Two talks

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Talk 1: Limit closures of some full subcategories

Abstract: John Kennison, Bob Raphael, and I have recently been working on the question of the limit closure of a full subcategory of a complete category inside that larger category. I will report on the results of these investigations in three cases:

1. The subcategory of metric spaces and uniformly continuous maps inside the category of separated uniform spaces.
2. The subcategory of integrally closed domains inside the category of commutative rings.
3. The subcategory of all integral domains inside the category of commutative rings.

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Limit closure

For our purposes, assume all subcategories full. $\mathcal{A}$ is a subcategory of the complete category $\mathcal{C}$. At least two definitions of limit closure are possible. Fortunately, they coincide.

1. $\mathcal{K}_1 = \bigcap\{\mathcal{B} \mid \mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B}$ is limit closed in $\mathcal{C}\}$
2. $\mathcal{K}_2 = \bigcup \mathcal{B}_\alpha$ taken over all ordinals α, where:

$$\mathcal{B}_\alpha = \begin{cases} \prod \mathcal{A} & \text{if } \alpha = 0 \\ \prod \bigcup_{\beta<\alpha} \mathcal{B}_\beta & \text{if } \alpha \text{ is a limit ordinal} \\ \mathrm{Eq}\,\mathcal{B}_\beta & \text{if } \alpha = \beta + 1 \end{cases}$$

Note, in connection with $\mathcal{K}_2$, that all the $\mathcal{B}_n$, for finite n, are product closed while $\bigcup \mathcal{B}_n$ is equalizer closed, but may have only finite products. All the $\mathcal{B}_\alpha$, for $\omega \leq \alpha < \Omega$, are product closed, while $\bigcup_{\alpha<\Omega}$ is equalizer closed, but may have only countable products, and so on.

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$\mathcal{K}_1 = \mathcal{K}_2$

If $\mathcal{B} \subseteq \mathcal{K}_1$, then both $\prod \mathcal{B}$ and $\mathrm{Eq}\,\mathcal{B}$ are still contained in $\mathcal{K}_1$, so $\mathcal{K}_2 \subseteq \mathcal{K}_1$. On the other hand $\mathcal{K}_2$ is a limit closed subcategory containing $\mathcal{A}$, so $\mathcal{K}_1 \subseteq \mathcal{K}_2$.
We denote either limit closure by $\mathcal{K}$.

1. Uniform spaces

Three definitions of uniform spaces, which we will always assume to be separated.

1. By entourages: subsets of the square containing the diagonal, subject to standard hypotheses. E.g., in a metric space, the ϵ neighbourhoods of the diagonal.
2. By uniform covers, again subject to standard hypotheses. E.g., in a metric space, covers by ϵ neighbourhoods of the points.
3. By a family of pseudometrics, again subject to standard hypotheses. E.g., in a metric space, the metric itself.
   A pseudometric is just like a metric except that the distance between distinct points may be 0. But to be separated requires that the family collectively distinguishes points.

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Our first guess

Our first hypothesis was that every (separated) uniform space was a limit of metric spaces. It is trivial that if you drop the separation hypothesis, then every uniform space is a limit of pseudometric spaces. So the interesting question is for separated uniform spaces. Not surprisingly, the third definition, in terms of pseudometrics is the most useful in answering this question.

Complete and Cooper complete spaces

A net $\{x_i\}$ in a uniform space is a Cauchy net if for all pseudometrics d and all $\epsilon > 0$ there is an i such that $j, k \geq i$ implies $d(x_j, x_k) < \epsilon$. The space is complete if every Cauchy net converges.

The net is **strongly Cauchy** if for all pseudometrics d there is an i such that $j \geq i$ implies $d(x_i, x_j) = 0$ (which implies that for all $j, k \geq i$, $d(x_j, x_k) = 0$). We call the space **Cooper complete** if every strongly Cauchy net converges.

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The theorem

Evidently, every metric space is Cooper complete. Since the condition is easily seen to be preserved under limit, it follows that every limit of metric spaces is Cooper complete. Therefore the following theorem is not surprising:

Theorem: *The limit completion of metric spaces is the subcategory of Cooper complete spaces.*

The proof, while not hard, requires more development than I have time for. If there is time at the end, I will sketch it.

Not every uniform space is Cooper complete

I will sketch this example, due to Cooper. The space is the set Ω of all countable ordinals with the uniformity inherited from $\Omega + 1$, the compact space of ordinals up to and including Ω. The net is the space itself, with $x_\alpha = \alpha$, which clearly lacks a limit. On the other hand, every pseudometric extends to $\Omega + 1$ since **R** is complete. But standard theorems of analysis imply that every real-valued function on $\Omega + 1$ is eventually constant, whence the strong Cauchy condition is satisfied.

