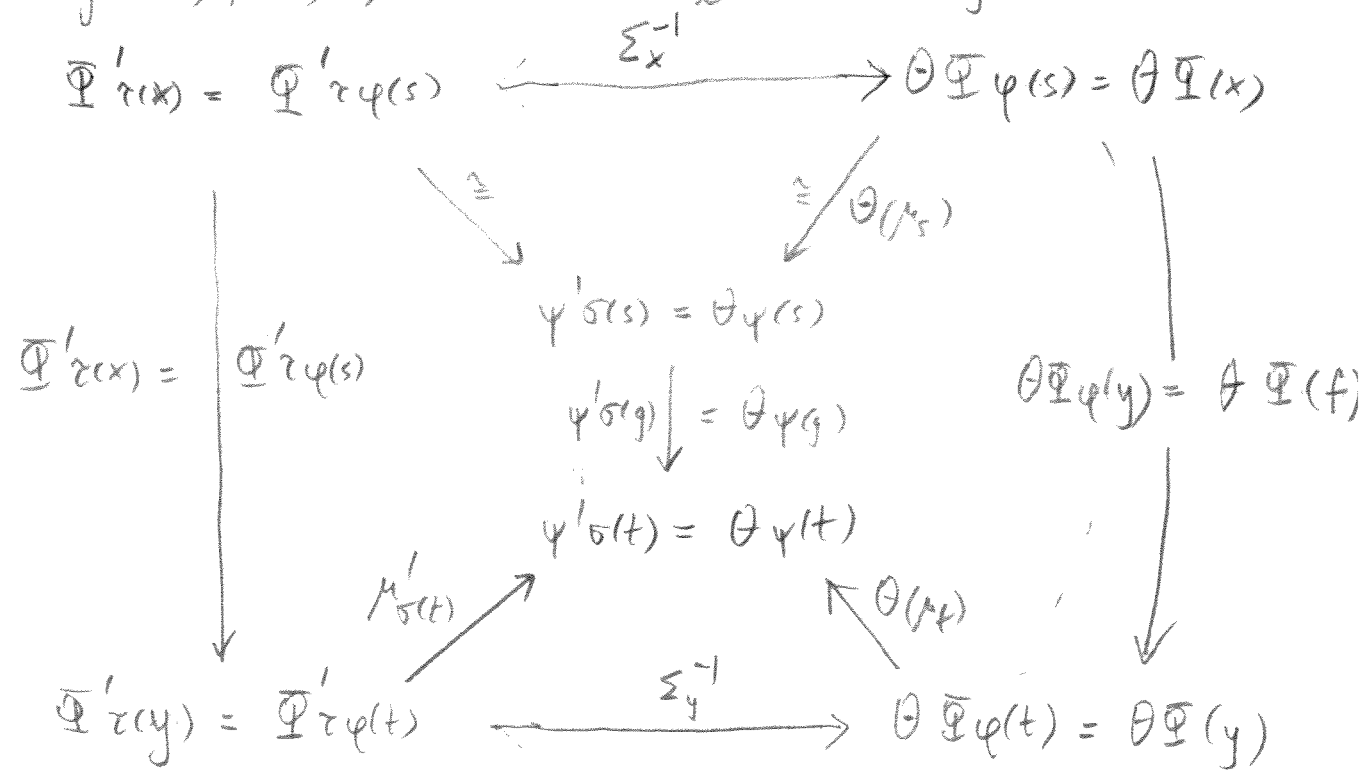
The right triangle #2 commutes by the naturality of μ tested by $i: s_1 \rightarrow s_2$, with an arrow application of θ . This proves the claim.

The claim allows us to define, for any given $x \in X$, the arrow Σ_x (see (1), p. 11) satisfying (2) ^{p. 11} for all $s \in |F|$ with $\varphi(s) = x$.

It remains to show that $\Sigma = (\Sigma_x)_{x \in X}$ so defined is a natural transformation $\Sigma: \Theta \Phi \rightarrow \Phi'$.

