Contractible simplicial objects

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Abstract

We examine the question of what it means for a simplicial object to be contractible. We look at three answers and then show by examples in Sets that the three answers really are different. We have also a discovered a new and useful way to look at simplicial homotopy.

Primer on simplicial objects

I include the next two slides for the benefit of people who wish to download and study them. They will be passed over during the presentation.

If $\mathcal{X}$ is a category, a simplicial object X in $\mathcal{X}$ consists of

1. A sequence $X_0, X_1, \ldots$ of objects of $\mathcal{X}$;
2. faces $d^i = d^i_n : X_n \longrightarrow X_{n-1}$ for all $0 \leq i \leq n$;
3. degeneracies $s^i = s^i_n : X_n \longrightarrow X_{n+1}$ for all $0 \leq i \leq n$.

subject to the equations

1. $d^i d^j = d^{j-1} d^i$ for $i < j$;
2. $s^i s^j = s^j s^{i-1}$ for $i > j$;
3. $d^i s^j = \begin{cases} s^{j-1} d^i & \text{for } i < j \\ s^j d^{i-1} & \text{for } i > j+1 \\ \mathrm{id} & \text{for } i = j, j+1 \end{cases}$

Primer on simplicial maps and homotopies

A simplicial map $f : X \longrightarrow Y$ is a sequence of morphisms $f_n : X_n \longrightarrow Y_n$ that commutes in the obvious ways with the faces and degeneracies. If $f, g : X \longrightarrow Y$ a homotopy $h : f \rightsquigarrow g$ consists of morphisms $h^i = h^i_n : X_n \longrightarrow Y_{n+1}$ satisfying the equations

1. $d^0 h^0 = f_n$ and $d^{n+1} h^n = g_n$
2. $d^i h^j = \begin{cases} h^{j-1} d^i & \text{if } 0 < i < j < n+1 \\ h^j d^{i-1} & \text{if } n+1 > i > j+1 > 0 \\ d^i h^{i-1} & \text{if } 0 < i = j < n+1 \end{cases}$
3. $s^i h^j = \begin{cases} h^j s^{i-1} & \text{if } i > j \\ h^{j+1} s^i & \text{if } i \leq j \end{cases}$

Contractible spaces

A topological space S is contractible to a point $s_0 \in S$ if there is a map $H : S \times I \longrightarrow S$ such that $H(s,0) = x$ and $H(s,1) = s_0$. For our purposes, this is a bit restrictive since it privileges the one element set. For our purposes, it is better to have a discrete subset $S_0 \in S$ and assume of H that $H(s,0) = s$ and $H(s,1) \in S_0$. This is equivalent to assuming that S is a disjoint union of sets each contractible to a point.

First definition: homotopic to a constant

Translating this into simplicial objects we begin by saying that a simplicial object is constant if all terms are the same and all face and degeneracy operations are the identity. For example, the singular simplicial set of a discrete space is constant.

Then we say that the simplicial object X is homotopic to a constant if there is a constant simplicial object C, maps $f : C \longrightarrow X$ and $g : X \longrightarrow C$, and a simplicial homotopy h such that $gf = \mathrm{id}$ and $h : \mathrm{id} \rightsquigarrow fg$.

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Extra degeneracies, take 1

A number of references define a simplicial object to be constractible if it has *extra degeneracies*. An extra degeneracy on X is described as a sequence of morphisms $t = t_n : X_n \longrightarrow X_{n+1}$ that "satisfies the equations of a degeneracy labeled -1". These equations are

1. $d^0 t = \mathrm{id}$;
2. $d^i t = t d^{i-1}$ for $i > 0$;
3. $s^i t = t s^{i-1}$ for $i > 0$;
4. $s^0 t = tt$.

For reasons about to be explained, this will be called a *strong extra degeneracy*.

Extra degeneracy, take 2

But when most of the sources actually write down the equations of an extra degeneracy, they omit the fourth equation $s^0 t = tt$ on the previous slide. We will say that an *extra degeneracy* on X is a series of morphisms $t = t_n : X_n \longrightarrow X_{n+1}$ that satisfy

1. $d^0 t = \mathrm{id}$;
2. $d^i t = td^{i-1}$ for $i > 0$;
3. $s^i t = ts^{i-1}$ for $i > 0$.

It should be mentioned that there is a "mirror" definition in which the extra degeneracy is at the top, that is like one numbered $n + 1$ in degree n. This differs only in the numbering from the situation we are considering.

Relation between these notions

The relations among the concepts of homotopic to a constant (HC), having an extra degeneracy (ED), and having a strong extra degeneracy (SED) are given by the following:

Theorem. *SED $\Rightarrow$ ED $\Rightarrow$ HC. Both implications are proper.*

The first implication is obvious. The second is not, but the proof is straightforward. The hard part is showing that the implications are proper. Both of them are done by starting with an example using partial or truncated simplicial sets defined only in low degrees and then extending to a full simplicial set using the so-called coskeleton. We also have:

Theorem. *A simplicial object satisfies ED if and only if it is a retract of a simplicial object that satisfies SED.*

The coskeleton of a simplicial object

Let Δ denote the category of finite ordinals and order preserving functions. Then the category of simplicial objects in $\mathcal{X}$ is the functor category $\mathcal{X}^{\Delta}$. If $[n] = \{0, 1, \ldots, n\}$, then d^i represents the injective mapping $[n-1] \longrightarrow [n]$ that omits the ith element i and s^i represents the surjective mapping $[n+1] \longrightarrow [n]$ that duplicates the ith element. If we let $\Delta_{(n)}$ be the full subcategory of Δ whose objects are $[0], [1], \ldots, [n]$, then a functor $\Delta_{(n)} \longrightarrow \mathcal{X}$ is called an n-truncated simplicial object. Assuming that $\mathcal{X}$ is sufficiently complete (in fact, only finite limits are required), the induced $\mathcal{X}^{\Delta} \longrightarrow \mathcal{X}^{\Delta_{(n)}}$ has a right Kan extension that extends an n-truncated simplicial object to a simplicial object called its coskeleton.