2. Integrally closed domains

An integral domain D is called **integrally closed** if every integral polynomial $a_0 + a_1x + \dots a_{n-1}x^{n-1} + x^n$ that has a root in its field F of fractions already has a root in D. This is a bit of *contre sens* but it is a long-used definition in ring theory.

(2,3)-closed

We begin with: A commutative ring is called **semiprime** if it has no non-zero nilpotents. Clearly any limit of domains is semiprime and we can thus limit the ambient category to the category of semiprime rings.

Next we note that if $b^3 = c^2 \neq 0$ is an element of D, then the element $a = c/b \in F$ is readily seen to be a root to the integral equations $x^2 - b = 0$ and $x^3 - c = 0$. The case that $b^3 = c^2 = 0$ trivially has a root.

This leads to the definition: We say that a commutative semi-prime ring R is **(2,3)-closed** if whenever $b^3 = c^2 \in R$, then there is a (provably unique) $a \in R$ such that $a^2 = b$ and $a^3 = c$. Thus integrally closed domains are (2,3)-closed

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Uniqueness

The uniqueness follows easily from showing that if a_1, a_2 are two solutions then $(a_1 - a_2)^3 = 0$.
Although we make no use of the fact, using the obvious definition of (k, ℓ)-closed, one can show that a semi-prime ring is (2,3)-closed if and only if it is (k, ℓ)-closed for some pair of relatively prime positive integers k, ℓ.

The main theorem

Theorem. *A semi-prime ring is in the limit closure of integrally closed domains if and only if it is (2,3)-closed.*

The "if" part of the proof is a long, rather involved argument that uses, for each ring R, a sheaf on $\mathrm{Spec}(R)$ whose stalk above the prime ideal P is the meet of all (2,3)-closed domains containing R/P.

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The "only if" part

An integrally closed domain is (2,3)-closed.

The conclusion then follows by using the transfinite inductive definition of the limit closure and then using the fact that there is an essentially algebraic theory whose models are the (2,3)-closed rings.

Start with the algebraic theory of commutative rings and add two partial operations. The first is a unary partial operation σ whose domain is $\{x \mid x^2 = 0\}$ and satisfies the equations $\sigma(x) = x$ and $\sigma(x) = 0$. This characterizes commutative semi-prime rings. The second, τ, has the domain $\{(x, y) \mid x^3 = y^2\}$ and satisfies the equations $\tau(x, y)^2 = x$ and $\tau(x, y)^3 = y$. This clearly characterizes (2,3)-closure.

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A (2,3)-closed domain that is not integrally closed.

Let $\mathbf{Z}[i,t]$ denote the ring of polynomials over the Gaussian integers. It is a UFD and therefore integrally closed. Let $R \subseteq \mathbf{Z}[i,t]$ denote the subring consisting of the polynomials whose constant term is real. Then

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \mathbf{Z}[i,t] \arrow[d] \\ \mathbf{Z} \arrow[r] & \mathbf{Z}[i] \end{tikzcd}$$

is a pullback so that R is (2,3)-closed. But the element $i = it/t$ in the field of fractions of R is obviously integral and does not belong to R.

3. Domains

Next I want to describe briefly the case that $\mathcal{A}$ consists of all domains. The main definition we need is:

The commutative semiprime ring R is **DL-closed** if, whenever $b, c \in R$ are such that $b^3 = c^2$ and also b is a square mod every prime ideal, then there is a (unique) $a \in R$ such that $a^2 = b$ and $a^3 = c$. Clearly every domain is DL-closed since 0 is a prime ideal.

Theorem. *A ring is in the limit closure of domains if and only if it is DL-closed.*

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The domain topology

For the "only if" part, we again show, somewhat surprisingly, that DL-closure can also be described by an essentially algebraic theory. We briefly explore this.

The domain (sometimes called co-Zariski) topology on $\mathrm{Spec}(R)$ takes as a subbase all sets $N(r) = \{P \mid r \in P\}$ for $r \in R$. It is a result of Hochster that this space is also compact. In fact, Hochster showed that the topology that takes all these sets and their complements as a subbase is even compact and Hausdorff.

Proposition. *If $b \in R$ is a square mod every prime ideal, then there are finitely many $a_1, a_2, \ldots, a_n$ such that $(b - a_1^2)(b - a_2^2) \cdots (b - a_n^2) = 0$.*

The proof uses the fact that for each prime ideal P, there is an element $a_P \in R$ such that $b - a_P^2 \in P$ and therefore the sets $N(b - a_P^2)$ cover $\mathrm{Spec}(R)$, whence a finite number do.

