(Marks)

(4) 1. For the function $f(x) = \sqrt[5]{x}$:

    (a) Find the Taylor series for $f(x)$ centered at $x = 32$.

    (b) Write the third degree Taylor polynomial $T_3(x)$ for the function $f(x)$ centered at $x = 32$.

    (c) Use $T_3(30)$ to estimate $\sqrt[5]{30}$, and use Taylor's Formula/Inequality or Lagrange's form of the remainder to estimate the accuracy of this estimate.

(5) 2. Let $g(x) = \int_0^x \dfrac{1 - e^{-t}}{t}\, dt$

    (a) Find the Maclaurin series for $g(x)$; express your answer in Σ notation. What is the interval of convergence for this series?

    (b) Use this series to approximate $g(0.2)$ to within 10^{-3}. Justify the correctness of your approximation.

(5) 3. (a) What is the Maclaurin series for $\dfrac{1}{1 + x^2}$?

    (b) What is the interval of convergence of this series?

    (c) Using this series, find the Maclaurin series of $\arctan(x)$.

    (d) What is the radius of convergence for this series?

    (e) Use this series to obtain the sum of the series $\displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n + 1)3^n \sqrt{3}} = \dfrac{1}{\sqrt{3}} - \dfrac{1}{9\sqrt{3}} + \dfrac{1}{45\sqrt{3}} \mp \cdots$

(8) 4. Consider the following polar curves: $r_1 = 1 + 2\cos\theta$ and $r_2 = 2$.

    (a) Sketch the graphs on the same axes.

    (b) Find all points of intersection (in Cartesian coordinates).

    (c) Find the area inside r_1 and outside r_2.

    (d) Set up, but do not evaluate, the integral needed to find the length of the inner loop of r_1.

(9) 5. Sketch and name each of the following surfaces in $\mathbb{R}^3$. Show all relevant work.

    (a) $x^2 - y^2 - z^2 = 4$ (b) $\rho = 2\sin\varphi\sin\theta$ (c) $z + r^2 = 1$

(8) 6. Given the curve with parametric equations $x = t^2 + 2t$, $y = 3t^2 - t^3$:

    (a) Find the x and y-intercepts.   (b) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. Simplify your answers.

    (c) Locate all points where the tangent is horizontal or vertical (identify which is which).

    (d) Sketch the curve showing all these points and the intercepts, and indicate with an arrow the direction of increasing t values (the orientation).

    (e) Set up the integral to find the area of the region in Quadrant II ($x \le 0, y \ge 0$) between the curve and the y-axis.

    (f) Set up the integral to find the arc length of that section of the curve (in Quadrant II).

(8) 7. A particle P moves along a curve $\boldsymbol{r}(t) = (t + \cos(t))\boldsymbol{i} + (t - \cos(t))\boldsymbol{j} + \sqrt{2}\sin(t)\,\boldsymbol{k}$.

    (a) Calculate the length of the curve from $t = 0$ to $t = 2\pi$.

    (b) Find the unit tangent vector $\boldsymbol{T}(t)$, the unit normal vector $\boldsymbol{N}(t)$, the curvature $\kappa(t)$, and the tangential and normal components a_T, a_N of acceleration.

(Marks)

(3)   8. The dimensions of a closed rectangular box are measured as 90 centimeters, 70 centimeters, and 80 centimeters, respectively, with the error in each measurement at most 0.2 centimeters. Use differentials to estimate the maximum error in calculating the surface area of the box.

(3)   9. Let $z = f(x, y)$ and $z = g(x, y)$ be two surfaces which intersect at the origin, so that f and g are differentiable at the origin. Show that the tangent planes to the two surfaces at the origin are perpendicular if and only if $\dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial y} = -1$ at the origin.

(6)   10. Calculate the following limits; if a limit does not exist, say so (and mention $\pm\infty$ if appropriate).

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2 y}{x^4 + y^2}$ (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^4 + y^2}$ (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - y^2}{x^2 - y}$

Be sure to justify your answers.

(3)   11. Suppose $z = f(u, v), u = x^2 - y^2, v = 2xy$, where f is a differentiable function.

Show that $x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} = 2(x^2 + y^2)\dfrac{\partial f}{\partial u}$.

(5)   12. Given the surface $\mathcal{S}$: $f(x, y, z) = 25x^2 + 4y^2 + 4z^2 = 33$ and the point $P(1, 1, 1)$, find:

(a) the directional derivative of f at the point P in the direction of $\boldsymbol{v} = \langle 1, 2, 3\rangle$;

(b) the equation of the tangent plane to the surface $\mathcal{S}$ at P;

(c) the minimum rate of change in f at P.