The naturality of Σ is shown by the commutativity of the outer square in the following diagram, where

$f: x \rightarrow y$ is an arbitrary arrow in X , $s, t \in |F|$ and $g: s \rightarrow t$ in $|F|$ are such that $\varphi(s) = x$, $\varphi(t) = y$ (by using 1), p. 2), and $\varphi(g) = f$ (by using 2), p. 2), and thus $\Sigma_x = s^{-1}$, $\Sigma_y = t^{-1}$:



where, on the two sides, the commutativities are ensured by the naturality of μ' , and μ , respectively.

2) "Given Σ , define σ : proof of R being full and faithful:

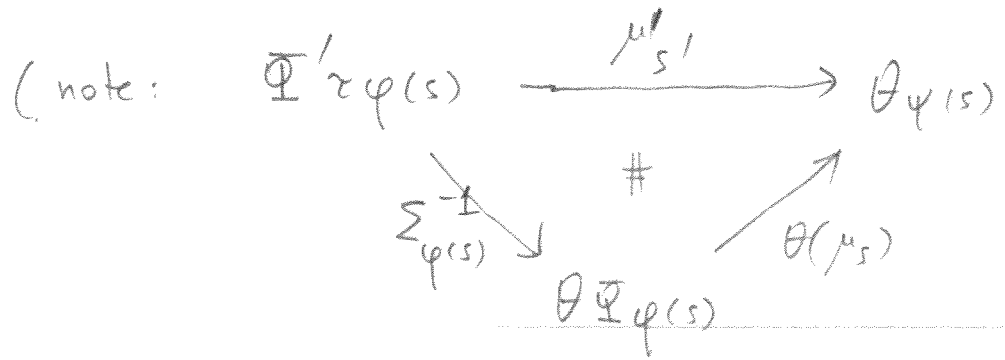
To define the functor $\sigma : |F| \rightarrow |F'|$,

first we define the object-function of σ . Let $s \in |F|$.

The object $s' \stackrel{\text{def}}{=} \sigma(s)$ of $|F'|$ has to satisfy the

following three equalities:

$$\left. \begin{aligned}
 \varphi'(s') &= \tau\varphi(s) \\
 \varphi'(s') &= \theta\varphi(s) \\
 \mu'_{s'} &= \theta(\mu_s) \circ \Sigma_{\varphi(s)}^{-1}
 \end{aligned} \right\} \begin{array}{l} \text{(see} \\ \text{see } (*), \text{ p } \textcircled{1}. \end{array} \quad \left. \vphantom{\begin{aligned} \varphi'(s') \\ \varphi'(s') \\ \mu'_{s'} \end{aligned}} \right\} (*)$$



is required by $(*)$, p $\textcircled{5}$)

We apply Lemma 1 (p 6) to $(F', \Phi', \mu') \in$

Sanap Fun. Where is a unique $s' \in |F'|$ satisfying the three equalities (*), previous page. The uniqueness of σ ^{the effect of σ on objects} immediately follows. What

remains is to define the effect of σ on arrows in $|F|$, and prove that σ is indeed a

functor, and that the equalities (*), p 3 hold

true for arrows in $|F|$ ^{as arguments for} (we have made sure that those equalities hold for objects in $|F|$; also that (*), p 5 holds; this latter equality involves only objects as arguments of σ .)

$\sigma(|F|)$

Let $g: s \rightarrow t$ be an arrow in $|F|$. Look at the

first equality in (*), p 3; this requires $\sigma(g) \stackrel{\text{def}}{=} g'$

that it satisfies

$$\begin{array}{ccc} \varphi' \sigma(s) & & \varphi' \sigma(t) \\ // & & // \\ \varphi'(g') & = & \tau \varphi(g) : \tau \varphi(s) \rightarrow \tau \varphi(t). \end{array}$$

According to condition 1), p 2, on F' , there is a unique sub arrow $g'/\sigma: \sigma(s) \rightarrow \sigma(t)$; this we take to be $\sigma(g)$. Thus, we have

for $s \xrightarrow{g} t$ in $|F|$, $\sigma(s) \xrightarrow{\sigma(g)} \sigma(t)$ in $|F'|$
such that

$$\varphi'(\sigma(g)) = \tau\varphi(g);$$

in particular, the first of the two equations (*)
on p. ③ is taken care of.

