PART 2

M. Barr's theorem

Let: $\mathbb{C}$ small regular category

We have: $i: \mathrm{Reg}[\mathbb{C}, Set] \longrightarrow [\mathbb{C}, Set]$

full inclusion $\qquad\qquad \mathcal{S} = Set$

$$\begin{tikzcd}[column sep=small]
 & {[[\mathbb{C}, \mathcal{S}], \mathcal{S}]} \arrow[dr, "i^*"] \arrow[d, phantom, "\parallel"] & \\
\mathbb{C} \arrow[ur, "\varepsilon_{\mathbb{C}}"] \arrow[rr, bend right, "ev_{\mathbb{C}}"'] & {} & {[\mathrm{Reg}[\mathbb{C}, \mathcal{S}], \mathcal{S}]}
\end{tikzcd}$$

(definition

(similarly to "Joyal's theorem", T_3 3)

<u>Theorem</u> $\quad ev_{\mathbb{C}}: \mathbb{C} \longrightarrow [\mathrm{Reg}[\mathbb{C}, \mathcal{S}], \mathcal{S}]$

is a regular full embedding.

The facts that the codomain is regular category, and that $ev_{\mathbb{C}}$ is a regular functor are automatic. To be proved: $ev_{\mathbb{C}}$ is <u>full and faithful</u>.

Duality properties of the category of sets

In any category:

<u>definition</u>: a QE (quotient of equivalence relation) diagram:

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{h} Z \qquad (*)$$

such that (f, g) is an equivalence relation (...) and h is a coequalizer of (f, g).

<u>Fact</u>: in Set, a (small) product $\prod\limits_{i \in I} \ldots$ of QE diagrams is QE again

(because $(*)$ is QE in Set iff (f, g) is a kernel-pair of h, and h is <u>surjective</u>)

Also: a filtered colimit of QE diagrams is a QE diagram in Set.

Denote: $\boxed{QE/FL[\mathbb{C}, \mathbb{S}]}$: full subcat of $[\mathbb{C}, \mathbb{S}]$ of the functors $\mathbb{C} \to \mathbb{S}$ that map QE / FL diagrams to QE / FL diagrams.

$\boxed{\Pi / FC[\mathbb{X}, \mathbb{S}]}$: functors $\mathbb{X} \to \mathbb{S}$ that map small-product diagrams in $\mathbb{X}$ to such in $\mathbb{S}$, and similarly for (small) filtered colimit diagrams.

For $\mathbb{S} = \text{Set}$, and for any $\mathbb{X}$, $\mathbb{C}$ $\in$ CAT we have a bijection

$$\frac{\mathbb{C} \longrightarrow \Pi / FC[\mathbb{X}, \mathbb{S}]}{\mathbb{X} \longrightarrow QE/FL[\mathbb{C}, \mathbb{S}]} \quad \Psi_{\mathbb{C}, \mathbb{X}}$$

defined by:

$$\begin{array}{ccccc}
\mathbb{C} & \xrightarrow{\Sigma} & \Pi/FC[\mathbb{X},\mathbb{S}] & \xrightarrow{\text{incl}} & [\mathbb{X},\mathbb{S}] \\
\hline
\mathbb{X} & \xrightarrow{\Gamma} & QE/FL[\mathbb{C},\mathbb{S}] & \xrightarrow{\text{incl}} & [\mathbb{C},\mathbb{S}]
\end{array}$$

$\Psi_{\mathbb{C},\mathbb{X}}$

$\varphi_{\mathbb{C},\mathbb{X}}$.

$$\Sigma \xmapsto{\Psi_{\mathbb{C},\mathbb{X}}} \Gamma$$

$$\Longleftrightarrow \quad \text{incl} \circ \Sigma \longmapsto \text{incl} \circ \Gamma$$

the exponential adjunction for CAT, Set.

This is purely formal, by said duality properties of Set.