Example that ED $\not\Rightarrow$ SED

We use the coskeleton of the 2-truncated augmented simplicial set:

	X_{-1}	X_0	X_1		X_2			
	α	β	γ	δ	ϵ	ζ	η	θ
d^0		α	β	β	γ	δ	γ	δ
d^1			β	β	γ	γ	γ	δ
d^2					γ	γ	δ	δ
s^0		δ	η	θ				
s^1			ζ	θ				
t	β	γ	ϵ	ζ				

t is an ED, but no SED exists. The coskeleton will have the same property.

Reduced homotopies

The coskeleton works for categories, but there is a problem with homotopies. I assume there is some kind of 2-Kan extension, but there is no reason to suppose it will have the properties we need. Instead, we found a different way to look at homotopies. Let me repeat the homotopy equations:

1. $d^0 h^0 = f_n$ and $d^{n+1} h^n = g_n$
2. $d^i h^j = \begin{cases} h^{j-1} d^i & \text{if } 0 < i < j < n+1 \\ h^j d^{i-1} & \text{if } n+1 > i > j+1 > 0 \\ d^i h^{i-1} & \text{if } 0 < i = j < n+1 \end{cases}$
3. $s^i h^j = \begin{cases} h^j s^{i-1} & \text{if } i > j \\ h^{j+1} s^i & \text{if } i \le j \end{cases}$

One of them says $d^i h^i = d^i h^{i-1}$ when $0 < i < n+1$, but it doesn't say exactly what it should be. In addition, we have $d^0 h^0 = f$ and $d^{n+1} h^n = g$. In degree n, these are maps $X_n \longrightarrow Y_n$.

Reduced homotopies, continued

We now define maps $r^i : X_n \longrightarrow Y_n$ by

$$r^i = \begin{cases} f & \text{if } i = 0 \\ d^i h^i = d^i h^{i-1} & \text{if } 0 < i < n+1 \\ g & \text{if } i = n+1 \end{cases}$$

These r^i satisfy

1. $r^0 = f_n$;
2. $r^{n+1} = g_n$;
3. $d^i r^j = \begin{cases} r^{j-1} d^i & \text{for } i < j \\ r^j d^i & \text{for } i \geq j \end{cases}$
4. $s^i r^j = \begin{cases} r^{j+1} s^i & \text{for } i < j \\ r^j s^i & \text{for } i \geq j \end{cases}$

We call such a system of r^i a *reduced homotopy*.

Reduced homotopies, continued

Theorem. *Let $f, g : X \longrightarrow Y$ be simplicial maps between simplicial objects. Then there is a one-one correspondence between homotopies and reduced homotopies between f and g.*

In one direction we have already indicated the correspondence. Conversely, given a reduced homotopy, we let $h^i = r^{i+1}s^i$. Thus the reduced homotopy encapsulates the same information as a homotopy. Moreover:

Theorem. *If $f, g : X \longrightarrow Y$ are maps between n-truncated simplicial objects and r is an n-truncated reduced homotopy between f and g, then r extends to a reduced homotopy between their coskeletons.*

Counter-example, I

This is a counter-example to show that HC $\not\Rightarrow$ ED. It has to be split between two slides. We use $u = r^1_1,\ v = r^1_2, w = r^2_2$.

	Y_0	Y_1			Y_2		
	$*$	$*$	α	$u^n\alpha$	$*$	β	γ
d^0		$*$	$*$	$*$	$*$	α	$*$
d^1		$*$	$*$	$*$	$*$	α	α
d^2					$*$	$*$	
s^0	$*$	$*$	β	$w^n\beta$			
s^1	$*$	$*$	γ	$v^n\gamma$			
r^0	$*$	$*$	α	$u^n\alpha$	$*$	β	γ
r^1		$*$	$u\alpha$	$u^{n+1}\alpha$	$*$	$v\beta$	$v\gamma$
r^2		$*$	$*$	$*$	$*$	$w\beta$	$w\gamma$
r^3					$*$	$*$	$*$

Counter-example, II

	Y_2					
d^0	α	$*$	$u^n\alpha$	$*$	$u^\ell\alpha$	$*$
d^1	$u^n\alpha$	$u^n\alpha$	$u^n\alpha$	$u^n\alpha$	$u^{k+\ell}\alpha$	$u^{k+\ell}\alpha$
d^2	$*$	$u^n\alpha$	$*$	$*$	$*$	$*$
s^0						
s^1						
r^0	$v^n\beta$	$v^n\gamma$	$w^n\beta$	$w^n\gamma$	$v^k w^\ell\beta$	$v^k w^\ell\gamma$
r^1	$v^{n+1}\beta$	$v^{n+1}\gamma$	$vw^n\beta$	$vw^n\gamma$	$v^{k+1}w^\ell\beta$	$v^{k+1}w^\ell\gamma$
r^2	$v^n w\beta$	$v^{n+1}w\gamma$	$w^{n+1}\beta$	$w^{n+1}\gamma$	$v^k w^{\ell+1}\beta$	$v^k w^{\ell+1}\gamma$
r^3	$*$	$*$	$*$	$*$	$*$	$*$