We turn to verifying the second equation in (*), p ③, for
an arrow-argument in $|F|$: for $g: s \rightarrow t$ in $|F|$, and $g' = \sigma(g)$,

$$(*) \quad \theta\varphi(g) \stackrel{?}{=} \varphi'(\sigma g') : \begin{array}{ccc} \theta\varphi(s) & \xrightarrow{\varphi(g)} & \theta\varphi(t) \\ \parallel & & \parallel \\ \varphi'\sigma(s) & \xrightarrow{\varphi'(g')} & \varphi'\sigma(t) \end{array}$$

For any $g': s' \rightarrow t'$ in $|F'|$, by the naturality of μ'
we have

$$\begin{array}{ccc} \Phi'\varphi'(s') & \xrightarrow{\Phi'\varphi'(g')} & \Phi'\varphi'(t') \\ \mu'_{s'} \downarrow \cong & \# & \cong \downarrow \mu'_{t'} \\ \varphi'(s') & \xrightarrow{\varphi'(g')} & \varphi'(t') \end{array}$$

moreover, $\varphi'(g')$ is the unique arrow $\varphi'(s') \rightarrow \varphi'(t')$
that makes the last diagram commute, since $\mu'_{s'}$ is an isomorphism.

Therefore, the questioned equality (*), previous page is equivalent to saying that the following commutes:

$$\begin{array}{ccc}
 \Phi' \varphi'(s') & \xrightarrow{\Phi' \varphi'(g')} & \Phi' \varphi'(t') \\
 \mu'_{s'} \downarrow & \#? & \downarrow \mu'_{t'} \\
 \psi(s') & \xrightarrow{\Theta \psi(g)} & \psi(t') ;
 \end{array}$$

here we've put $s' = \sigma(s)$, $t' = \sigma(t)$, $g' = \sigma(g)$.

We bring in the defining equations for s' and t' : see (*), p (15). Our diagram is the pasting of the following!

$$\begin{array}{ccc}
 \Phi' \varphi'(s') = \Phi' \tau \varphi(s) & \xrightarrow{\Phi' \tau \varphi(g) = \Phi' \varphi'(g')} & \Phi' \tau \varphi(t) \\
 \downarrow \Sigma_{\varphi(s)}^{-1} & \textcircled{\#1} & \downarrow \Sigma_{\varphi(t)}^{-1} \\
 \Theta \Phi \varphi(s) & \xrightarrow{\Theta \Phi \varphi(g)} & \Theta \Phi \varphi(t) \\
 \downarrow \Theta(\mu_s) & \textcircled{\#2} & \downarrow \Theta(\mu_t) \\
 \Theta \psi(s) & \xrightarrow{\Theta \psi(g)} & \Theta \psi(t)
 \end{array}$$

The diagram is annotated with additional elements:

- A large left curly bracket labeled $\mu'_{s'}$ spans from $\Theta \Phi \varphi(s)$ to $\Theta \psi(s)$, with a circled $\#3$ next to it.
- A large right curly bracket labeled $\mu'_{t'}$ spans from $\Theta \Phi \varphi(t)$ to $\Theta \psi(t)$, with a circled $\#4$ next to it.

whose cells commute, by the naturality of Σ (for #1), the naturality of μ (#2), and the third of the equations (*), p (15). This proves (*), p (17).