Let $\underline{QE/FL}$ denote the 2-category of all categories having the QE and FL operations defined on them, with 1-cells the QE/FL functors and 2-cells: all nat. transf's

Similarly: $\underline{\Pi/FC}$.

We obtain, purely formally, the 2-adjunction

$$
\boxed{
\begin{tikzcd}[column sep=huge]
\underline{QE/FL}^{op} \arrow[r, shift right=2, "(-)^* = \Pi/FC{[-, Set]}"'] \arrow[r, phantom, "\perp"] & \underline{\Pi/FC} \arrow[l, shift right=2, "(-)^\# = QE/FL{[-, Set]}"']
\end{tikzcd}
}
$$

Counit: $\varepsilon_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{C}^{*\#}$

<u>Theorem</u> (MM, 1988) For small Barr-exact category $\mathbb{C}$, $\varepsilon_{\mathbb{C}}$ is an equivalence of categories.

NB: this $\varepsilon_{\mathbb{C}}$ is like the one on T_3 6.2: the same function in fact, with codomain restricted

The theorem says: in Barr's theorem the image of the full embedding

$$ev_{\mathbb{C}} : \mathbb{C} \longrightarrow [\,Reg\,[\mathbb{C}, \mathbb{S}], \mathbb{S}]$$

(see: T_3 8)
consists of the Π / FC preserving functors $Reg\,[\mathbb{C}, \mathbb{S}] \longrightarrow \mathbb{S}$.

<u>Definition</u> A <u>dual</u>-<u>regular</u> category $\mathbb{X}$ is one for which there is a small regular category $\mathbb{C}$ such that

$$\mathbb{X} \simeq Reg\,[\,\mathbb{C}, Set].$$

Let <u>Exact</u> be the 2-category of small exact categories, and regular functors; <u>Dual Reg</u> the full sub-2-category of <u>Π/FC</u> on the dual regular categories

We obtain a <u>biequivalence</u>

$$\begin{tikzcd} \underline{\mathrm{Exact}} \arrow[r, shift right, "G = (-)^*"'] & \underline{\mathrm{Dual{\cdot}Reg}} \arrow[l, shift right, "F = (-)^{\#}"'] \end{tikzcd}$$

$$FG \xrightarrow{\simeq} 1 \qquad 1 \xrightarrow{\simeq} GF$$

$$\varepsilon : FG \xrightarrow{\simeq} 1, \qquad \eta : 1 \xrightarrow{\simeq} GF$$

<u>Corollary</u> (John Bourke)

The "limit theorem" holds for <u>Dual Reg</u>: any $\boxed{\text{weighted pseudo-limit}}$ of dual regular categories and Π/FC functors between them is again dual regular.

? : see later!

Pseudo-natural transformations

(psnt's)

In what follows,

	$\mathbb{X}$,	$\mathbb{A}$,	$\mathbb{M}$: 2-categories
0-cells ;	$X, Y, \dots$	$A, B, \dots$	$M, N, \dots$

2-functor: $\mathbb{X} \xrightarrow{F} \mathbb{A}$: structure preserving ("on the nose", strictly)

psnt:

$$\begin{tikzcd} \mathbb{X} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\downarrow\varphi"] & \mathbb{A} \end{tikzcd} :$$

$$\varphi = (\langle\varphi_X\rangle, \langle\varphi_f\rangle_{f: X\to Y}) \quad \text{where}$$

$X \in \mathbb{X}_0$: $\varphi_X : FX \longrightarrow GX$ (in $\mathbb{A}_1$, of course)

and $\varphi_f : (Gf)\circ\varphi_X \xrightarrow{\cong} \varphi_Y \circ (Ff)$ ($\in \mathbb{A}_2$)

$$\begin{array}{c} X \\ f\downarrow \\ Y \end{array} \in \mathbb{X}_1 : \quad \begin{tikzcd} FX \arrow[r, "\varphi_X"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\cong\ \swarrow\varphi_f"] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\varphi_Y"'] & GY \end{tikzcd}$$

