Limit closure of categories of domains

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http://www.math.mcgill.ca/barr/papers

Abstract

This continues (under a better title) the talk I gave three weeks ago. I give some of the proofs of claims made then.

• \mathcal{A} is a category of domains.

- Every domain can embedded into a field that belongs to A.
- *K* is the limit closure of *A* in commutative rings.
- $\mathcal B$ consists of the domains in $\mathcal K$.
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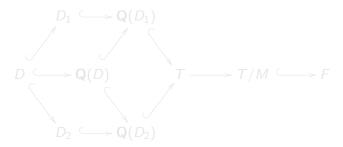
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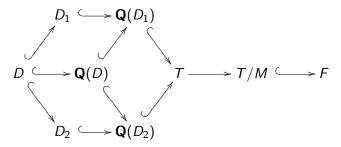
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As a result, we can compute G(D) by embedding D in a field $F \in \mathcal{A}$ and letting G(D) be the meet of all \mathcal{B} -subobjects of F that contain D.

The inclusion $D \longrightarrow G(D)$ is epic in SPR.

For suppose that $G(D) \xrightarrow{f}_{g} R$ are two maps that agree on D. Can suppose R is a field and even a field in \mathcal{A} , whence the equalizer of f and g will be a smaller \mathcal{B} -subobject that contains D

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Proof: Factor the map as $F \longrightarrow F_1 \longrightarrow F_2 \longrightarrow E$ into a pure transcendental extension followed by a purely inseparable extension followed by a separable extension.

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Epimorphisms of fields, cont'd

If F_1 were a proper extension of F, then it has many automorphisms that fix F. Let $\sigma \neq 1$ be one. For any $a \in F_2$, there is an integer k s.t. $a^{p^k} \in F_1$.

Then there is a unique element, call it $\overline{\sigma}(a) \in F_2$ s.t. $(\overline{\sigma}(a))^{p^k} = \sigma(a^{p^k})$. Then $\overline{\sigma}$ is an automorphism of F_2 that extends σ and therefore fixes F, a contradiction so that $F_1 = F$. For any domain D, G(D) is a subdomain of the perfect closure of $\mathbf{Q}(D)$. In characteristic 0, G(D) is a subdomain of $\mathbf{Q}(D)$.

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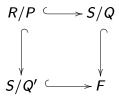
Suppose $f : R \hookrightarrow S$ is epic in SPR. Then Spec(f) is injective. Proof. Assume Q, Q' are primes of S s.t. $P = Q \cap R = Q' \cap R$. Use the amalgamation property to form the diagram



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Properties of K

Canonical $R \longrightarrow K(R)$ is injective and epic in SPR. Proof: Embed R into a product $\prod F_i$ of fields, which can be assumed to lie in \mathcal{A} . The embedding $R \hookrightarrow \prod F_i$ factors through K(R), by adjointness, and a first factor of an injection is an injection.

If $f, g: K(R) \longrightarrow S$ are two maps into a semiprime ring that agree on R, we can easily reduce to the case that S is a field in \mathcal{A} and then it follows by adjointness.

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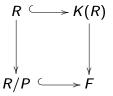
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Note, however, that this bijection is not an order isomorphism in general. Example: $\mathbf{Z} \longrightarrow \mathcal{K}_{\mathrm{fld}}(\mathbf{Z})$.

If $P \subseteq R$, let $P^{\mathfrak{G}} \subseteq K(R)$ be the kernel of $K(R) \longrightarrow K(R/P)$. Since K(R/P) is not generally a domain, there is no reason for $P^{\mathfrak{G}}$ to be prime.

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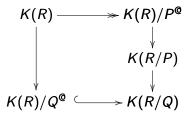
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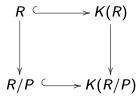


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DL-closed rings

We will be sketching the proof that \mathcal{K}_{dom} is the category of DL-closed rings. One thing that makes this case much easier is that when R is DL-closed and $P \subseteq R$ is a prime ideal, then R/P is DL-closed.

The proof that \mathcal{K}_{ic} is the category of (2,3)-closed rings is much harder and I will not attempt to do it in this talk.

The sheaf

The following construction is for the red case. There is a more more complicated one in the black case with less satisfactory results.

So suppose \mathcal{A} is a red category of domains, \mathcal{B} and \mathcal{K} as before. Given a ring R (always assumed semiprime), we build a sheaf as follows. The base is $\operatorname{Spec}(R)$ with the domain topology and the stalk above the prime P is G(R/P) = K(R/P).