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DL-closed is an essentially equational condition

At first glance, it doesn't even appear to be first order. But armed with the above result, we let, for each $n > 0$, an $(n+2)$-ary operation τ_n whose domain is

$$\{x_1, x_2, \ldots, x_n, y, z \mid (y - x_1^2)(y - x_2^2)\cdots(y - x_n^2) = y^3 - z^2 = 0\}$$

This is subject to the conditions $\tau_n(x_1, \ldots, x_n, y, z)^2 = y$ and $\tau_n(x_1, \ldots, x_n, y, z)^3 = z$. The equations defining commutative rings, the set of all these τ_n, together with the σ defined above to force semiprimeness are a near equational description of DL-closed rings.

Sketch of the proof for uniform spaces

If X is a uniform space and d a pseudometric, let E_d be the equivalence relation defined by xE_dy when $d(x,y) = 0$. Then $X_d = X/E_d$ is a metric space and the quotient map is uniform. If X is separated, this embeds it into embeds $Y = \prod_d X_d$. If this embedding is closed, then it can be shown, perhaps surprisingly, that the space Y/X gotten by shrinking X to a point is still a separated uniform space and can therefore be embedded into a product of metric spaces, rendering X as an equalizer of two maps between two maps of products of metric spaces. Of course, such an equalizer is always closed.

The converse implication

It therefore suffices to show that when X is Cooper complete, then the embedding $X \longrightarrow Y$ is always closed.
Since the sup of two pseudometrics is a pseudometric, we can and will assume that the set of pseudometrics is directed by numerical order. Let $q_d : X \longrightarrow X_d$ be the quotient mapping, $p_d : Y \longrightarrow X_d$ be the product projection. When $d \leq e$, there is also a quotient mapping $q_{de} : X_e \longrightarrow X_d$ such that $q_{de} q_e = q_d$. Since $p_d | X = q_d$ and $p_e | X = q_e$, we also have that $d \leq e$ implies that $q_{de} p_e | X = p_d | X$ which, since all the spaces are separated, implies that $q_{de} p_e | \mathrm{cl}(X) = p_d | \mathrm{cl}(X)$.

The converse, continued

To repeat, $q_{de} p_e | \mathrm{cl}(X) = p_d | \mathrm{cl}(X)$.
Suppose $y \in \mathrm{cl}(X)$. Since every q_d is surjective, we have, for each d an element $x_d \in X$ such that $q_d(x_d) = p_d(y)$. For $d \leq e$, we have

$$q_d(x_e) = q_{de} q_e(x_e) = q_{de} p_e(y) = p_d(y) = q_d(x_d)$$

which is possible if and only if $d(x_d, x_e) = 0$. Since this is true whenever $d \leq e$, it follows that the net of x_d is strongly Cauchy and hence converges in X. Since it also converges to y and the spaces are separated, it follows that $y \in X$ and so X is closed in Y.

Some other results

In all three cases below, $\mathcal{A}$ is a subcategory of uniform spaces consisting of a single object and its endomorphisms.

1. If the object is the two point discrete space, $\mathcal{C}$ consists of the totally disconnected compact Hausdorff spaces with their unique uniformity.
2. If the object is the unit interval, $\mathcal{C}$ consists of all compact Hausdorff spaces with their unique uniformity.
3. If the object is the real line with its usual metric, $\mathcal{C}$ consists of all complete uniform spaces.

Talk 2: Completions of subcategories of domains

Abstract: We have been studying the limit completion, in the category of commutative rings, of various subcategories of integral domains. Since any limit of domains is a semiprime ring (only nilpotent is 0), we will concentrate on the limit closure in that subcategory.

Some subcategories of domains

- $\mathcal{A}_{\mathrm{dom}}$, the category of domains;
- $\mathcal{A}_{\mathrm{fld}}$, the category of fields;
- $\mathcal{A}_{\mathrm{pfld}}$, the category of perfect fields;
- $\mathcal{A}_{\mathrm{per}}$, the category of perfect domains;
- $\mathcal{A}_{\mathrm{ic}}$, the category of integrally closed domains;
- $\mathcal{A}_{\mathrm{ica}}$, the category of absolutely integrally closed domains;
- $\mathcal{A}_{\mathrm{icp}}$, the category of perfect integrally closed domains;
- $\mathcal{A}_{\mathrm{qrat}}$, the category of quasi-rational domains;
- $\mathcal{A}_{\mathrm{noe}}$, the category of Noetherian domains;
- $\mathcal{A}_{\mathrm{bez}}$, the category of Bézout domains;
- $\mathcal{A}_{\mathrm{ufd}}$, the category of unique factorization domains.