<u>Satisfying</u>:

1)

$$\left(\begin{tikzcd} X \arrow[d, "f"'] \arrow[d, phantom, shift left=3, "\xrightarrow{m}"] & \\ Y \end{tikzcd} \!\!\downarrow g \right) \in \mathcal{X}_2 :
\qquad
\begin{tikzcd}[column sep=large]
FX \arrow[rr, "\varphi_X"] \arrow[d, bend right, "Ff"'] \arrow[d, bend left, "Fg"] \arrow[d, phantom, "\xrightarrow{Fm}"] & & GX \arrow[d, bend right, "Gf"'] \arrow[d, bend left, "Gg"] \arrow[d, phantom, "\xrightarrow{Gm}"] \\
FY \arrow[rr, "\varphi_Y"'] & & GY
\end{tikzcd}$$

commutes: $\boxed{(Gm)\varphi_X = \varphi_Y(Fm)}$

2)

$$\left(\begin{tikzcd} X \arrow[d, "f"'] \\ Y \arrow[d, "g"'] \\ Z \end{tikzcd}\right) :
\quad
\begin{tikzcd}
FX \arrow[r, "\varphi_X"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\overset{\cong}{\Swarrow}\varphi_f"] & GX \arrow[d, "Gf"] \\
FY \arrow[r, "\varphi_Y"] \arrow[d, "Fg"'] \arrow[dr, phantom, "\overset{\cong}{\Swarrow}\varphi_g"] & GY \arrow[d, "Gg"] \\
FZ \arrow[r, "\varphi_Z"'] & GZ
\end{tikzcd}
\;=\;
\begin{tikzcd}
FX \arrow[r, "\varphi_X"] \arrow[d, "F(gf)"'] \arrow[dr, phantom, "\overset{\cong}{\Swarrow}\varphi_{gf}"] & GX \arrow[d, "G(gf)"] \\
FZ \arrow[r, "\varphi_Z"'] & FZ
\end{tikzcd}$$

$$\boxed{\varphi_{gf} = ((Gg)\varphi_f) \circ (\varphi_g(Ff))}$$

end of definition

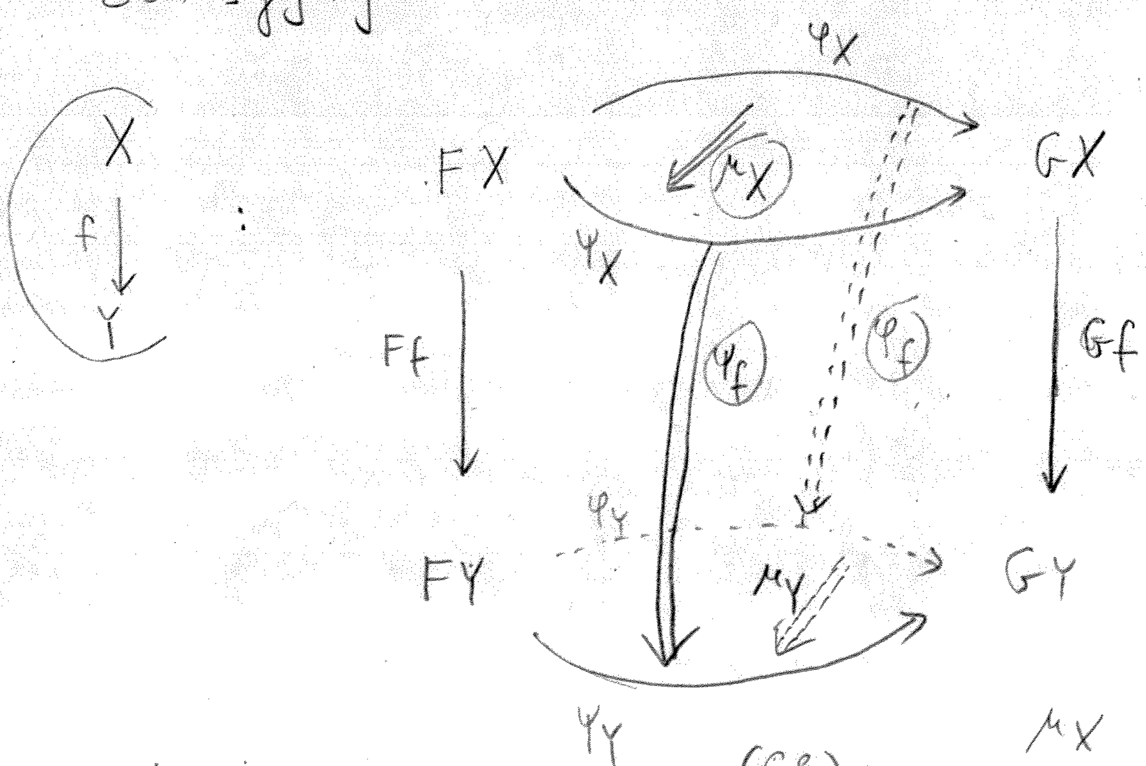
of psnt <u>of</u> 2-functors (!)