For σ being a functor: immediate from the fact that, for $s \xrightarrow{g} t \xrightarrow{h} u$, $\sigma(g): \sigma(s) \rightarrow \sigma(t)$ is the unique arrow $g': \sigma(s) \rightarrow \sigma(t)$ for which $\varphi'(g') = \tau\varphi(g)$

(namely, if $s \xrightarrow{g} t \xrightarrow{h} u$ in $|F|$, we can show that the composite $\sigma(h) \circ \sigma(g): \sigma(s) \rightarrow \sigma(u)$, abbreviated $k': \sigma(s) \rightarrow \sigma(u)$, satisfies

$$\begin{aligned} \varphi'(k') &\stackrel{?}{=} \tau\varphi(hog) \\ &\parallel \\ \varphi'(\sigma(h) \circ \sigma(g)) &\parallel \tau\varphi(h) \circ \tau\varphi(g) \\ &\parallel \varphi'\sigma(h) \circ \varphi'\sigma(g) \\ &\parallel \varphi'\sigma(hog) \end{aligned}$$

therefore, $k' = \sigma(hog)$. A similar, but simpler argument shows that $\sigma(1_s) = 1_{\sigma(s)}$.

This completes the proof of Lemma 2.

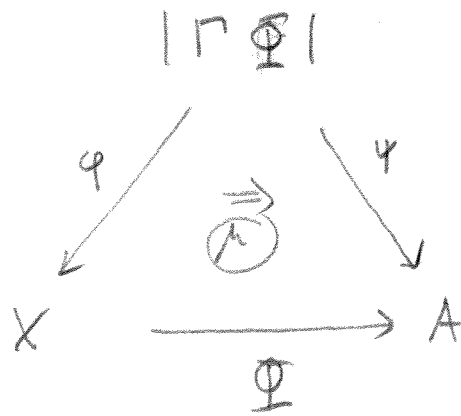
□ Lemma 2

I am returning to page 6, to defining the functor $\hat{\Gamma}$, and proving statements 1) and 2) on that page.

3. The definition of $\hat{\Gamma}: p\text{Fun} \rightarrow \text{SmapFun}$.

Given any functor $\Phi: X \rightarrow A$, we define

$\hat{\Gamma}(\Phi) = (\Gamma\Phi, \Phi, \mu[\Phi])$, where $\Gamma\Phi = (|\Gamma\Phi|, \varphi[\Phi], \psi[\Phi])$,



$\varphi = \varphi[\Phi]$
 $\psi = \psi[\Phi]$
 $\mu = \mu[\Phi]$

as follows:

object of $|\Gamma\Phi|$: triple (x, a, ν) ,

where $x \in X, a \in A, \nu: \Phi x \xrightarrow{\cong} a$;

arrow of $|\Gamma\Phi|$:

$$(x, a, \nu) \xrightarrow{(f, g)} (y, b, \rho)$$

where $f: x \rightarrow y$, $g: a \rightarrow b$ such that

$$\begin{array}{ccc}
 \Phi(x) & \xrightarrow{v} & a \\
 \Phi(f) \downarrow & \# & \downarrow g \\
 \Phi(y) & \xrightarrow{f} & b
 \end{array} ; \quad (*)$$

composition: component-wise.

Φ and Ψ : obvious forgetful functors

$$\mu: \Phi \Psi \xrightarrow{\cong} \Psi \text{ has components}$$

$$\mu_{(x, a, v)} = v: \Phi x \xrightarrow{\cong} a.$$

This is a very general construction, defining the so-called (here) pseudo-graph of any functor.

The main point is that

$$\Gamma \Phi = (\Gamma \Phi, \Psi, \Psi) \in \text{Sana},$$

i.e., $\Gamma \Phi$ satisfies conditions 1), 2) and 3)

on page 2. Condition 1) holds, since in the diagram (*), there always is a unique $g: a \rightarrow b$

once the other items are given.

Concerning condition 2): for $x \in X$, $b \in A$,

we now have $|F|(x, b) = \{(x, b, \nu) : \nu \in \text{Iso}(\Phi_x, b)\}$.

For $s = (x, a, \mu)$, $\text{Iso}(\psi(s), b) = \text{Iso}(a, b)$.