(5)   13. Use the method of Lagrange Multipliers to find the maximum and minimum values of the function $2x + 4y + 3z$ subject to the constraint $x^2 + 4y^2 + 2z^2 = 800$.

(5)   14. Find and classify the critical points of $f(x, y) = 2x^3 - 6xy + 3y^2$.

(9)   15. (a) Evaluate $\displaystyle\int_0^1 \int_0^{\arccos y} \sin x \, e^{\sin^2 x} \, dx \, dy$ (b) Evaluate $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2 - y^2} \, dy \, dx$

(c) Evaluate $\displaystyle\iiint_{\mathcal{H}} \sqrt{x^2 + y^2 + z^2} \, dV$ where $\mathcal{H}$ is the region $x^2 + y^2 + z^2 \leq 4, x \geq 0$.

(4)   16. Let $\mathcal{R}$ be the region $0 \leq z \leq 3, 0 \leq y \leq 4, 0 \leq x \leq \sqrt{16 - y^2}$. Set up appropriate integrals to calculate the volume of $\mathcal{R}$ in (a) Cartesian coordinates, and (b) cylindrical coordinates. (You do not have to evaluate the integrals.)

(5)   17. Find the volume of the region bounded below by the cone $z = \sqrt{3x^2 + 3y^2}$ and bounded above by the sphere $x^2 + y^2 + z^2 = 16$.

(5)   18. Let $\mathcal{D}$ be the region bounded by $x - 2y = -2, x + 2y = 2, x + 2y = -2$, and $x - 2y = 2$. Evaluate $\displaystyle\iint_{\mathcal{D}} (3x + 6y)^2 \, dx dy$.   *Hint: Use the change of variables $u = x + 2y$ and $v = x - 2y$.*

1. (a) $2\left(1 + \dfrac{x-32}{5\cdot 32} + \dfrac{(\frac{1}{5})(-\frac{4}{5})}{2!\cdot 32^2}(x-32)^2 + \dfrac{(\frac{1}{5})(-\frac{4}{5})(-\frac{9}{5})}{3!\cdot 32^3}(x-32)^3 + \cdots\right)$

   (b) $T_3 = 2 + \frac{1}{80}(x-32) - \frac{1}{6400}(x-32)^2 + \frac{3}{1024000}(x-32)^3$

   (c) $\sqrt[5]{30} \simeq 2 - \frac{1}{40} - \frac{1}{1600} - \frac{3}{128000} = 1.9743516 \pm 1.31 \times 10^{-6}$

2. (a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n\cdot n!}$,   I of C: $(-\infty, \infty)$

   (b) $g(0.2) \simeq 0.19 \pm 0.00044$ by the AST form of the remainder: $|R_2(0.2)| < \frac{0.2^3}{3\cdot 3!} \simeq 0.00044$.

3. (a) $1 - x^2 + x^4 - x^6 \pm \cdots = \displaystyle\sum_{n=0}^{\infty}(-1)^n x^{2n}$ (b) $(-1, 1)$

   (c) $x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 \pm \cdots = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$
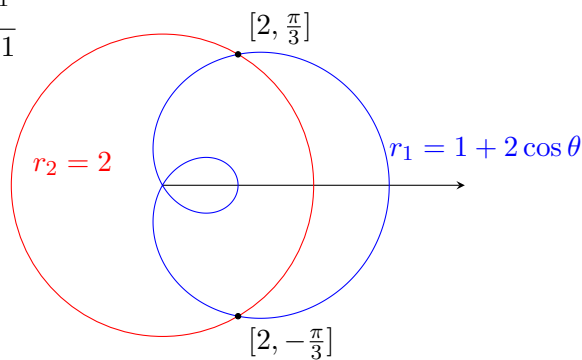
   (d) $R = 1$ (e) $\arctan(1/\sqrt{3}) = \pi/6$
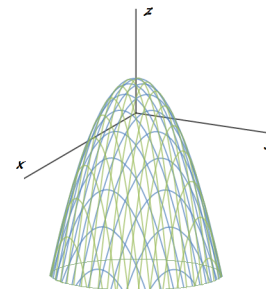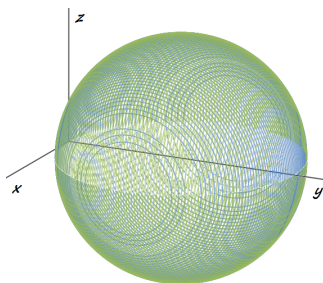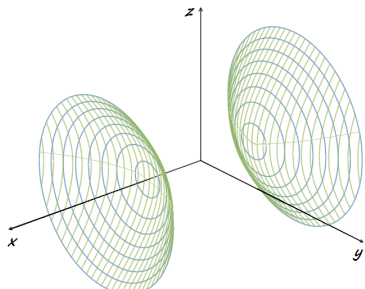
4. (a) Graph at right