<u>modification</u> :

$$\begin{tikzcd} \mathbb{X} \arrow[rr, bend left, "F"] \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\varphi\Downarrow \xrightarrow{\mu} \Downarrow\psi"] & & \mathbb{A} \end{tikzcd} :$$

$$\mu = \langle \varphi_X \rangle_X \quad , \quad \mu_X : \varphi_X \to \psi_X \quad \in \mathbb{X}_2$$

satisfying



commutes; i.e.:

$$\begin{tikzcd} (Gf)\varphi_X \arrow[r, "\mu_X"] \arrow[d, "\varphi_f"'] \arrow[dr, phantom, "\equiv"] & (Gf)\psi_X \arrow[d] \\ \varphi_Y(Ff) \arrow[r, "\mu_Y"'] & \psi_Y(Ff) \end{tikzcd}$$

GRAY (after John Gray)

is a 3-dimensional category, with the underlying 3-graph, with

0-cells: 2-categories

1-cells: 2-functors

2-cells: psnt's

3-cells: modifications

— more on the def'n of GRAY: see later.

— · —

Why we need psnt's:

Recall limits in categories:

Let: $\mathbb{C}$, $\mathbb{G}$: cat's, $\mathbb{G}$ small; e.g. $\mathbb{G} = \boxed{\begin{tikzcd} & 1 \arrow[d, "3"] \\ 2 \arrow[r, "4"'] & 0 \end{tikzcd}}$;

a $\mathbb{G}$-type diagram $\Gamma: \mathbb{G} \to \mathbb{C}$ in $\mathbb{C}$;

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd}$$ in the example

X : object in $\mathbb{C}$.

<u>Definition</u> ρ is a <u>cone</u> on Γ with vertex X:

$$\rho : \ulcorner X \urcorner \longrightarrow \Gamma \qquad \text{natural transf}$$

$$\begin{tikzcd} G \arrow[r, bend left, "\ulcorner X \urcorner"] \arrow[r, bend right, "\Gamma"'] \arrow[r, phantom, "\Downarrow \rho"] & \mathbb{C} \end{tikzcd}$$

where $\ulcorner X \urcorner$ is the constant-X functor.