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Global sections and K

$\Gamma(E) \in \mathcal{K}$. That is, the ring of global sections is in the limit closure.

I will not prove it, but it comes down to showing that the global sections of a sheaf can be constructed as a complicated limit starting with the stalks. Since the stalks belong to \mathcal{K} so does the ring of global sections.

 $\Gamma(E) \cong K(R)$ in such a way that the map $\zeta : R \longrightarrow \Gamma(E)$, defined by $\zeta(r)(P) = r + P$, is the adjunction morphism.

This holds in the red case and, under certain additional conditions, in others. It is not so easy, however.

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Before beginning the proof sketch, we must look more closely at the sheaf. If $\gamma \in \Gamma(E)$ and U is a compact open subset of $\operatorname{Spec}(R)$, we will say that the element $r \in R$ represents γ on U if for all $P \in U$, $\gamma(P) = r + P$, that is the image of r in R/P.

We say that $\gamma = r$ if r on U represents γ on U and that $\gamma = r$ if r represents γ on all of Spec(R).

We say that $(r_1, \ldots, r_n; U_1, \ldots, U_n)$ represents γ on $U = \bigcup U_i$ if r_i represents γ on U_i . Among other things, this requires that the U_i be compact and open.

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Local representations, cont'd

If $(U_1, U_2, \ldots, U_n; r_1, r_2, \ldots, r_n)$ represents γ on $U = U_1 \cup U_2 \cup \cdots \cup U_n$, then for all sufficiently large w there is an $a_w \in R$ such that $(U_1 \cup U_2, \ldots, U_n; a_w, (r_3 - r_1)^w, \ldots, (r_n - r_1)^w)$ represents $(\gamma - r_1)^w$ on U.

The relevance is this. We are going to be showing that when R is DL-closed, every global section is representable. Well, γ is representable iff $\gamma - r_1$ is. Second, just consider the case n = 2. Then this says that all sufficiently large powers of $\gamma - r_1$ are representable. If R is DL-closed, θ representable mod every prime and if θ^2 and θ^3 are representable, then θ is. But if θ^w is representable for all sufficiently large w, then so are $\theta^{2(w-1)}$ and $\theta^{3(w-1)}$ and hence θ^{w-1} and eventually θ .

An important observation is that since open sets are up-closed and the set of P for which $\gamma(P) = r$ is open, it follows that $P \subseteq Q$ implies $\gamma(Q) = r$.

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Proof: Assume that $r_1 = 0$. For i = 1, 2, let $J_i = \bigcap \{P \mid P \in U_i\}$. We claim that $r_2^w \in J_1 + J_2$ for all sufficiently large w. This is equivalent to showing that the image of r_2 belongs to every prime of $R/(J_1 + J_2)$ or equivalently, that r_2 belongs to every prime of Rthat contains both J_1 and J_2 .

So suppose that Q is such a prime. We can show (handout) that there exist $P_1 \in U_1$ and $P_2 \in U_2$ with $P_1 \subseteq Q$ and $P_2 \subseteq Q$. But $\gamma(P_1) = r_1 = 0$, which implies that $\gamma(Q) = 0$ since sections will agree on an open set and every open set is up-closed in the domain topology. Similarly $\gamma(P_2) = r_2$ which implies that $\gamma(Q) = r_2 = 0$ and thus $r_2 \in Q$, as claimed. For sufficiently large w, we can write $r_2^w = a_w + b_w$ with $a_w \in J_1$ and $b_w \in J_2$. If $P \in U_1$, we have that $0 = \gamma^w(P) = a_w$ since $a_w \in J_1 \subseteq P$. If $P \in U_2$, then $\gamma^w(P) = a_w + b_w = a_w$ since $b_w \in J_2 \subseteq P$ and so we see that $\gamma^w = a_w$ on all of $U_1 \cup U_2$ as required.

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• DL-1 \Rightarrow DL-2: This is the essence of the preceding development

- DL-2 ⇒ DL-1: Assume that a, b ∈ R are such that a has a square root mod every prime ideal and that a³ = b². Then mod every prime ideal P, there exists a unique c_P such that mod P, we have c_P² = a and c_P³ = b. Since Z(c_P² a) ∩ Z(c_P³ b) is open in the domain topology, these equations hold in a neighbourhood of P and the elements c must agree on overlaps by uniqueness. So they determine a section γ. But by DL-2, γ ∈ R and so R satisfies DL-1.
- DL-2 ⇒ DL-3 ⇒ DL-4: The first is obvious, while the second is a consequence of the fact that the ring of global sections is in the limit closure of the stalks.
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