   (b) $(1, \pm\sqrt{3})$ (in polar at right:)

   (c) $\frac{5}{2}\sqrt{3} - \frac{1}{3}\pi$

   (d) $\displaystyle\int_{2\pi/3}^{4\pi/3}\sqrt{4\sin^2\theta + (1+2\cos\theta)^2}\,d\theta$

   (Figure: $r_2 = 2$, $r_1 = 1 + 2\cos\theta$, points $[2, \frac{\pi}{3}]$ and $[2, -\frac{\pi}{3}]$)

5. (a) Hyperboloid of 2 sheets   (b) sphere (radius 1, center $(0,1,0)$)   (c) circular paraboloid
   Graphs:

6. (a) $(0,0), (15,0), (0,20)$

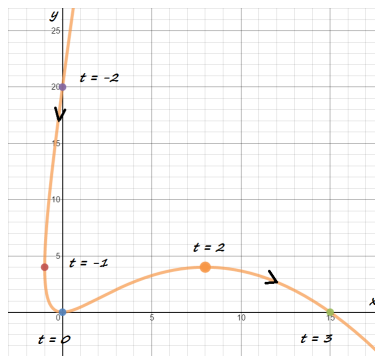   (b) $\dfrac{dy}{dx} = \dfrac{3t(2-t)}{2(1+t)}$, $\dfrac{d^2y}{dx^2} = \dfrac{(6-6t)(2t+2) - 2(6t-3t^2)}{(2t+2)^3}$

   (c) HT at $(0,0), (8,4)$; VT at $(-1,4)$

   (d) Graph at right

   (e) $\int_{y=0}^{y=20}(-x)\,dy = \int_0^{-2}-(t^2+2t)(6t-3t^2)\,dt$
       $= \int_{-2}^0 (t^2+2t)(6t-3t^2)\,dt$

   (f) $\int_{-2}^0 \sqrt{(2t+2)^2 + (6t-3t^2)^2}\,dt$

7. (a) 4π

   (b) $\boldsymbol{v} = \langle 1-\sin t, 1+\sin t, \sqrt{2}\cos t\rangle$, $v = 2$, $\boldsymbol{T} = \frac{1}{2}\langle 1-\sin t, 1+\sin t, \sqrt{2}\cos t\rangle$,
   $\boldsymbol{T}' = \frac{1}{2}\langle -\cos t, \cos t, -\sqrt{2}\sin t\rangle$, $|\boldsymbol{T}'| = \frac{1}{\sqrt{2}}$, $\boldsymbol{N} = \frac{1}{\sqrt{2}}\langle -\cos t, \cos t, -\sqrt{2}\sin t\rangle$,
   $\kappa = \frac{\sqrt{2}}{4}$, $a_T = 0$, $a_N = \sqrt{2}$

8. Max error $= 192 \text{cm}^2$

9. The two normals are $\langle f_x, f_y, -1 \rangle, \langle g_x, g_y, -1 \rangle$ and are perpendicular if and only if their dot product is 0, so: $f_x g_x + f_y g_y + 1 = 0$ (qed).

10. (a) The limit does not exist. Two paths give different limits (*e.g.* along $y = x^2$ and $x = 0$).
    (b) The limit $= 0$, using the squeeze theorem.
    (c) The limit $= 0$, simply factoring, cancelling, and letting $(x, y) = (0, 0)$.

11. Use $\frac{\partial z}{\partial x} = 2x \frac{\partial f}{\partial u} + 2y \frac{\partial f}{\partial v}$, $\frac{\partial z}{\partial y} = -2y \frac{\partial f}{\partial u} + 2x \frac{\partial f}{\partial v}$.

12. (a) $90/\sqrt{14}$ (b) $25x + 4y + 4z = 33$ (c) $-\sqrt{2628} = -6\sqrt{73}$

13. Max 100, at $(16, 8, 12)$, and Min -100, at $(-16, -8, -12)$.

14. Saddle at $(0, 0)$, local min at $(1, 1)$.

15. (a) $\frac{1}{2}(e - 1)$ (b) $\frac{\pi}{4}(1 - e^{-4})$ (c) 8π

16. (a) $\displaystyle \int_0^4 \int_0^{\sqrt{16-y^2}} \int_0^3 dz\, dx\, dy$

    (b) $\displaystyle \int_0^{\pi/2} \int_0^4 \int_0^3 r\, dz\, dr\, d\theta$

17. $\frac{64}{3}\pi(2 - \sqrt{3})$

18. $\displaystyle \int_{-2}^2 \int_{-2}^2 \frac{9}{4} u^2\, du\, dv = 48$