<u>example:</u> when G is as above, Γ and ρ look like this:

$$\begin{tikzcd} & X \arrow[d, "1_X"] \arrow[drr, "\rho_1"] & & \\ X \arrow[r, "1_X"] \arrow[dr, "\rho_2"'] & X \arrow[drr, "\rho_0"] \arrow[d, phantom, "\equiv"] \arrow[r, phantom, "\equiv"] & & B \arrow[d, "f"] \\ & C \arrow[rr, "g"'] & & A \end{tikzcd} \equiv \begin{tikzcd} X \arrow[r, "\rho_1"] \arrow[d, "\rho_2"'] \arrow[dr, "\rho_0"] & B \arrow[d] \\ C \arrow[r, "g"'] & A \end{tikzcd}$$

$$\equiv \begin{tikzcd} X \arrow[r, "\rho_1"] \arrow[d, "\rho_2"'] \arrow[dr, phantom, "\equiv"] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd}$$

(ρ_0 is then uniquely recovered)

<u>'pullback-shaped diagram'</u>

For fixed Γ and X, the set of cones-on-Γ-with vertex-X,

$$\mathrm{Cone}_\Gamma(X) \underset{\text{def}}{=} \mathrm{Nat}(\ulcorner X \urcorner, \Gamma) \quad (\text{a } \underline{\text{set}}).$$

For fixed Γ, but variable X-in-$\mathbb{C}$, we have the <u>functor</u>:

$$\mathrm{Cone}_\Gamma : \mathbb{C}^{op} \longrightarrow \mathbf{Set}$$

$$\begin{tikzcd}
X & & \mathrm{Cone}_\Gamma(X) & & \ulcorner X \urcorner \arrow[r, "\rho"] \arrow[d, mapsto] & \Gamma \\
Y \arrow[u, "f"] & \arrow[r, mapsto] & \mathrm{Cone}_\Gamma(Y) \arrow[from=u, "f^*"] & \ulcorner Y \urcorner \arrow[r, "\ulcorner f \urcorner"'] \arrow[rr, bend right, "\rho \circ \ulcorner f \urcorner"'] & \ulcorner X \urcorner \arrow[r, "\rho"] & \Gamma
\end{tikzcd}$$

<u>Definition</u> Γ has a limit in $\mathbb{C}$ if

$$\mathrm{Cone}_\Gamma : \mathbb{C}^{op} \longrightarrow \mathbf{Set}$$

is <u>representable</u>:

<u>there is</u> (L, π)

$$L \in \mathbb{C}, \quad \pi \in \mathrm{Cone}_\Gamma(L)$$

such that the natural transformation

$$\theta_{U,u} : \mathbb{C}(-, L) \longrightarrow \mathrm{Cone}_\Gamma$$

$$\begin{cases} (\theta_{U,u})_X : \mathbb{C}(X, L) \longrightarrow \mathrm{Cone}_\Gamma(X) \quad (X \in \mathbb{C}) \\ X \xrightarrow{f} L \longmapsto \underbrace{\pi \circ \ulcorner f \urcorner} \end{cases}$$

$$\begin{cases} \mathrm{Cone}_\Gamma(L) \xrightarrow{f^*} \mathrm{Cone}_\Gamma(X) \\ \pi \longmapsto \pi \circ \ulcorner f \urcorner \end{cases}$$

is an isomorphism (as a nat. transf., equivalently, for each X separately).

L is the <u>limit</u>, π is the <u>limit cone</u>.

Returning to the example: $G = \boxed{\begin{array}{ccc} & & 1 \\ & & \downarrow 3 \\ 2 & \xrightarrow[4]{} & 0 \end{array}}$, and ρ T_3 10.4

$$\pi : \ulcorner L \urcorner \longrightarrow \Gamma$$

i.e.

$$\begin{tikzcd}
L \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\equiv"] & B \arrow[d, "f"] \\
C \arrow[r, "g"'] & A
\end{tikzcd}$$

is a limit cone iff for all X in $\mathbb{C}$ and

for all

$$\begin{tikzcd}
X \arrow[r, bend left, "\rho_1"] \arrow[dr, bend right, "\rho_2"'] \arrow[r, phantom, shift right=6, "\equiv"] & B \arrow[d, "f"] \\
& C \arrow[r, "g"'] & A
\end{tikzcd}$$

$\boxed{\text{there is unique}}$ $X \xrightarrow{f} L$ such that

$$\begin{tikzcd}
X \arrow[dr, "f"] \arrow[drr, bend left, "\rho_1"] \arrow[ddr, bend right, "\rho_2"'] & & \\
& L \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & B \arrow[d, "f"] \\
& C \arrow[r, "g"'] & A
\end{tikzcd}$$

CAT as a category has all small limits and colimits (as a locally finitely presentable category) — the limits computed in an expected manner (simply) (colimits: not simple at all!)