For $s = (x, a, \mu)$ and $t \in |F|(x, b)$, $t = (x, b, \nu)$,

the arrow $g_{s, t, \pm\psi(s)}$ is the arrow $(\mathbb{1}_x, g) : s \rightarrow t$

where g is defined by the commutativity

$$\begin{array}{ccc}
 & \mu \rightarrow a & \\
 \Phi_x & \xrightarrow{\cong} & \downarrow g \\
 & \nu \rightarrow b &
 \end{array}
 \quad (*)$$

The mapping $F_{s, b} : F(\psi(s), b) \rightarrow \text{Iso}(\psi(s), b)$

maps (x, b, ν) to the arrow g determined as in $(*)$ preceding. $F_{s, b}$ is a bijection, since if $s = (x, a, \mu)$ is given and if g is any

isomorphism $g : a \xrightarrow{\cong} b$, then there is a unique $\nu : \Phi_x \xrightarrow{\cong} b$,

an isomorphism, and hence a unique $t = (x, b, \nu) \in |F|(x, b)$

such that $F_{s, b}(t) = g$.

Condition 3): given $x \in X$, we have $s = (x, \Phi_x, \text{id}_{\Phi_x}) \in |\Gamma\Phi|$,

such that $\varphi(s) = x$.

Remark: $\mathcal{P}\mathcal{Q}$ is the saturated anafunctor associated with the functor \mathcal{Q} , according to the paper cited on p. 16.

We have defined the object function $\hat{\Gamma} : \text{Ob}(\text{pFun}) \rightarrow \text{Ob}(\text{Sana pFun})$ such that, with functor

$R : \text{Sana pFun} \rightarrow \text{pFun}$, we have that the composite

$$R \circ \hat{\Gamma} : \text{Ob}(\text{pFun}) \rightarrow \text{Ob}(\text{pFun})$$

is the identity. Given that R is full and faithful, it immediately follows that there is a unique

functor $\hat{\Gamma} : \text{pFun} \rightarrow \text{Sana pFun}$ with the

given object function such that $R \circ \hat{\Gamma} : \text{pFun} \rightarrow \text{pFun}$

is the identity functor. Moreover, $\hat{\Gamma}$ is the quasi-inverse of R , i.e. $\hat{\Gamma} \circ R \cong \text{Id}_{\text{Sana pFun}}$, as is immediately

seen by using the fact that R is full and faithful.

again

23.1

Conclusion: $R: \text{Simp Fun} \longrightarrow \text{pFun}$

is an equivalence of categories, with
quasi-inverse the canonical (choice-free)

functor $\hat{R}: \text{pFun} \longrightarrow \text{Simp Fun}$

Lemma 3

$S : \text{SanaFun} \rightarrow \text{Sana}$
is surjective on objects

Proof. Let $F \in \text{Sana}$; we use the notation started on p. ①. We are to define $\Phi : X \rightarrow A$ and μ as in (*), p. 4.

Let, for any $x \in X$, $s_x \in |F|$ be chosen so that $\varphi(s_x) = x$ (see condition 3), p. ②).

Define $\Phi(x) \stackrel{\text{def}}{=} \psi(s_x)$.

For $f : x \rightarrow y$ in X , we have

$g_f \stackrel{\text{def}}{=} g_{s_x, s_y, f}$ (see p. ②, condition 1)); we

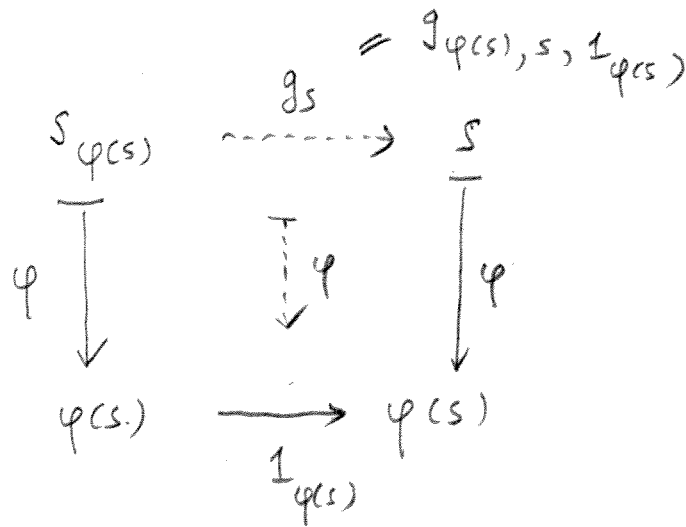
let $\Phi(f) \stackrel{\text{def}}{=} \psi(g_f) : \Phi(x) \rightarrow \Phi(y)$. The

uniqueness part in condition 1), p. ②, ensures

that if $x \xrightarrow{f} y \xrightarrow{e} z$, then $g_{ef} = g_e g_f$, and $g_{1_x} = 1_{s_x}$,

which ensures that $\Phi : X \rightarrow A$ is a functor.