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Some subcategories of domains

- $\mathcal{A}_{\mathrm{dom}}$, the category of domains;
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- $\mathcal{A}_{\mathrm{pfld}}$, the category of perfect fields;
- $\mathcal{A}_{\mathrm{per}}$, the category of perfect domains;
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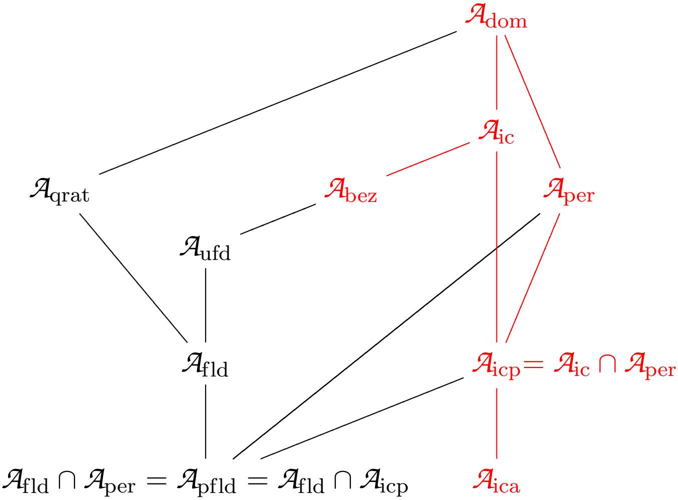
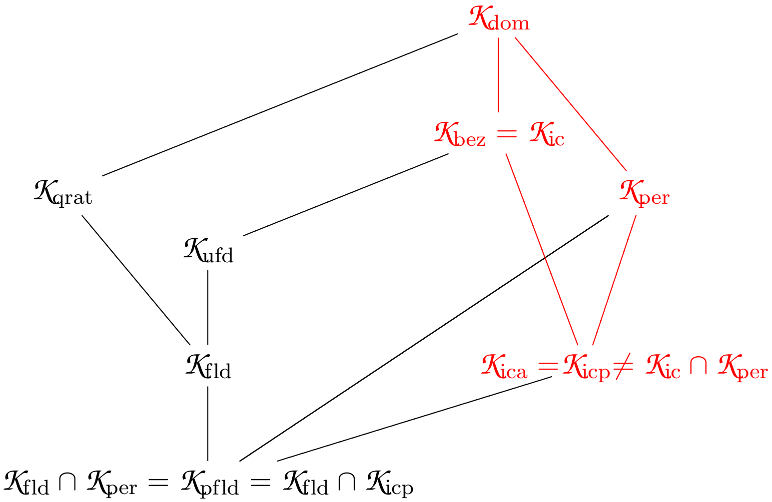


Relations among their limit closures.



Basic assumptions

- $\mathcal{A}$ is a category of domains (such as one of the above).
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Construction of K and G

$K : \mathcal{SPR} \longrightarrow \mathcal{K}$ is the adjoint to the inclusion of $\mathcal{K}$ into the category of semiprime rings, easily shown to exist.

G is more interesting. Let $\mathcal{B} \subseteq \mathcal{K}$ consist of all domains in $\mathcal{K}$. In most cases it is larger than $\mathcal{A}$.

Example: Let D be the pullback of three UFDs

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D \arrow[r] \arrow[d] & \mathbf{Z}[t] \arrow[d] \\
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\end{tikzcd}$$

it consists of polynomials in t whose odd order terms are even. $D \in \mathcal{B}_{\mathrm{ic}}$ but is not integrally closed since $t = 2t/2 \notin D$ satisfies the integral equation $x^2 - t^2$ with coefficients in D.

For a domain D we let $G(D)$ denote the intersection of all objects of $\mathcal{B}$ that contain D. There is at least one since there is a field in $\mathcal{A}$ that contains D.

• Suppose $D \subseteq F \in \mathcal{A}$ with F a field. Then $G(D)$ is the intersection of all $\mathcal{B}$-subobjects of F that contain D.

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Diagram of logical inferences

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\text{6. kernel prime} \arrow[d, Rightarrow] & \text{3. } G(D) \text{ to } G(D/P) \arrow[r, Rightarrow] & \text{4. Spec surj on } G \arrow[u, Rightarrow] \\
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Why is it important that K be domain invariant?

Or that G be a functor?

Or that $G = K$, etc.?

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