With

$$\begin{tikzcd} & \mathbb{B} \arrow[d, "F"] \\ \mathbb{C} \arrow[r, "G"'] & \mathbb{A} \end{tikzcd}$$

categories and functors, the pull back $\mathbb{P}$ has: $P_0 = \{(B, C) : F(B) = F(C)\}$

$P_1 = \dots$ as expected

However: with $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ having terminal objects, F, G preserving them (taking a terminal to a terminal) — it does not follow that $\mathbb{P}$ has a terminal object: for $1_{\mathbb{B}}$, $1_{\mathbb{C}}$ the terminals in $\mathbb{B}$, $\mathbb{C}$, $F(1_{\mathbb{B}})$ and $F(1_{\mathbb{C}})$ are terminals in $\mathbb{A}$, but may not be <u>equal</u>. (Explicit example is easy.)

Thus, the 2-cats <u>Reg</u>, <u>Exact</u>, <u>Dual Reg</u>, ...

do not have ordinary pullbacks — however, they have

small <u>pseudo-limits</u> and <u>pseudo-colimits</u>.

! { <u>Categorification</u> of the notion of limit

(Max Kelly and Ross Street) :

Let : $\mathbb{C}$: 2-category

G : 2-category (or: just a category)

$\Gamma : G \to \mathbb{C}$: 2-functor

$W : G \to \mathrm{Cat}$: 2-functor ("weight")

(special case : $W = \ulcorner \mathbb{1} \urcorner$, constant $\mathbb{1}$ (terminal cat) functor, with $\mathbb{1}$ the terminal 2-cat).

<u>Definition</u> ρ is a W-weighted pseudo-cone on Γ with vertex X if

$$\rho : W \xrightarrow[\text{pseudo-nat}]{} \mathbb{C}(X, \Gamma(-))$$

$$\begin{tikzcd}[column sep=huge] G \arrow[r, bend left, "W"] \arrow[r, bend right, "\mathbb{C}(X,\Gamma(-))"'] \arrow[r, phantom, "\text{psnt}\ \Downarrow\rho"] & \mathrm{CAT}\end{tikzcd}$$

where

$$\begin{tikzcd} G \arrow[r, "W"'] \arrow[rr, bend left, "\mathbb{C}(X,\Gamma(-))"] & \mathbb{C} \arrow[r, "\underbrace{\mathbb{C}(X,-)}_{\text{representable 2-functor}}"'] & \mathrm{CAT}\end{tikzcd}$$

$$Y \longmapsto \underbrace{\mathbb{C}(X,Y)}_{\text{a category}}$$

When $W = \ulcorner 1 \urcorner$, above reduces to

$$\begin{tikzcd}[column sep=huge] G \arrow[r, bend left, "\ulcorner X \urcorner"] \arrow[r, bend right, "\Gamma"'] \arrow[r, phantom, "\text{psnt}\ \Downarrow\rho"] & \mathbb{C}\end{tikzcd}$$

The category $\mathrm{Cone}^W_\Gamma(X)$ has objects these ρ's, arrows : modifications.