For $s \in |F|$, we have:



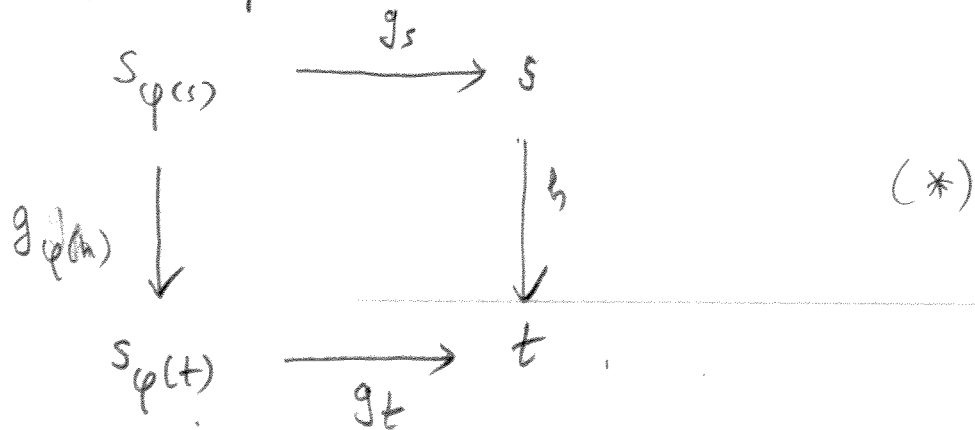
Thus, $\psi(g_s) : \psi(s_{\varphi(s)}) \longrightarrow \psi(s)$

is an arrow, an isomorphism,

$$\psi(g_s) = \mu_s : \underline{\varphi(s)} \xrightarrow{\cong} \varphi(s)$$

Moreover, μ_s is natural in s : let $h: s \rightarrow t$ in $|F|$,

and consider the diagram



The functor $\varphi: |F| \rightarrow X$ maps this to the commutative diagram

$$\begin{array}{ccc}
 \varphi(s) & \xrightarrow{I_{\varphi(s)}} & \varphi(cs) \\
 \varphi(h) \downarrow & & \downarrow \varphi(h) \\
 \varphi(t) & \xrightarrow{I_{\varphi(t)}} & \varphi(ct)
 \end{array}$$

Therefore $(*)$, previous page, is commutative (by 1), p(2)

It's image under $\psi: |F| \rightarrow A$,

$$\begin{array}{ccc}
 \psi(s_{\varphi(cs)}) & \xrightarrow{\psi(g_s)} & \psi(s) \\
 \psi(g_{\varphi(h)}) \downarrow & \# & \downarrow \psi(h) \\
 \psi(s_{\varphi(ct)}) & \xrightarrow{\psi(g_t)} & \psi(t)
 \end{array}$$

is the same as

$$\begin{array}{ccc}
 \Phi_{\varphi(s)} & \xrightarrow{M_s} & \psi(s) \\
 \Phi_{\varphi(h)} \downarrow & \# & \downarrow \psi(h) \\
 \Phi_{\varphi(t)} & \xrightarrow{M_t} & \psi(t)
 \end{array}$$

We have shown that $\mu = (\mu_s)_{\text{self}} : \Phi \xrightarrow{\cong} \Psi$
 is a natural isomorphism, and this completes
 the proof that the functor \mathcal{L} is surjective
 on objects.

□ Lemma 3

Lemma 2 (p. 9), the Conclusion on p. 23.1,
 and Lemma 3, p. 24 establish the Theorem,
 and the Corollary, p. 3.