$\mathrm{Cone}^W_\Gamma(X)$ as a function of $X \in \mathbb{C}$

is a <u>2-functor</u>:

$$\mathrm{Cone}^W_\Gamma : \mathbb{C}^{op} \longrightarrow \mathrm{CAT}$$

$$\begin{tikzcd} X \\ Y \arrow[u, "f"] \end{tikzcd} \longmapsto \begin{tikzcd} \mathrm{Cone}^W_\Gamma(X) \arrow[d, "f^*"] \\ \mathrm{Cone}^W_\Gamma(Y) \end{tikzcd} \qquad \begin{tikzcd} W \arrow[r, "\rho"] & \mathbb{C}(X, \Gamma(-)) \arrow[d, mapsto] & \\ W \arrow[r, "\rho"'] \arrow[rr, bend right] & \mathbb{C}(X, \Gamma(-)) \arrow[r, "(-)\circ f"'] & \mathbb{C}(Y, \Gamma(-)) \end{tikzcd}$$

$$\begin{tikzcd} X \\ Y \arrow[u, bend left, "f"] \arrow[u, bend right, "g"'] \arrow[u, phantom, "\overset{m}{\Rightarrow}"] \end{tikzcd} \longmapsto \text{etc}$$

<u>Definition</u> For $\begin{tikzcd}[row sep=small] & \mathrm{Cat} \\ G \arrow[ur, "W"] \arrow[dr, "\Gamma"'] & \\ & \mathbb{C} \end{tikzcd}$, Γ <u>has</u> W-<u>weighted</u> <u>pseudo</u> <u>limit</u> <u>in</u> $\mathbb{C}$ if $\mathrm{Cone}^W_\Gamma : \mathbb{C}^{op} \to \mathrm{CAT}$

is (up-to-iso) representable.

T_3 10.12

Examples:

1) Let: $G = $ $\begin{tikzcd} & 1 \arrow[d, "3"] \\ 2 \arrow[r, "4"'] & 0 \end{tikzcd}$

and $W = \ulcorner 1 \urcorner$; the limit: <u>pseudo pullback</u>;

exists in CAT : the pseudo-pullback of

$$\begin{tikzcd} & \mathbb{B} \arrow[d, "F"] \\ \mathbb{C} \arrow[r, "G"'] & \mathbb{A} \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{P} \arrow[r] \arrow[d] & \mathbb{B} \arrow[d, "F"] \arrow[dl, phantom, "\Swarrow"] \\ \mathbb{C} \arrow[r] & \mathbb{A} \end{tikzcd}$$ (slight cheating)

is given by:

$$\mathbb{P}_0 = \{ (B, C, \varphi) : F(B) \xrightarrow[\cong]{\varphi} G(C) \}$$

$\mathbb{P}_1$: $(B, C, \varphi) \longrightarrow (B', C', \varphi')$ is: any

$(b, c) = (b : B \to B', \ c : C \to C')$

such that

$$\begin{tikzcd} F(B) \arrow[r, "\varphi"] \arrow[d, "F(b)"'] \arrow[dr, phantom, "\equiv"] & G(C) \arrow[d, "G(c)"] \\ F(B') \arrow[r, "\varphi'"'] & G(C') \end{tikzcd} .$$

Reg, Exact, Dual Reg, ... have

pseudo pullbacks

computed as in CAT

— for Dual Reg: not obvious!

(John Bourke's "corollary" above)

but for Reg, Exact, ... it is 'obvious' (easy).

2) Let: $\mathcal{G}$ = the terminal 2-cat $\mathbb{1}$

(one object, one arrow, one 2-cell)

with a small cat T; $W = \ulcorner T \urcorner : \mathbb{1} \longrightarrow CAT$

T_3 10.17

$$\Gamma = \ulcorner X \urcorner : \mathbb{1} \longrightarrow \mathbb{C}$$

$$(X \in \mathbb{C}_0)$$

The W-limit of Γ is called the

cotensor

$$T \pitchfork X \qquad \text{(if it exists)}$$

In $\mathbb{C} = CAT$, (small) cotensors exist:

$$T \pitchfork X = [T, X]$$

Reg, Exact, ..., Dual Reg inherit

cotensors from CAT

(for Dual Reg: not obvious)